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Reflexivity, the dual Radon-Nykodym property, and continuity of adjoint semigroups

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In this note for certain Banach spaces we give characterizations of reflexivity and the dual Radon-Nykodym property in terms of continuity of adjoint semigroups. Some applications outside the realm of semigroup theory are given.

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0. INTRODUCTION

Let $T(t)$ be a C_0-semigroup on a Banach space X. It is well-known that the adjoint semigroup $T^*(t) = (T(t))^*$ need not be strongly continuous on X^*. However, if X is reflexive, it is; this is a theorem of R.S. Phillips [15]. In this note we will prove a converse. The idea is as follows. First, it is shown that on every infinite-dimensional Banach space with a Schauder decomposition, a C_0-semigroup with an unbounded generator can be constructed in a canonical way. Next, every nonreflexive space with a finite-dimensional Schauder decomposition (FFD) also has a nonshrinking FDD. This provides us with elements $x^* \in X^*$ with certain properties that can be used to show that the canonical semigroup mentioned above has no strongly continuous adjoint.
To be precise, we have

Theorem A. *Let X be a Banach space with an FDD. The following statements are equivalent:*

(1) X is reflexive;
(2) X is a Grothendieck space;
(3) Every adjoint semigroup on X^* is strongly continuous.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold in every Banach space.*

In general Banach spaces, Theorem A can be combined with the well-known fact that a Banach space is reflexive if and only if each of its subspaces with a Schauder basis is reflexive. In particular, Theorem A states that Grothendieck spaces with an FDD are reflexive. More generally, W. B. Johnson [11] proved that a Grothendieck space with a Markusevich basis is reflexive, hence in particular separable Grothendieck spaces are reflexive.
It follows from Theorem A that Grothendieck spaces with the Dunford-Pettis property cannot have a Schauder decomposition. This was first observed by D.W. Dean [4]; see also [13]. Using the same techniques, we give a very simple proof of the well-known fact (e.g., see [4]) that weak Schauder decompositions are if fact strong decompositions.

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The Radon-Nykodym property is in many ways a close analogue of reflexivity. Here we will show that a weak*-continuous semigroup on a dual Banach space with the Radon-Nykodym property is strongly continuous for $t > 0$. In this setting it turns out to be useful to consider Banach spaces with an unconditional basis, since on them C_0-semigroups can be constructed in a canonical way such that, when X^* is nonseparable, the adjoint semigroup fails to be strongly continuous even for $t > 0$. These observations, together with the fact that separable duals have the Radon-Nykodym property, indicate what ideas lie behind the following theorem.

Theorem B. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^{\infty}$. The following statements are equivalent:*

(1) X^* is separable;
(2) X^* has the Radon-Nykodym property;
(3) Every adjoint semigroup on X^* is strongly continuous for $t > 0$.

The implications $(1) \Rightarrow (2) \Rightarrow (3)$ *hold in every Banach space.*

In fact, if $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis for X, we will show that (1) – (3) hold if and only if $\{x_n\}_{n=1}^{\infty}$ is shrinking. By a theorem of R.C. James (see [12]), this is the case if and only if X does not contain a subspace isomorphic to l^1. More generally, H.P. Lotz proved that for Banach lattices X, X^* has the Radon-Nykodym property if and only if X does not contain a subspace isomorphic to l^1; see [8].
It should be noted that a Banach space with an unconditional FDD is isomorphic with a space with an unconditional basis [12]; therefore no extra generality is gained by introducing FDDs in the setting of Theorem B.

This note is organized as follows. In paragraph 1 we will give some definitions and standard results which will be used afterwards. After that, paragraphs 2 and 3 are concerned with Theorems A and B, respectively. In paragraph 4 our results are applied to bases in c_0.

1. PRELIMINARIES

A one-parameter family $\{T(t)\}_{t\geq 0}$ (briefly, $T(t)$) of bounded linear mappings from a Banach space X into itself is called a *semigroup* if the following two conditions are satisfied:

(1) $T(0) = I$ (I the identity map of X);

(2) $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$.

A *strongly continuous* semigroup (also called a C_0-semigroup) is a semigroup that satisfies

(3) $\|T(t)x - x\| \to 0 \quad (t \downarrow 0) \quad$ for all $x \in X$.

The *generator* A of a C_0-semigroup $T(t)$ is defined by

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x) \text{ exists } \};$$

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x) \quad (x \in D(A)).$$

A semigroup $T^*(t)$ on a dual space X^* is called an *adjoint* semigroup if there is a C_0-semigroup $T(t)$ on X such that $(T(t))^* = T^*(t)$ for all $t \geq 0$. An adjoint semigroup need not be strongly continuous. Therefore it makes sense to define

$$X^{\odot} = \{x^* \in X^* : \|T^*(t)x^* - x^*\| \to 0 \quad (t \downarrow 0)\}.$$

Of course, in general $X^{\odot}$ depends on the particular semigroup under consideration.

We will need the following properties of C_0-semigroups and their adjoints [3,10,16].

Proposition 1.1. *Let $T(t)$ be a C_0-semigroup on a Banach space X.*

(1) There exist real constants $M \geq 1$ and ω such that $\|T(t)\| \leq Me^{\omega t}$.

(2) The adjoint semigroup $T^*(t) = (T(t))^*$ is weak-continuous, that is,*

$$\langle T^*(t)x^* - x^*, x\rangle \to 0 \quad (t \downarrow 0)$$

for all $x \in X$.

(3) $X^{\odot}$ is a norm-closed, weak-dense subspace of X^*.*

Proposition 1.2. *Let $T(t)$ be a semigroup on a Banach space X.*

(1) If the map $t \to T(t)x$ is measurable for all $x \in X$ then $T(t)$ is strongly continuous for $t > 0$.

(2) If $T(t)$ is weakly continuous (that is, $\langle x^*, T(t)x - x\rangle \to 0$ $(t \downarrow 0)$ for all $x^* \in X^*$) then $T(t)$ is strongly continuous.

A countable collection of closed subspaces $\{X_n\}_{n=1}^{\infty}$ of a Banach space X is called a *Schauder decomposition* of X if for every $x \in X$ there is a unique sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x = \sum_{n=1}^{\infty} x_n$ and for each n, $x_n \in X_n$. If $\dim X_n < \infty$ for each n, then $\{X_n\}_{n=1}^{\infty}$ is called a *finite-dimensional* Schauder decomposition (briefly, FDD). A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a *Schauder basis* (briefly, basis) if for every $x \in X$ there exists a unique sequence $\{\alpha_n\}_{n=1}^{\infty}$ of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. A basis $\{x_n\}_{n=1}^{\infty}$ is called *normalized* if $\|x_n\| = 1$ for all n.

A basis $\{x_n\}_{n=1}^{\infty}$ is called *unconditional* if for every $x \in X$ the expansion $\sum_{n=1}^{\infty} \alpha_n x_n$ of x converges unconditionally, that is, for every permutation σ of the positive integers, $\sum_{n=1}^{\infty} \alpha_{\sigma(n)} x_{\sigma(n)}$ converges.

$\{x_n\}_{n=1}^{\infty}$ is called *shrinking* if $\lim_{N\to\infty} \|x^*|_{[x_N, x_{N+1}, \ldots]}\| = 0$ for every $x^* \in X^*$. Here $x^*|_{[x_N, x_{N+1}, \ldots]}$ denotes the restriction of x^* to the closed linear span $[x_N, x_{N+1}, \ldots]$ of $\{x_n\}_{n=N}^{\infty}$. A basis is called *boundedly complete* if the following holds: whenever $\{\|\sum_{n=1}^{N} \alpha_n x_n\|\}_N$ is bounded, then $\sum_{n=1}^{N} \alpha_n x_n$ actually converges to some $x \in X$ as $N \to \infty$. Analogous definitions apply to Schauder decompositions.

As an example, note that the standard unit vector basis of c_0 is shrinking but not boundedly complete.

Proposition 1.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space X.*

(1) The coordinate functionals x_n^* defined by $\langle x_n^*, \sum_{n=1}^{\infty} \alpha_n x_n\rangle = \alpha_n$ are continuous. The maps π_N defined by

$$\pi_N \sum_{n=1}^{\infty} \alpha_n x_n = \sum_{n=1}^{N} \alpha_n x_n$$

are projections and $C = \sup_N \|\pi_N\| < \infty$. Hence if $\{x_n\}_{n=1}^{\infty}$ is normalized, then $\|x_n^*\| \leq 2C$ for all $n = 1, 2, \ldots$;

(2) $\{x_n\}_{n=1}^{\infty}$ is shrinking if and only if the coordinate functionals $\{x_n^*\}_{n=1}^{\infty}$ form a basis of X^*;

(3) If $\{x_n\}_{n=1}^{\infty}$ is unconditional, then there is a constant $K > 0$ such that for every $t \in l^{\infty}$ and $x = \sum_{n=1}^{\infty} \alpha_n x_n \in X$,

$$\|\sum_{n=1}^{\infty} t_n \alpha_n x_n\| \leq K(\sup_n |t_n|)\ \|\sum_{n=1}^{\infty} \alpha_n x_n\|.$$

The constant C in (1) is called the *basis constant* of $\{x_n\}_{n=1}^{\infty}$. Analogues for (1), (2) and (3) hold if X has a Schauder decomposition, in which case the constant in (1) will be called the *decomposition constant.* Proofs may be found in [12].

A Banach space X is called a *Grothendieck space* if weak*-sequential convergence and weak sequential convergence in X^* coincide. Every reflexive space is trivially Grothendieck.

A Banach space is said to have the *Dunford-Pettis property* if the following holds: whenever $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ are sequences in X and X^* respectively, such that $x_n \to 0$ weakly and $x_n^* \to 0$ weakly, then $\langle x_n^*, x_n \rangle \to 0$.

Let (Ω, Σ, μ) be a finite measure space. A Banach space X is said to have the *Radon-Nykodym property with respect to* (Ω, Σ, μ) if for every μ-continuous vector-valued measure $G : \Sigma \to X$ of bounded variation there exists $g \in L^1(\mu; X)$ such that

$$G(E) = \int_E g d\mu$$

for all $E \in \Sigma$. X has the *Radon-Nykodym property* if it has the Radon-Nykodym property with respect to every finite meaure space.
A bounded linear operator $S : L^1[0,1] \to X$ is called *Riesz-representable* if there exists a $g \in L^{\infty}([0,1]; X)$ such that

$$Sf \ = \int_0^1 fgdm \quad \text{for all } f \in L^1[0,1].$$

We will need the following result [5, Thm III.1.5; Cor. V.3.8].

Proposition 1.4. *X has the Radon-Nykodym property if and only if each bounded linear operator $S :\ L^1[0,1] \to X$ is Riesz-representable.*

2. REFLEXIVITY AND SCHAUDER DECOMPOSITIONS

In this section we will prove Theorem A. We start with a general existence theorem.

Theorem 2.1. *Every infinite-dimensional Banach space with a Schauder decomposition $\{X_n\}_{n=1}^\infty$ admits a C_0-semigroup with an unbounded generator, which satisfies $\limsup_{t\downarrow 0}\|T(t)\| \leq C$, where C is the decomposition constant of $\{X_n\}_{n=1}^\infty$.*

Proof:

We will define such a semigroup in a somewhat greater generality than is needed at the present stage. Let $0 = N_0 < N_1 < \dots$ be any increasing sequence of integers. Define $\epsilon_m = 1/(N_m \cdot 2^m)$ $(m = 1, 2, \dots)$. Put $k_1 = 1$. Let $t_1 > 0$ be defined by

$$e^{-k_1 t_1} = 1 - \epsilon_1.$$

Choose $k_2 \in \mathbb{N}$, $k_2 \geq k_1 + 1$ such that

$$\frac{e^{-k_2 t_1}}{1 - e^{-t_1}} < 1.$$

Let t_2 be defined by

$$e^{-k_2 t_2} = 1 - \epsilon_2.$$

Continue as follows. Suppose $k_1, k_2, \dots, k_{m-1}$ and $t_1, t_2, \dots, t_{m-1}$ have been chosen. Choose $k_m \in \mathbb{N}$, $k_m \geq k_{m-1} + 1$ such that

$$\frac{e^{-k_m t_{m-1}}}{1 - e^{-t_{m-1}}} < \frac{1}{2^{m-2}}.$$

Let t_m be defined by

$$e^{-k_m t_m} = 1 - \epsilon_m.$$

Observe that $t_1 > t_2 > \dots \to 0$. We will now construct a semigroup $T(t)$ on X for the case that X has a *basis* $\{x_n\}_{n=1}^\infty$. When X has a Schauder decomposition, the construction is entirely similar. For $t \geq 0$ define operators $T(t)$ by

$$T(t)x_n = e^{-k_m t} x_n,$$

where $N_{m-1} < n \leq N_m$. Using the conditions $k_m \geq k_{m-1} + 1$ it is easily seen that $T(t)x$ converges for all $x \in X$, that is, $T(t)$ is well-defined on X. Noting that the coordinate functionals corresponding to $\{x_n\}_{n=1}^\infty$ are continuous, it follows that the closed graph theorem applies and hence for every $t \geq 0$ the operator $T(t)$ is bounded.

Let C be the basis constant of $\{x_n\}_{n=1}^\infty$. Fix some $x = \sum_{n=1}^\infty \alpha_n x_n \in X$, $\|x\| = 1$. Let $t > 0$ be very small such that $t_{M+1} < t \leq t_M$. Then

$$\|T(t)x - \sum_{n=1}^{N_M} \alpha_n x_n \; - \; e^{-k_{M+1} t} \sum_{n=N_M+1}^{N_{M+1}} \alpha_n x_n\| \leq$$

$$N_M \cdot (\max_{n=1,\dots,N_M} \|\alpha_n x_n\|) \cdot (1 - e^{-k_M t_M}) + \| \sum_{m=M+2}^{\infty} e^{-k_m t} \sum_{n=N_{m-1}+1}^{N_m} \alpha_n x_n\| \leq$$

$$N_M \cdot 2C \cdot \epsilon_M + \sum_{m=M+2}^{\infty} e^{-k_m t} \cdot 2C \leq \frac{C}{2^{M-1}} + \sum_{j=0}^{\infty} e^{-k_{M+2+j} t_{M+1}} \cdot 2C =$$

$$\frac{C}{2^{M-1}} + \frac{2C \cdot e^{-k_{M+2} t_{M+1}}}{1 - e^{-t_{M+1}}} = \frac{C}{2^{M-2}}.$$

But $\sum_{n=1}^{N_M} \alpha_n x_n + e^{-k_{M+1}t} \sum_{n=N_M+1}^{N_{M+1}} \alpha_n x_n$ has norm $\leq C$, being a convex combination of $\sum_{n=1}^{N_M} \alpha_n x_n$ and $\sum_{n=1}^{N_{M+1}} \alpha_n x_n$, who both have norm $\leq C$, by the definition of the basis constant. This proves that

$$\limsup_{t \downarrow 0} \|T(t)\| \leq C.$$

At the same time, it is obvious from the above calculation that

$$\|T(t)x - x\| \to 0 \quad (t \downarrow 0),$$

which proves that $T(t)$ is a C_0-semigroup. Finally, the generator A of $T(t)$, being defined by

$$Ax_n = -k_m x_n$$

for $N_{m-1} < n \leq N_m$, is obviously unbounded.

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This theorem states that in a Banach space X with a basis (or Schauder decomposition), there is always a C_0-semigroup which is, to an arbitrary degree of accuracy, a continuous 'interpolation' of the expansion of elements $x \in X$ in terms of the basis vectors. In Corollaries 2.3 and 4.2, we will give examples how information on bases may be derived in this way from the 'corresponding' semigroups.

Corollary 2.2. *Grothendieck spaces with the Dunford-Pettis property do not admit a Schauder decomposition.*

Proof:

By a theorem of H.P. Lotz [13], a Grothendieck space with the Dunford-Pettis property admits C_0-semigroups with *bounded* generators only.

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A countable collection of closed subspaces $\{X_n\}_{n=1}^{\infty}$ of a Banach space X is called a *weak* Schauder decomposition of X if for every $x \in X$ there is a unique sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $x_n \in X_n$, such that $x = \sum_{n=1}^{\infty} x_n$, where the convergence is with respect to the weak topology of X.

Corollary 2.3. *Every weak Schauder decomposition of a Banach space is a Schauder decomposition.*

Proof:

Let $\{X_n\}_{n=1}^{\infty}$ be a weak Schauder decomposition of X. In Theorem 2.1 let $N_i = i$ and consider the semigroup defined by

$$T(t)x_n = e^{-k_n t} x_n \quad (x_n \in X_n).$$

Fix some $x = weak - \lim_N \sum_{n=1}^{N} x_n \in X$. Reasoning as in Theorem 2.1, it is easy to see that $T(t)x \to x$ weakly as $t \downarrow 0$, that is, $T(t)$ is a weakly continuous semigroup. By Proposition 1.2 (2), $T(t)$ is a C_0-semigroup. Also, since weakly convergent sequences are norm-bounded, $\|x_n\| \leq C$ for some constant C. Using this, straightforward estimates show that

$$\|x - \sum_{n=1}^{N} x_n\| \leq \|T(t_N)x - x\| + \|T(t_N)x - \sum_{n=1}^{N} x_n\| \to 0 \quad (N \to \infty).$$

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It is a result of M. Zippin [17] that a Banach space X with a basis is reflexive if and only if every basis of X is shrinking. In the case X has an FDD there is an analogous result. First, in this case X is reflexive if and only if this FDD is shrinking and boundedly complete [12]. Recently, M.A. Ariño [2] proved that in a Banach space with a (finite-dimensional) Schauder decomposition, every (finite-dimensional) Schauder decomposition is shrinking if and only if every (finite-dimensional) Schauder decomposition is boundedly complete. Combining these facts, we get

Proposition 2.4. *A nonreflexive Banach space X with an FDD has a nonshrinking FDD.*

Proof of Theorem *A*: (1) $\Rightarrow$ (2) is trivial, whereas (2) $\Rightarrow$ (3) follows from Prop. 1.2 (2). We have to prove (3) $\Rightarrow$ (1). Suppose X is nonreflexive. Let $\{X_n\}_{n=1}^{\infty}$ be a nonshrinking Schauder decomposition of X. Again we assume without loss of generality that actually we have a *basis* $\{x_n\}_{n=1}^{\infty}$. Choose inductively a sequence of integers $0 = N_0 < N_1 < ..$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ of norm-1 vectors as follows. First let $x_0^* \in X^*$, $\|x_0^*\| = 1$ and $0 < \epsilon < 1$ be such that

$$\lim_N \|x_0^*|_{[x_N, x_{N+1}, \ldots]}\| > \epsilon.$$

Let $z_1 = \sum_{n=1}^{\infty} \alpha_{1n} x_n$ be any norm-1 vector such that

$$|\langle x_0^*, z_1\rangle| > \epsilon.$$

Choose N_1 sufficiently large such that

$$|\langle x_0^*, \sum_{n=1}^{N_1} \alpha_{1n} x_n\rangle| > \epsilon.$$

Put $y_1 = \sum_{n=1}^{N_1} \alpha_{1n} x_n$. We may, by choosing N_1 large enough, multiply y_1 with an appropriate scalar so as to make a norm-1 vector of it without affecting the above inequality. Choose $z_2 = \sum_{n=N_1+1}^{\infty} \alpha_{2n} x_n \in [x_{N_1+1}, x_{N_1+2}, \ldots]$ of norm 1 such that

$$|\langle x_0^*, z_2\rangle| > \epsilon.$$

Choose N_2 such that

$$|\langle x_0^*, \sum_{n=N_1+1}^{N_2} \alpha_{2n} x_n\rangle| > \epsilon.$$

Define $y_2 = \sum_{n=N_1+1}^{N_2} \alpha_{2n} x_n$ and again assume without loss of generality that y_2 has norm 1. Continue in this way. For $N_{m-1} < n \leq N_m$ define $T(t)$ by

$$T(t)x_n = e^{-k_m t} x_n,$$

where the numbers k_m are chosen as in the proof of Theorem 2.1. This defines a C_0-semigroup on X. Now fix $t > 0$. Upon choosing m sufficiently large, we get that

$$|\langle x_0^*, T(t)y_m\rangle| = e^{-k_m t}|\langle x_0^*, y_m\rangle| \leq \frac{\epsilon}{2}.$$

Hence

$$\|T^*(t)x_0^* - x_0^*\| \geq |\langle x_0^*, T(t)y_m - y_m\rangle| \geq$$
$$|\langle x_0^*, y_m\rangle| - |\langle x_0^*, T(t)y_m\rangle| \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

This shows that $T^*(t)$ is not strongly continuous at $t = 0$ and proves Theorem A.

Theorem A does not hold for arbitrary Banach spaces. For instance, let $X = L^\infty[0,1]$. Since X is a Grothendieck space with the Dunford-Pettis property, every C_0-semigroup on X has a bounded generator. From this it is obvious that the adjoint of such a semigroup is strongly continuous and has a bounded generator as well. In fact, for X we may take any infinite-dimensional Grothendieck space with the Dunford-Pettis property. Note that these spaces always are nonseparable [11]. One still may ask whether Theorem A holds for arbitrary *separable* Banach spaces X, since not every separable Banach space has an FDD [7]. For instance, it is known [12] that c_0 and l^1 contain subspaces Y without an FDD. In these two cases however the answer is easy, since Y contains a complemented subspace Z isomorphic to c_0 or l^1 respectively [12]. On Z we may construct a C_0-semigroup whose adjoint is not strongly continuous; this semigroup can be extended to Y by putting it identically 1 on the complement of Z. Hence, Theorem A holds for closed subspaces of c_0 and l^1.

By a theorem of A. Pelczynski [14] a Banach space is reflexive if and only is every closed subspace with a basis is. This, in combination with Theorem A, gives the following corollary.

Corollary 2.5 . *A Banach space X is reflexive if and only if for every closed subspace Y of X, every C_0-semigroup $T(t)$ on Y has a strongly continuous adjoint $T^*(t)$ on Y^*.*

3. THE RADON-NYKODYM PROPERTY AND UNCONDITIONAL BASES

Lemma 3.1. *Every weak*-continuous semigroup $T(t)$ on a dual Banach space X^* with the Radon-Nykodym property is strongly continuous for $t > 0$.*

Proof:

Fix an arbitrary $x^* \in X^*$. By the uniform boundedness theorem, there is an $M < \infty$ such that $\|T(t)x^*\| \leq M$ for all $t \in [0,1]$. Define $S : L^1[0,1] \to X^*$ by

$$Sg = weak^* \int_0^1 g(t)T(t)x^* dt.$$

Since $\langle T(t)x^*, x\rangle$ is continuous for each $x \in X$, it follows that $\langle g(t)T(t)x^*, x\rangle \in L^1[0,1]$ for all $x \in X$, and the above integral is well-defined. S is bounded:

$$\|Sg\| = \sup_{\|x\|=1} \left| \int_0^1 \langle g(t)T(t)x^*, x\rangle dt \right| \leq \sup_{\|x\|=1} \int_0^1 |g(t)| \, |\langle T(t)x^*, x\rangle| dt \leq M \, \|g\|_1.$$

Since X^* has the Radon-Nykodym property, by Proposition 1.4 there is an $h \in L^\infty([0,1]; X^*)$ such that

$$Sg = \int_0^1 g(t)h(t)dt$$

for all $g \in L^1[0,1]$. For $0 \leq t < 1$ and $\epsilon > 0$ small enough, let $E = [t, t+\epsilon]$ and put $g = \frac{1}{\epsilon}\chi_E$, where χ is the characteristic function. It follows that

$$weak^* \int_t^{t+\epsilon} \frac{1}{\epsilon} T(\tau)x^* d\tau = \int_t^{t+\epsilon} \frac{1}{\epsilon} h(\tau) d\tau.$$

By the Lebesgue differentiation theorem, for almost all $t \in [0,1)$ the right-hand side converges to $h(t)$ as $\epsilon \to 0$. Hence, for such t we have

$$\frac{1}{\epsilon}\int_t^{t+\epsilon} \langle T(\tau)x^*, x\rangle d\tau \to \langle h(t), x\rangle \quad (\epsilon \to 0)$$

for all $x \in X$. But the integrand on the left-hand side is continuous, and therefore the integral converges to $\langle T(t)x^*, x\rangle$. So $T(t)x^* = h(t)$ a.e. In particular, $T(t)x^*$ is measurable on $[0,1]$, hence on $[0,\infty)$. It follows from Prop. 1.2 (1) that $T(t)$ is strongly continuous for $t > 0$.

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If $T(t)$ in Lemma 3.1 is an *adjoint* semigroup, the above result is implicit in W. Arendt [1], where it is obtained by an entirely different method of proof.
Every nonreflexive Banach space X with a basis (or FDD) admits a C_0-semigroup whose adjoint is strongly continuous *precisely* for $t > 0$. In fact, the semigroup from the proof of Theorem A will do, as is easily seen from its construction. However, this is a rather non-constructive example. The following example is constructive. It is adapted from [1], where it is credited to H.P. Lotz.

Example 3.2.
Let J be the James space consisting af all sequences of scalars $x = (a_1, a_2, ...)$ for which

$$\|x\| = \sup\, [(a_{p_1} - a_{p_2})^2 + (a_{p_2} - a_{p_3})^2 + ... + (a_{p_{m-1}} - a_{p_m})^2 + (a_{p_m} - a_{p_1})^2]^{1/2} < \infty$$

and

$$\lim_{n\to\infty} a_n = 0,$$

where the sup is taken over all possible choices of integers m and $p_1 < p_2 < .... < p_m$. Let e_n denote the nth unit vector. Then $\{e_n\}_{n=1}^{\infty}$ is a shrinking basis for J and consequently the unit vectors e_n^* of J^* form a basis for J^*. On J define a C_0-semigroup $T(t)$ by

$$(T(t)x)_n = e^{-nt}(x)_n$$

where $(x)_n$ denotes the nth coordinate of x. It is obvious that each e_n^* belongs to $J^{\odot}$, hence also each linear combination of them. Since $J^{\odot}$ is closed, it follows that $J^{\odot} = J^*$. So $T^*(t)$ is a C_0-semigroup on J^*. Now $\dim J^{**}/J = 1$; consequently J^{**} is separable and therefore has the Radon-Nykodym property. Hence $T^{**}(t)$ is strongly continuous for $t > 0$ by Lemma 3.1. One can show that J^{**} is isomophic to $J \oplus \mathbb{C}e$, where $e = (1,1,.....)$. Under this isomophism, we may regard $T^{**}(t)$ as a weak*-continuous semigroup on $J \oplus \mathbb{C}e$. In [1] it is shown that $e \notin J^{*\odot}$. Therefore $T^{**}(t)$ is not strongly continuous at $t = 0$.

◇

This example is interesting for another reason. There are many examples of C_0-semigroups on Banach spaces X such that $\dim X^*/X^{\odot} = \infty$. The above example shows that $X^{\odot}$ can also have any *finite* codimension in X^*:

Corollary 3.3. *For each $n \in \mathbb{N}$ there exists a Banach space X and a C_0-semigroup $T(t)$ on X such that* $\dim X^*/X^{\odot} = n$.

Proof:

If $n = 0$, let $T(t)$ be any C_0-semigroup on a reflexive space. Otherwise, consider the C_0-semigroup $T^*(t)$ on J^* from Example 3.2. Since $J^{**} = J \oplus \mathbb{C}e = J^{*\odot} \oplus \mathbb{C}e$ we see that $\dim J^{**}/J^{*\odot} = 1$. Let $X = J^* \times J^* \times ... \times J^*$, n times, together with the 'product' semigroup obtained from n copies of $T^*(t)$.

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Proof of Theorem *B*: The implication (1) $\Rightarrow$ (2) is a classical theorem of N. Dunford and B.J. Pettis [6], whereas (2) $\Rightarrow$ (3) follows from Lemma 3.1. It therefore remains to be shown that (3) $\Rightarrow$ (1) holds. In view of Proposition 1.3 (2) it suffices to show that the unconditional basis $\{x_n\}_{n=1}^{\infty}$ of X is shrinking. Suppose the contrary is true. Exactly as in the proof of Theorem A one can construct a sequence of integers $0 = N_0 < N_1 < ..$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ of norm-1 vectors, $y_k \in [x_{N_{k-1}+1}, x_{N_{k-1}+2}, ..., x_{N_k}]$, together with an $x_0^* \in X^*$, $\|x_0^*\| = 1$ and $0 < \epsilon < 1$, such that for all n,

$$|\langle x_0^*, y_n\rangle| > \epsilon.$$

For $N_{m-1} < n \leq N_m$ define

$$T(t)x_n \;=\; e^{imt}x_n,$$

where x_n is the nth basis vector. By Prop 1.3 (3), there is a $K > 0$ such that $\|T(t)\| \leq K$ for all $t \geq 0$. From this it is easy to see that $T(t)$ is a C_0-semigroup on X. Now let $t > 0$ be arbitrary and fixed. We will show that $T^*(t)x_0^* \notin X^{\odot}$. Let $m \in \mathbb{N}, m \geq 1$. By the irrationality of the number π, we can find a positive integer k such that

$$|1 - e^{i\frac{k}{m}}| > 2 - \epsilon.$$

We have the following estimates.

$$\begin{aligned} \|T^*(t+\frac{1}{m})x_0^* - T^*(t)x_0^*\| \;&\geq\; |\langle T^*(t+\frac{1}{m})x_0^* - T^*(t)x_0^*, y_k\rangle| \;= \\ |e^{ik(t+\frac{1}{m})} - e^{ikt}| \cdot |\langle x_0^*, y_k\rangle| \;&\geq\; (2-\epsilon)\cdot\epsilon. \end{aligned}$$

This proves Theorem B.

It is natural to ask whether an analogue of Corollary 2.5 holds for Banach spaces whose dual have the Radon-Nykodym property. H.P. Lotz's theorem on l^1 in Banach lattices [8] shows that for Banach *lattices* this is indeed the case: If the dual of a Banach lattice does not have the Radon-Nykodym property, then X contains a copy of l^1; on l^1 we have a C_0-semigroup whose adjoint is not strongly continuous for $t > 0$ by Theorem B. For general Banach spaces we remark that J. Hagler [9] proved that a separable Banach space with a nonseparable dual has a subspace with a basis whose dual is nonseparable. Therefore it would be enough to prove Theorem B, (3) $\Rightarrow$ (1), without the assumption that the basis of X should be unconditional. (note that we made a rather crude step at this stage in just using that the basis of a space with nonseparable dual necessarily must be nonshrinking). The following theorem shows that in order to solve this problem, it suffices to construct a C_0-semigroup on X whose adjoint has a nonseparable orbit.

Theorem 3.4. *Let $T(t)$ be a C_0-semigroup on a Banach space X. Let $x^* \in X^*$. The orbit $\{T^*(t)x^* : t \geq 0\}$ is separable if and only if $t \to T^*(t)x^*$ is strongly continuous for $t > 0$ if and only if $t \to T^*(t)x^*$ is weakly continuous for $t > 0$.*

Proof:

It is obvious that strong continuity implies weak continuity. If $t \to T^*(t)x^*$ is weakly continuous for $t > 0$ then it is certainly weakly separable, which is the same as strongly separable. Suppose $\{T^*(t)x^* : t \geq 0\}$ is separable. The proof that the map $t \to T^*(t)x^*$ is strongly continuous for $t > 0$ is a slight modification of the argument given in [10, Thm 10.3.2]. Choose numbers $0 < \alpha < \tau < \beta < \xi$ and let η be so small that $\beta < \xi - \eta$. Now $T^*(\xi)x^* = T^*(\tau)T^*(\xi - \tau)x^*$ is independent of τ, hence certainly integrable on $[\alpha, \beta]$ with respect to τ. Therefore

$$(\beta - \alpha)\,[T^*(\xi \pm \eta) - T^*(\xi)]x^* = \int_\alpha^\beta T^*(\tau)[T^*(\xi \pm \eta - \tau) - T^*(\xi - \tau)]x^* d\tau.$$

The norm of the integrand is majorized by $2M\|x^*\|$, where M is such that $\|T^*(t)\| = \|T(t)\| \leq M$ on $[0, \xi + \eta]$. Since $\tau \to [T^*(\xi \pm \eta - \tau) - T^*(\xi - \tau)]x^*$ is measurable (by Pettis' measurability theorem), so is $\|[T^*(\xi \pm \eta - \tau) - T^*(\xi - \tau)]x^*\|$. This gives

$$(\beta - \alpha)\,\|[T^*(\xi \pm \eta) - T^*(\xi)]x^*\| \leq M \int_{\xi-\beta}^{\xi-\alpha} \|[T^*(\sigma \pm \eta) - T^*(\sigma)]x^*\| d\sigma \to 0 \quad (\eta \to 0);$$

see [10, Thm 3.8.3].

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Theorem 3.5. *Let $T(t)$ be a C_0-semigroup on a Banach space X. Let $x^* \in X^*$. Then $t \to T^*(t)x^*$ is strongly continuous for $t \geq 0$ if and only if $t \to T^*(t)x^*$ is weakly continuous for $t \geq 0$.*

Proof:

We only have to prove the 'if' part. If $T^*(t)$ is an adjoint semigroup, then there is a positive M such that $\|T^*(t)\| \leq M$ in a neighbourhood of $t = 0$ (since such an estimate holds for its predual $T(t)$). Now the proof can be finished in exactly the same way as in [16, Ch. IX,1].

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These two theorems can be considered as the 'orbitwise' analogues for adjoint semigroups of Prop. 1.2. The point of their proofs is that we have bounds on $T^*(t)$ beforehand, since we are dealing with *adjoint* semigroups.

4. NONSHRINKING BASES IN c_0

Theorem *A* guarantees the existence of a C_0-semigroup without strongly continuous adjoint on the nonreflexive space c_0 (and, more generally, on every separable Banach space containing c_0, since by A. Sobczyk's theorem [12], c_0 is complemented in such spaces). The following theorem shows that it can be hard to give an explicit example of such a semigroup.

Theorem 4.1. *Let $T(t)$ be a C_0-semigroup on c_0; $\|T(t)\| \leq Me^{\omega t}$. If $M < 2$, then $T^*(t)$ is strongly continuous on l^1.*

Proof:

Put $K = M - 1$. Pick an arbitrary $\epsilon > 0$ and choose $\epsilon_1, \epsilon_2, ... > 0$ such that

$$\prod_{i=1}^{n}(K + \epsilon_i) < K^n + \epsilon \qquad \forall n \in \mathbb{N}.$$

Now let $x_0 = \sum_n \alpha_n e_n \in l^1$ (e_n denoting the nth unit vector of l^1); $\|x_0\| = 1$. Let N be such that $\sum_{n=N+1}^{\infty} \alpha_n e_n < \epsilon_1/5$. Choose $t_1 > 0$ so small that $\|T^*(t_1)x_0\| \leq M + \epsilon_1/5$ and $|(T^*(t_1)x_0 - x_0)_n| \leq \epsilon_1/(5N)$ $(n = 1, 2, ..., N)$. Such t_1 exists by the weak*-continuity of the map $t \to T^*(t)x_0$ and by the estimate $\|T(t)\| \leq Me^{\omega t}$. We have

$$\sum_{n=1}^{N} |(T^*(t_1)x_0)_n| \geq \sum_{n=1}^{N} |(x_0)_n| - \sum_{n=1}^{N} |(T^*(t_1)x_0 - x_0)_n| \geq$$
$$1 - \frac{\epsilon_1}{5} - N \cdot \frac{\epsilon_1}{5N} = 1 - \frac{2\epsilon_1}{5}.$$

Therefore

$$\|x_0 - T^*(t_1)x_0\| = \sum_{n=1}^{N} |(T^*(t_1)x_0 - x_0)_n| + \sum_{n=N+1}^{\infty} |(T^*(t_1)x_0 - x_0)_n| \leq$$
$$\frac{\epsilon_1}{5} + \sum_{n=N+1}^{\infty} |(T^*(t_1)x_0)_n| + \sum_{n=N+1}^{\infty} |(x_0)_n| \leq$$
$$\frac{\epsilon_1}{5} + (\|T^*(t_1)x_0\| - (1 - \frac{2\epsilon_1}{5})) + \frac{\epsilon_1}{5} \leq M - 1 + \epsilon_1.$$

Put $x_1 = x_0 - T^*(t_1)x_0$. In the same way, there is an $t_2 > 0$ such that

$$\|x_1 - T^*(t_2)x_1\| \leq (M - 1 + \epsilon_2)\|x_1\| \leq (M - 1 + \epsilon_1)(M - 1 + \epsilon_2).$$

Put $x_2 = x_1 - T^*(t_2)x_1$. Proceed with the construction inductively in the obvious way. After n steps, we have $t_1, t_2, ..., t_n > 0$ and vectors $x_1, x_2, ..., x_n$ such that

$$\|x_n\| = \|x_{n-1} - T^*(t_n)x_{n-1}\| =$$
$$\|x_0 - T^*(t_1)x_0 - T^*(t_2)x_1 - ...\| \leq \prod_{i=1}^{n}(M - 1 + \epsilon_i) < (M - 1)^n + \epsilon.$$

Since $M - 1 < 1$, upon taking n sufficiently large, we find $\|x_n\| \leq 2\epsilon$. Since l^1 is a separable dual space, it has the Radon-Nykodym property and therefore, by Lemma 3.1, $T^*(t_i)x_{i-1} \in (c_0)^{\odot}$ for all $i = 1, 2, ...$ We have proved that x_0 is in the closure of $(c_0)^{\odot}$. By 1.1 (3), $(c_0)^{\odot}$ is closed and therefore $x_0 \in (c_0)^{\odot}$. Hence $(c_0)^* = l^1 = (c_0)^{\odot}$, as was to be shown.

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We noted that the standard unit vector basis of c_0 is shrinking. Of course, this basis has basis constant $C = 1$. By M. Zippin's theorem we are told that there exists a nonshrinking basis for c_0, since c_0 is nonreflexive. What can be said of the basis constant of such a basis?

Corollary 4.2. *Every nonshrinking basis of c_0 has basis constant $C \geq 2$.*

Proof:

Let $\{x_n\}_{n=1}^{\infty}$ be nonshrinking basis of c_0 with basis constant C. Let $T(t)$ be the C_0-semigroup, defined with respect to $\{x_n\}_{n=1}^{\infty}$, as in Theorem A. Then $T^*(t)$ is not strongly continuous. Let $\epsilon > 0$ be arbitrary. By Theorem 2.1, there is a $t_0 > 0$ such that $\|T(t)\| \leq C + \epsilon$ for $t \in [0, t_0]$. Hence (this is easy to verify) there is an ω such that $\|T(t)\| \leq (C+\epsilon)e^{\omega t}$ $(t \geq 0)$. By Theorem 4.1, $C + \epsilon \geq 2$.

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The results of Theorem 4.1 and Corollary 4.2 are optimal: let z_i denote the ith unit vector of c_0 and put $y_n = \sum_{i=1}^{n} z_i$, then the basis $\{y_n\}_{n=1}^{\infty}$ is nonshrinking and has basis constant 2.

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