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Almost Disturbance Decoupling by Measurement Feedback : a Frequency Domain Analysis

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In this paper the almost disturbance decoupling problem by measurement feedback is formulated for systems described by transfer matrices. Necessary and sufficient conditions in frequency domain terms for the solvability of the problem are given. The conditions are derived using frequency domain techniques only.

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1. Introduction

In a series of papers J.C. Willems developed a theory around the concept of almost invariance for linear systems in state space form (cf. [4-6]). The concept was introduced to formulate and solve a number of important control problems that all had in common the feature of *almost decoupling*. In the present paper one of these problems, the so-called *almost disturbance decoupling problem by measurement feedback* , will be studied again, now from a frequency domain point of view.

In Willems [6] the almost disturbance decoupling problem by measurement feedback was introduced using the parameters of a linear system in state space form. The conditions for the solvability of the problem were formulated in terms of (almost invariant) subspaces in the state space that satisfy certain subspace inclusions. However, for the derivation of the conditions a combination was needed of state space (= time domain) techniques on the one hand and frequency domain techniques on the other hand. In addition, the algorithm extracted from the derivations was a mixture of computations in the time domain and the frequency domain.

In the present paper we present a straightforward and constructive approach to the almost disturbance decoupling problem by measurement feedback completely in terms of the frequency domain. This means that we describe the system and formulate the problem completely in terms of transfer matrices and rational matrices. Also the conditions for the solvability of the problem, the derivation of these conditions and the resulting algorithm, are specified completely in terms of transfer matrices and rational matrices. We do not claim that the conditions are new. In fact, they are just slightly generalized frequency domain versions of the conditions derived in [6]. However, we do think that our approach to the problem is more uniform than the approach described in [6].

The outline of the paper is as follows. In section 2 we introduce the linear system that we are interested in. Furthermore, in section 2 we formulate the almost disturbance decoupling problem by measurement feedback. In section 3 we introduce some notions concerning the various types of solvability for rational matrix equations, whereupon we state the main theorem of this paper. The theorem

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states how the different types of solvability are related. From the theorem necessary and sufficient conditions for the solvability of our version of the almost disturbance decoupling problem by meaurement feedback can be derived easily. A proof of our main result is given in section 4. It is preceeded by some preliminary results. In section 5 we describe in detail a new algorithm for the computation of a controller which, if it exists, achieves almost decoupling upto a specified accuracy.

2. Problem Formulation

In this paper we are dealing with *controlled* systems that can be pictured as in figure 1.

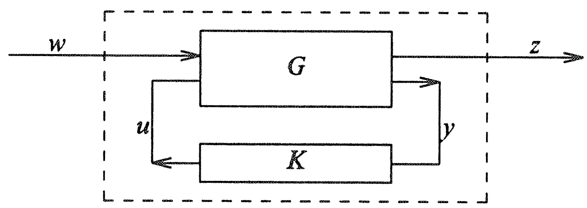

Figure 1.

In figure 1. w,u,z and y are vector-valued variables and G,K are transfer matrices. The variable w is called the *exogenous*, or *disturbance input*, u the *control input*, z the *exogenous*, or *to-be-controlled output* and y the *measurement output*. The matrix G is called the *plant*, the part of the controlled system that is fixed, and the matrix K is called the *controller*, the part of the controlled system that has to be designed.

Throughout this paper we adopt the following notation. The set of real numbers will be denoted by $\mathbb{R}$. $\mathbb{C}$ and $\mathbb{C}^-$ denote the complex plane and the open left half of the complex plane, respectively (i.e. $\mathbb{C}^- = \{s \in \mathbb{C} \mid \mathrm{Re}\, s < 0\}$, where $\mathrm{Re}\, s$ denotes the real part of the complex variable s). $\mathbb{R}[s]$ denotes the set of polynomials with real coefficients. The set of rational functions with real coefficients, also called the set of real rational functions, is denoted by $\mathbb{R}(s)$ (i.e. $\mathbb{R}(s) = \{f/g \mid f,g \in \mathbb{R}[s] \text{ with } g \neq 0\}$). $\mathbb{R}_0(s)$ and $\mathbb{R}_+(s)$ denote the set of proper real rational functions and the set of strictly proper real rational functions, respectively (i.e. $\mathbb{R}_0(s) = \{f \in \mathbb{R}(s) \mid$ there is a real number c such that $\lim_{s\to\infty} f(s) = c\}$ and $\mathbb{R}_+(s) = \{f \in \mathbb{R}(s) \mid \lim_{s\to\infty} f(s) = 0\}$). If S is an arbitrary set, then $M(S)$ denotes the set of matrices with entries in S. The dimensions of any element of $M(S)$ will always be determined by the context in which it is used.

In this paper we assume that the matrices G and K are proper real rational matrices (i.e. $G,K \in M(\mathbb{R}_0(s))$). Corresponding to the inputs and outputs, we partition the matrix G as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

with $G_{11}, G_{21}, G_{12}, G_{22} \in M(\mathbb{R}_0(s))$. The plant in figure 1. is then described by

$$z = G_{11}w + G_{12}u, \tag{1.a}$$

$$y = G_{21}w + G_{22}u \tag{1.b}$$

and the controller is described by

$$u = Ky. \tag{2}$$

If $I - G_{22}K$ is invertible as a rational matrix, then we can describe the controlled system by

$$z = Tw = (G_{11} + G_{12}XG_{21})w \tag{3}$$

where $X = K(I - G_{22}K)^{-1}$. A condition to guarantee that $I - G_{22}K$ is invertible is that G_{22} is a strictly proper real rational matrix. Indeed, if $G_{22} \in M(\mathbb{R}_+(s))$ then $I - G_{22}K$ is a bicausal real rational matrix (cf. Hautus & Heymann [2]). A bicausal real rational matrix is a proper real rational matrix that has a proper real rational inverse. It can be proved easily that a proper real rational matrix is bicausal if and only if its determinant, which is a proper real rational function, does not vanish at infinity. Thus it follows that $X \in M(\mathbb{R}_0(s))$ and it can be proved easily that $K=(I+XG_{22})^{-1}X$. Therefore, to assure that the controlled system can be described by (3), we assume henceforth that $G_{11}, G_{12}, G_{21} \in M(\mathbb{R}_0(s))$ and $G_{22} \in M(\mathbb{R}_+(s))$.

For the formulation of the problem treated in this paper and for the development of the results we need some notions concerning complex and rational matrices. If $A \in M(\mathbb{C})$ then $\bar{\sigma}(A)$ denotes the largest singular value of the complex matrix A. If $A \in M(\mathbb{R}_0(s))$ is a proper real rational matrix with only poles in $\mathbb{C}^-$, then we define the H_∞-norm of A, denoted $\|A\|_\infty$, as

$$\|A\|_\infty := \sup_{\mathrm{Re}\, s>0} \bar{\sigma}(A(s)).$$

Note that due to the properness of A and the fact that A has only poles in $\mathbb{C}^-$, it follows from the maximum principle that

$$\|A\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(A(i\omega)).$$

In the sequel, whenever we use $\|A\|_\infty$ or say that $\|A\|_\infty$ exists, we implicitelly assume that $A \in M(\mathbb{R}_0(s))$ and A only has poles in $\mathbb{C}^-$. For an introductory book on the H_∞-norm and related control theory, we refer to Francis [1]. This reference also contains a number of references to more advanced textbooks on H_∞-theory. If $A \in M(\mathbb{R}_0(s))$ then $A(\infty)$ denotes the constant term in the power series of A around infinity. As this notation suggests, $A(\infty) = \lim_{s\to\infty} A(s)$. It can be proved easily that $\bar{\sigma}(A(\infty)) \leqslant \|A\|_\infty$. For $A \in M(\mathbb{R}(s))$ with $A \neq 0$ we define the *degree* of A, denoted ∂A, as

$$\partial A := \min\{j \in \mathbb{Z} \mid s^{-j}A(s) \in M(\mathbb{R}_0(s))\}$$

where $\mathbb{Z}$ denotes the set of integers. In the sequel, whenever we use ∂A we implicitely assume that $A \neq 0$. Note that $A \in M(\mathbb{R}_0(s))$ and $A \neq 0$ if and only if $\partial A \leqslant 0$. Also note that polynomials with real coefficients are real rational functions and that the above notion of degree coincides with the usual notion of the degree for nonzero polynomials and polynomial matrices. Using these conventions we can now formulate the following problem, given the plant (1).

PROBLEM
Given $\epsilon > 0$, find a controller (2) such that the transfer matrix T of the closed loop system (3) satisfies $\|T\|_\infty \leqslant \epsilon$.

A controller (2) that solves the problem is said to achieve *almost decoupling* with an error less than or equal to ϵ. If the above problem can be solved for all $\epsilon > 0$, we say that the *almost disturbance decoupling problem by measurement feedback* is solvable. We refer to this problem as ADDPM. Hence, ADDPM is solvable, if for all $\epsilon > 0$ there is a controller (2) such that the transfer matrix T of the closed loop system (3) satisfies $\|T\|_\infty \leqslant \epsilon$. For alternative state space oriented formulations of ADDPM we refer to Willems [6]. We conclude the present section by the next corollary which is an immediate consequence of the foregoing discussion.

COROLLARY 2.1. ADDPM *is solvable if and only if for all* $\epsilon > 0$ *there exists a proper real rational matrix* $X \in M(\mathbb{R}_0(s))$ *such that* $\|G_{11} + G_{12}XG_{21}\|_\infty \leqslant \epsilon$.

3. Results

To derive conditions by which we can check whether or not ADDPM is solvable we consider the rational matrix equation

$$UXV+W=0, \tag{RME}$$

where $U, V, W \in M(\mathbb{R}_0(s))$ are known proper real rational matrices and $X \in M(\mathbb{R}(s))$ is an unknown real rational matrix. The matrix W is sometimes called the inhomogeneous part of RME.

If there is a real rational matrix $X \in M(\mathbb{R}(s))$ that satisfies RME, we say that RME is *solvable over* $\mathbb{R}(s)$. We say that RME is *approximately solvable over* $\mathbb{R}_0(s)$ if for all $\epsilon > 0$ there exists a proper real rational matrix $X_\epsilon \in M(\mathbb{R}_0(s))$ such that $\|UX_\epsilon V+W\|_\infty \leq \epsilon$. If there exists a real matrix $X_r \in M(\mathbb{R})$ such that $U(\infty)X_r V(\infty)+W(\infty)=0$, we say that RME is *solvable at infinity*. Note that if RME is solvable over $\mathbb{R}(s)$, this does not imply that RME is solvable at infinity. Also the converse is not true. For instance, if $U(s)=1/s, V(s)=1$ and $W(s)=1$ then RME is solvable over $\mathbb{R}(s)$, but not solvable at infinity. On the other hand, if $U(s)=1, V(s)=[s/(s+1), 0]$ and $W(s)=[1, 1/s]$ then RME is solvable at infinity, but not over $\mathbb{R}(s)$. Using the above conventions we can state our main result which is a generalization of lemma 3 in Willems [6]. The proof of the result will be given in section 4.

Theorem 3.1. RME *is approximately solvable over* $\mathbb{R}_0(s)$ *if and only if* RME *is solvable over* $\mathbb{R}(s)$ *and* RME *is solvable at infinity.*

The relevance of this theorem in connection with the solvability of ADDPM is obvious by corollary 2.1. As announced a proof of theorem 3.1 is given in the next section. In the present section we continue with the derivation of *verifiable* conditions for the solvability of ADDPM. To this end, we let $\mathbb{F}$ denote an arbitrary field. If $A \in M(\mathbb{F})$ is a given matrix, we say that rank$A=q$ if there exists a q^{th} order minor of A unequal to $0(\in\mathbb{F})$, while every $(q+1)^{th}$ order minor of A is equal to $0(\in\mathbb{F})$. Now if $A, B, C \in M(\mathbb{F})$ are given matrices of suitable dimensions, we have the following result.

Lemma 3.2. *The following statements are equivalent :*
(1) There exists a matrix $Y \in M(\mathbb{F})$ *such that* $AYB+C=0$.
(2) Rank A = *rank* $[A, C]$ *and rank* B = *rank* $\begin{bmatrix} B \\ C \end{bmatrix}$.
(3) $PC=0$ *for all* $P \in M(\mathbb{F})$ *such that* $PA=0$, *and* $CQ=0$ *for all* $Q \in M(\mathbb{F})$ *such that* $BQ=0$.

The proof of this lemma is straightforward and follows by elementary results from matrix theory (cf. Lancaster & Tismentski [3], and Willems [6], appendix B). Now using corollary 2.1, theorem 3.1 and lemma 3.2 we obtain the following theorem.

Theorem 3.3. ADDPM *is solvable if and only if*

$$\text{rank } G_{12} = \text{rank } [G_{11}, G_{12}], \text{ rank } G_{21} = \text{rank} \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix}, \text{ and}$$

$$\text{rank } G_{12}(\infty) = \text{rank } [G_{11}(\infty), G_{12}(\infty)], \text{ rank } G_{21}(\infty) = \text{rank} \begin{bmatrix} G_{11}(\infty) \\ G_{21}(\infty) \end{bmatrix}.$$

Observe that in the first condition of theorem 3.3 the notion of rank is associated to the field of real rational functions $\mathbb{R}(s)$, whereas the notion of rank in the second condition of theorem 3.3 is associated to the field of real numbers. Theorem 3.3 clearly provides *verifiable* necessary and sufficient conditions for the solvability of ADDPM. However, once we have established that ADDPM is solvable, we do not yet have a controller that achieves almost decoupling with an error less than or equal to an in advance given $\epsilon > 0$. An algorithm for the calculation of such a controller is presented in section 5. The algorithm is extracted from the proof of theorem 3.1 which is given in the next section.

4. PROOF OF THEOREM 3.1.
For the proof of theorem 3.1. we need three somewhat technical results.

LEMMA 4.1. *Consider* RME. *If* RME *is solvable at infinity, then there exists a proper real rational matrix* $X \in M(\mathbb{R}_0(s))$ *such that* $UXV+W \in M(\mathbb{R}_+(s))$ *(i.e.* $UXV+W$ *is strictly proper).*

PROOF. Let $X_r \in M(\mathbb{R})$ be such that $U(\infty)X_rV(\infty)+W(\infty)=0$, and let $X' \in M(\mathbb{R}_+(s))$ be arbitrary. Define $X \in M(\mathbb{R}_0(s))$ as $X(s)=X_r+X'(s)$. Then, since $\lim_{s\to\infty} U(s)X(s)V(s)+W(s) = U(\infty)X_rV(\infty) + W(\infty)=0$, it is clear that $UXV+W \in M(\mathbb{R}_+(s))$. □

For our next preliminary result we assume that the inhomogeneous part of RME is strictly proper.

LEMMA 4.2. *Consider* RME *and assume that* $W \in M(\mathbb{R}_+(s))$. *If* RME *is solvable over* $\mathbb{R}(s)$, *then there exists a proper real rational matrix* $X \in M(\mathbb{R}_0(s))$ *such that* $UXV+W \in M(\mathbb{R}_+(s))$ *and* $\|UXV+W\|_\infty \leqslant \infty$.

PROOF. Let $X' \in M(\mathbb{R}(s))$ be such that $UX'V+W=0$. If $W=0$ then define $X:=0 \in M(\mathbb{R}_0(s))$ and the lemma is proved. If $W \neq 0$ then $X' \neq 0$ and define $\tau := \partial X'$. If $\tau \leqslant 0$ take $X:=X' \in M(\mathbb{R}_0(s))$ and again the lemma is proved. Next suppose that $W \neq 0$ and $\tau > 0$. Let $N \in M(\mathbb{R}[s])$ and $d \in \mathbb{R}[s]$ be such that $W=N/d$ and define $\delta := \partial d$. Because $0 \neq W \in M(\mathbb{R}_+(s))$ it is clear that $0 \leqslant \partial N < \delta$. Now let $q \in \mathbb{R}[s]$ be a polynomial such that $\partial q = \delta + \tau$ and q has only zeros in $\mathbb{C}^-$. Calculate polynomials $p, r \in \mathbb{R}[s]$ such that $q = dp + r$ with either $0 \leqslant \partial r < \delta$ or $r=0$. Note that $\partial q = \delta + \partial p$. Now define the real rational function $f \in \mathbb{R}(s)$ and the real rational matrix $X \in M(\mathbb{R}(s))$ as $f := r/q$ and $X := fX'$, respectively. Then it follows that $X \in M(\mathbb{R}_0(s))$ and $UXV+W = fUX'V+W = (1-f)W = ((q-r)/q)(N/d) = (p/q)N$. Hence, $UXV+W \in M(\mathbb{R}_+(s))$, since $\partial p + \partial N < \partial p + \delta = \partial q$. Furthermore, $UXV+W$ has only poles in $\mathbb{C}^-$, since q has only zeros in $\mathbb{C}^-$. Therefore, it follows that $\|UXV+W\|_\infty \leqslant \infty$. □

For our last preliminary result we assume that the inhomogeneous part of RME is strictly proper and has a nonzero finite H_∞-norm.

LEMMA 4.3. *Consider* RME *and assume that* $W \in M(\mathbb{R}_+(s))$ *and* $0 < \|W\|_\infty < \infty$. *Let* $\epsilon \in \mathbb{R}$ *be a real number such that* $0 < \epsilon < \|W\|_\infty$ *and let* $X' \in M(\mathbb{R}(s))$ *be a real rational matrix such that* $X' \neq 0$ *and* $\|UX'V+W\|_\infty < \infty$. *Then there exists a real rational matrix* $X \in M(\mathbb{R}(s))$ *such that* $\partial X = \partial X' - 1$ *and* $\|UXV+W\|_\infty \leqslant \|UX'V+W\|_\infty + \epsilon$.

PROOF. Since $W \in M(\mathbb{R}_+(s))$ it follows that $\lim_{|\omega|\to\infty} W(i\omega)=0$. Because $0<\epsilon<\sigma := \|W\|_\infty$ there exists a positive real number R such that $\sup_{|\omega|>R} \bar{\sigma}(W(i\omega)) < \epsilon$. Let $\lambda \in \mathbb{R}$ be such that $0 < \lambda < \frac{\epsilon}{R}(\sigma^2-\epsilon^2)^{-\frac{1}{2}}$ and define $f(s) := \frac{1}{s\lambda+1}$. Note that $f \in \mathbb{R}_+(s)$ and f has only poles in $\mathbb{C}^-$. Hence, $\|f\|_\infty$ exists. In fact, $\|f\|_\infty = \sup_{\omega\in\mathbb{R}} |f(i\omega)| = \sup_{\omega\in\mathbb{R}} (\sqrt{\lambda^2\omega^2+1})^{-1} = 1$. Define the real rational matrix $X \in M(\mathbb{R}(s))$ as $X := fX'$ then it is clear that $\partial X = \partial X' - 1$. Furthermore, it follows that $UXV+W = fUX'V+W = (1-f)W + f(UX'V+W)$. Now both $(1-f)W$ and $f(UX'V+W)$ are strictly proper real rational matrices and only have poles in $\mathbb{C}^-$, since $\|W\|_\infty < \infty$ and $\|UX'V+W\|_\infty < \infty$. This implies that $\|UXV+W\|_\infty$ is well-defined and satisfies

$$\|UXV+W\|_\infty \leqslant \|(1-f)W\|_\infty + \|f(UX'V+W)\|_\infty \leqslant$$

$$\|(1-f)W\|_\infty + \|f\|_\infty \|(UX'V+W)\|_\infty \leqslant \|(1-f)W\|_\infty + \|UX'V+W\|_\infty .$$

where the submultiplicative property of the H_∞-norm is used (cf. [1]). Now observe that

$$\|(1-f)W\|_\infty = \sup_{\omega\in\mathbb{R}} \bar{\sigma}((1-f(i\omega))W(i\omega)) = \sup_{\omega\in\mathbb{R}} (|1-f(i\omega)|\bar{\sigma}(W(i\omega))) =$$

$$\max\left[\sup_{|\omega|>R}(|1-f(i\omega)|\bar{\sigma}(W(i\omega))),\ \sup_{|\omega|\leqslant R}(|1-f(i\omega)|\bar{\sigma}(W(i\omega)))\right] \leqslant$$

$$\max\left[\sup_{|\omega|>R}\left(\sqrt{\frac{\lambda^2\omega^2}{\lambda^2\omega^2+1}}\,\epsilon\right),\ \sup_{|\omega|\leqslant R}\left(\sqrt{\frac{\lambda^2\omega^2}{\lambda^2\omega^2+1}}\,\sigma\right)\right] \leqslant \max(\epsilon, \frac{\epsilon}{\sigma}\sigma) = \epsilon.$$

Hence, $X\in M(\mathbb{R}(s))$ satisfies $\partial X = \partial X' - 1$ and $\|UXV+W\|_\infty \leqslant \|UX'V+W\|_\infty + \epsilon$. □

Using lemmas 2.2, 4.1, 4.2 and 4.3 we can give the following proof of theorem 3.1.

PROOF OF THEOREM 3.1.

(if-part). Assume that RME is solvable over $\mathbb{R}(s)$ and is solvable at infinity, and let $\epsilon>0$ be arbitrary. Then to prove the (if part) of theorem 3.1, we construct a proper real rational matrix $X_\epsilon\in M(\mathbb{R}_0(s))$ such that $\|UX_\epsilon V+W\|_\infty \leqslant \epsilon$.

Let $X\in M(\mathbb{R}(s))$ be such that $UXV+W=0$. Because RME is solvable at infinity it follows from lemma 4.1 that there is a proper real rational matrix $\hat{X}\in M(\mathbb{R}_0(s))$ such that $U\hat{X}V+W\in M(\mathbb{R}_+(s))$. Denote $X' := X-\hat{X}$ and $W' := U\hat{X}V+W$. Clearly $UX'V+W'=0$ with $X'\in M(\mathbb{R}(s))$ and $W'\in M(\mathbb{R}_+(s))$. By lemma 4.2 there is a matrix $\bar{X}\in M(\mathbb{R}_0(s))$ such that $U\bar{X}V+W'\in M(\mathbb{R}_+(s))$ and $\|U\bar{X}V+W'\|_\infty<\infty$. Denote $X_0 := X'-\bar{X}$ and $W_0 := U\bar{X}V+W'$ then $UX_0V+W_0=0$ with $X_0\in M(\mathbb{R}(s))$, $W_0\in M(\mathbb{R}_+(s))$ and $\|W_0\|_\infty<\infty$.

If $X_0=0$, then define $X_\epsilon := X$ and it follows easily that $X_\epsilon\in M(\mathbb{R}_0(s))$ and $\|UX_\epsilon V+W\|_\infty = 0\leqslant\epsilon$. Hence, in case $X_0=0$ the proof is completed. If $X_0\neq 0$, then ∂X_0 is defined and set $\tau := \partial X_0$ and $\sigma := \|W_0\|_\infty$. If $\sigma\leqslant\epsilon$, define $X_\epsilon := \bar{X}+\hat{X}$ and it follows that $X_\epsilon\in M(\mathbb{R}_0(s))$ and $\|UX_\epsilon V+W\|_\infty = \|W_0\|_\infty \leqslant \epsilon$. If $\tau\leqslant 0$, define $X_\epsilon := X_0+\bar{X}+\hat{X}$ and it follows that $X_\epsilon\in M(\mathbb{R}_0(s))$ and $\|UX_\epsilon V+W\|_\infty = 0\leqslant\epsilon$. So, if $X_0\neq 0$ the proof has been completed in the cases that $\sigma\leqslant\epsilon$ or $\tau\leqslant 0$.

Remains to consider the case for $X_0\neq 0$ that $\sigma>0$ and $\tau>0$. In this case repeated application of lemma 4.3 proves the existence of real rational matrices $X_1, X_2, \ldots, X_\tau\in M(\mathbb{R}(s))$ such that $\partial X_{i+1} = \partial X_i - 1$ and $\|UX_{i+1}V+W_0\|_\infty \leqslant \|UX_iV+W_0\|_\infty + (\epsilon/\tau)$ for all $i=0,1,\ldots,\tau-1$. Define $X_\epsilon := X_\tau+\bar{X}+\hat{X}$. Then it follows that $X_\epsilon\in M(\mathbb{R}_0(s))$ because $\partial X_\tau = \partial X_0-\tau = 0$. Furthermore, $\|UX_\epsilon V+W\|_\infty = \|UX_\tau V+W_0\|_\infty \leqslant \|UX_0V+W_0\|_\infty + \tau(\epsilon/\tau) = 0+\epsilon = \epsilon$. Hence, for $X_0\neq 0$ the proof has been completed also in the case that $\sigma>\epsilon$ and $\tau>0$.

(only if-part). From lemma 2.2 it follows that for proving the solvability of RME over $\mathbb{R}(s)$ it suffices to prove that $PW=0$ for all $P\in M(\mathbb{R}(s))$ that satisfy $PU=0$ and $WQ=0$ for all $Q\in M(\mathbb{R}(s))$ that satisfy $VQ=0$. Using suitable scalar premultiplication, it follows that it even suffices to prove that $PW=0$ for all $P\in M(\mathbb{R}_0(s))$ that satisfy $PU=0$, $\|P\|_\infty<\infty$, and $WQ=0$ for all $Q\in M(\mathbb{R}_0(s))$ that satisfy $VQ=0$, $\|Q\|_\infty<\infty$. Therefore assume that $P\in M(\mathbb{R}_0(s))$ is such that $PU=0$ and $\|P\|_\infty<\infty$, and assume that for all $\epsilon>0$ there is an $X_\epsilon\in M(\mathbb{R}_0(s))$ such that $\|UX_\epsilon V+W\|_\infty\leqslant\epsilon$. Then for all $\epsilon>0$ we have that $\|PW\|_\infty = \|P(UX_\epsilon V+W)\|_\infty \leqslant \|P\|_\infty\|UX_\epsilon V+W\|_\infty \leqslant \epsilon\|P\|_\infty$. Thus, for all $\epsilon>0$ we have that $\|PW\|_\infty \leqslant \epsilon\|P\|_\infty$. Hence, $PW=0$. Completely dual we can prove that $WQ=0$ for all $Q\in M(\mathbb{R}(s))$ such that $VQ=0$.

To prove that RME is solvable at infinity, again by lemma 2.2, it suffices to prove that $P'W(\infty)=0$ for all $P'\in M(\mathbb{R})$ that satisfy $P'U(\infty)=0$ and $W(\infty)Q'=0$ for all $Q'\in M(\mathbb{R})$ that satisfy $V(\infty)Q'=0$. Therefore assume that $P'\in M(\mathbb{R})$ is such that $P'U(\infty)=0$ and assume that for all $\epsilon>0$ there is a matrix $X_\epsilon\in M(\mathbb{R}_0(s))$ such that $\|UX_\epsilon V+W\|_\infty\leqslant\epsilon$. Then it follows that $\bar{\sigma}(P'W(\infty)) = \bar{\sigma}(P'(U(\infty)X_\epsilon(\infty)V(\infty)+W(\infty))) \leqslant \|P'(UX_\epsilon V+W)\|_\infty \leqslant \bar{\sigma}(P')\|UX_\epsilon V+W\|_\infty \leqslant \epsilon\bar{\sigma}(P')$ for all $\epsilon>0$. Hence, $\bar{\sigma}(P'W(\infty))=0$, or $P'W(\infty)=0$. Here we have used that $\bar{\sigma}(P') = \|P'\|_\infty$ for all $P'\in M(\mathbb{R})$ and that $\bar{\sigma}(A(\infty)) \leqslant \|A\|_\infty$ for all $A\in M(\mathbb{R}_0(s))$. Completey dual we can prove that $W(\infty)Q'=0$ for all $Q'\in M(\mathbb{R})$ that satisfy $V(\infty)Q'=0$. □

5. Algorithm

In the present section we describe an algorithm by which we can decide whether or not ADDPM is solvable. Furthermore, if ADDPM is solvable and $\epsilon>0$ is given, then the algorithm enables us to determine the transfer matrix $K \in M(\mathbb{R}_0(s))$ of a controller (2) such that the H_∞-norm of the transfer matrix T of the controlled system (3) is less than or equal to ϵ. The algorithm given below is obtained by a careful examination of the proof of the if-part of theorem 3.1.

Algorithm

Data : $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ with $G_{11}, G_{21}, G_{12} \in M(\mathbb{R}_0(s))$, and $G_{22} \in M(\mathbb{R}_+(s))$, and $\epsilon>0$.

1. If $G_{11}=0$, set $K:=0$, and it is clear that $\|T\|_\infty = \|G_{11}\|_\infty \leqslant \epsilon'$ for all $\epsilon'>0$. Hence, ADDPM is solvable and the algorithm can stop after a trivial step. If $G_{11} \neq 0$, the algorithm continues as follows.

2. Check if $\operatorname{rank} G_{12} = \operatorname{rank}[G_{11}\,,\, G_{12}]$, $\operatorname{rank} G_{21} = \operatorname{rank}\begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix}$, and

$$\operatorname{rank} G_{12}(\infty) = \operatorname{rank}[G_{11}(\infty)\,,\, G_{12}(\infty)]\,,\ \operatorname{rank} G_{21}(\infty) = \operatorname{rank}\begin{bmatrix} G_{11}(\infty) \\ G_{21}(\infty) \end{bmatrix}.$$

If all four rank conditions hold then ADDPM is solvable (see theorem 3.3); if not, there is an $\epsilon'>0$ such that $\|(G_{11} + G_{12}K(I-G_{22}K)^{-1}G_{12}\|_\infty > \epsilon'$ for all $K \in M(\mathbb{R}_0(s))$. If ADDPM is solvable the algorithm continues as follows.

3. Compute a real rational matrix $X \in M(\mathbb{R}(s))$ and a real matrix $X_r \in M(\mathbb{R})$ such that $G_{11}+G_{12}XG_{21}=0$ and $G_{11}(\infty)+G_{12}(\infty)X_rG_{21}(\infty)=0$. The existence of these matrices follows from lemma 3.2. Because $G_{11} \neq 0$, it follows that $X \neq 0$. Denote $\tau := \partial X$. If $\tau \leqslant 0$, then $X \in M(\mathbb{R}_0(s))$. Set $K:=(I+XG_{22})^{-1}X$, and it follows that $0=\|G_{11}+G_{12}XG_{21}\|_\infty = \|G_{11}+G_{12}K(I-G_{22}K)^{-1}G_{21}\|_\infty = \|T\|_\infty \leqslant \epsilon$. Hence, the algorithm can stop. If $\tau>0$ the algorithm proceeds as follows.

4. Denote $X'(s):=X(s)-X_r$ and $G'_{11}(s):=G_{11}(s)+G_{12}(s)X_rG_{21}(s)$. Note that $\partial X'=\tau$ since $\tau>0$. Now if $G'_{11}=0$ set $K(s):=(I+X_rG_{22}(s))^{-1}X_r$. Then $0=\|G_{11}(s)+G_{12}(s)X_rG_{21}(s)\|_\infty = \|G_{11}+G_{12}K(I-G_{22}K)^{-1}G_{21}\|_\infty = \|T\|_\infty \leqslant \epsilon$ and the algorithm can stop. If $G'_{11} \neq 0$ then the algorithm proceeds with the next step.

5. Determine $d \in \mathbb{R}[s]$ and $N \in M(\mathbb{R}[s])$ such that $G'_{11}=N/d$ and denote $\delta:=\partial d$. Since $0 \neq G'_{11} \in M(\mathbb{R}_+(s))$ it follows that $0 \leqslant \partial N < \delta$. Next, determine $q \in \mathbb{R}[s]$ such that $\partial q=\tau+\delta$ and q has only zeros in $\mathbb{C}^-$ (for instance, take $q(s)=(s+1)^{\tau+\delta}$). Calculate $p,r \in \mathbb{R}[s]$ such that $q=pd+r$ with either $0 \leqslant \partial r<\delta$ or $r=0$, and define $\hat{X} \in M(\mathbb{R}(s))$ as $\hat{X}:=(r/q)X'$. Denote $X_0:=X'-\hat{X}$ and $\hat{G}_{11}:=G'_{11}+G_{12}\hat{X}G_{21}$. Note that $\partial X_0 = \partial X' = \tau>0$. Furthermore, by the proof of lemma 4.2 it follows that $\hat{G}_{11} \in M(\mathbb{R}_+(s))$ and $\|\hat{G}_{11}\|_\infty<\infty$. Denote $\sigma:=\|\hat{G}_{11}\|_\infty$. If $\sigma \leqslant \epsilon$, set $K(s) := (I+(\hat{X}(s)+X_r)G_{22}(s))^{-1}\,(\hat{X}(s)+X_r)$, then $\|\hat{G}_{11}\|_\infty = \|G_{11}+G_{12}K(I-G_{22}K)^{-1}G_{21}\|_\infty = \|T\|_\infty \leqslant \epsilon$ and the algorithm can stop. If $\sigma>\epsilon$, the algorithm continues as follows.

6. Determine $R \in \mathbb{R}$ such that $0<R<\infty$ and $\sup\limits_{|\omega|>R} \bar{\sigma}(\hat{G}_{11}(i\omega)) < \epsilon/\tau$. Let $\lambda \in \mathbb{R}$ be such that $0<\lambda< \frac{\epsilon}{\tau R}(\sigma^2-\frac{\epsilon^2}{\tau^2})^{-\frac{1}{2}}$ and define $X_\tau(s):=(s\lambda+1)^{-\tau}X_0(s)$. From the proof of theorem 3.1 it follows that $X_\tau \in M(\mathbb{R}_0(s))$ and $\|\hat{G}_{11}+G_{12}X_\tau G_{21}\|_\infty \leqslant \epsilon$. Define $X''(s):=X_\tau(s)+\hat{X}(s)+X_r$ and $K:=(I+X''G_{22})^{-1}X''$. Then it follows that $\|G_{11}+G_{12}X''G_{21}\|_\infty = \|\hat{G}_{11}+G_{12}X_\tau G_{21}\|_\infty = \|G_{11}+G_{12}K(I-G_{22}K)^{-1}G_{21}\|_\infty = \|T\|_\infty \leqslant \epsilon$. Hence, the algorithm can stop.

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