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Department of Operations Research, Statistics, and System Theory Report BS-R8913 July

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Probabilistic Analysis of the Minimum Weighted Flowtime Scheduling Problem

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The minimum weighted flow time scheduling problem is studied from a probabilistic point of view. A probability distribution is specified over its problem instances, and the asymptotics of the optimal solution value are derived. Rewriting this value as a *U*-statistic perturbed by a small term allows us to use results from the well-established theory on these statistics. We derive a law of large numbers, a law of the iterated logarithm and a central limit theorem. As a byproduct we obtain a proof of asymptotic optimality almost surely of a greedy heuristic (the shortest weighted processing time first rule) for the solution of the *NP*-complete problem with more than one machine.

Key Words & Phrases: Minimum weighted flowtime schedule, strong law of large numbers, law of the iterated logarithm, central limit theorem, *U*-statistic, asymptotic optimality.
1980 Mathematics Subject Classification: 60F05, 60G50, 90B35

Introduction

In this paper we derive convergence properties of the optimal solution value of the *scheduling* problem with a *minimum weighted flowtime* objective. For this problem we are given n jobs with *processing times* p_j and *weights* w_j, $j=1,...,n$, that are to be scheduled on m identical parallel machines. Let C_j be the completion time of job j, $j=1,...,n$, under a feasible schedule. Then the weighted flow time of this schedule is $\Sigma_{j=1}^{n} w_j C_j$.

This problem, in scheduling standard notation $m\|\Sigma_{j=1}^{n} w_j C_j$, is *NP-hard* for $m \geqslant 2$ (see [Garey & Johnson 1979, p. 240]). For the single machine version ($m=1$) an optimal schedule is obtained by the (polynomial) *shortest weighted processing time first* rule (SWPT) [Smith 1956]: the jobs are scheduled on the machine in order of increasing p_j/w_j - ratio.

In Section 2 we assume a probability distribution over the class of problem instances and analyse the asymptotic behaviour of the optimal solution value of the single machine problem. More specifically, we derive *a strong law of large numbers* which shows that the optimal solution value of problem instances with a growing number of jobs converges with probability 1 (*wp* 1) to a constant, if properly normalized. Moreover we present a rate of convergence in the form of a *law of the iterated logarithm*, and a *central limit theorem*. The derivation of these results rely heavily on the transcription of the optimal solution value as a *U*-statistic perturbed by a term that is asymptotically negligible.

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The results follow then easily from the well-established theory on these statistics.

In Section 3 we consider the m-machine problem. We derive upper and lower bounds on the optimal solution value. For the upper bound we use the SWPT-rule as an approximation algorithm. Once established these bounds, the asymptotics of the optimal solution value of the m-machine problem appear to come straightforward from those of the single machine problem. As a byproduct we obtain a proof of asymptotic optimality (*wp* 1) of the SWPT-rule.

Some concluding remarks follow in Section 4. Throughout the paper random variables are indicated by boldface characters.

2. The single machine problem

Let us assume that the processing times of the jobs $\mathbf{p}_1, \mathbf{p}_2, \ldots$ are positive, independent and identically distributed (i.i.d.) random variables and that the weights $\mathbf{w}_1, \mathbf{w}_2, \ldots$ are positive i.i.d. random variables. The optimal solution value of a minimum weighted flowtime problem is then a random variable denoted by $\mathbf{z}^*_{m,n}$.

In this section we study convergence properties of $\mathbf{z}^*_{1,n}$, the optimal value of the single machine problem. This value can be characterized through the SWPT-rule (see Section 1). Let $(\mathbf{w}_{(1)}, \mathbf{p}_{(1)}), (\mathbf{w}_{(2)}, \mathbf{p}_{(2)}), \ldots, (\mathbf{w}_{(n)}, \mathbf{p}_{(n)})$ be order statistics of n observations of the weight-processing time pair, where the order is dictated by increasing $\mathbf{p}_j/\mathbf{w}_j$ - ratio's, i.e.,

$$\frac{\mathbf{p}_{(1)}}{\mathbf{w}_{(1)}} \leqslant \frac{\mathbf{p}_{(2)}}{\mathbf{w}_{(2)}} \leqslant \ldots \leqslant \frac{\mathbf{p}_{(n)}}{\mathbf{w}_{(n)}}.$$

The completion time of the j-th job in this sequence is equal to $\Sigma_{k=1}^{j} \mathbf{p}_{(k)}$, and the optimal value is given by

$$\mathbf{z}^*_{1,n} = \Sigma_{j=1}^{n} \mathbf{w}_{(j)} \Sigma_{k=1}^{j} \mathbf{p}_{(k)}. \tag{2.1}$$

To avoid analytical difficulties we assume that the ratio $\mathbf{p}/\mathbf{w}$ has a continuous distribution. In Section 2.1 we show that the behaviour of $\mathbf{z}^*_{1,n}$, properly normalized, is essentially that of a so-called U-statistic. Given this knowledge, in Section 2.2 asymptotic properties of $\mathbf{z}^*_{1,n}$ are derived almost straightforwardly from the theory on U-statistics.

2.1. Towards a U-statistic

Let $h(x_1, \ldots, x_l)$ be a symmetric real-valued function, called a *kernel*. Its domain is the l-fold Cartesian product of the spaces where the x_i's live on. For any such a kernel, the corresponding U-statistic, based on a sample of n independent observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$ on some distribution, with $n \geqslant l$, is obtained by averaging the kernel h over the observations:

$$\mathbf{U}_n = \frac{1}{\binom{n}{l}} \sum_{i_1, \ldots, i_l} h(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_l}),$$

where the summation is over all the $\binom{n}{l}$ possible combinations of l out of n observations. For a survey of the theory on U-statistics we refer to [Serfling 1980, Chapter 5].

In our specific situation we define the kernel function $h: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$h((w,p), (w', p')) = \begin{cases} wp'/2 & \text{if } p/w > p'w' \\ w'p/2 & \text{if } p/w < p'w' \\ 0 & \text{if } p/w = p'/w'. \end{cases} \tag{2.2}$$

The corresponding U-statistic is given by

$$\mathbf{U}_n = \frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} h((\mathbf{w}_i, \mathbf{p}_i), (\mathbf{w}_j, \mathbf{p}_j)).$$

It is not hard to see that $\mathbf{z}^*_{1,n}$ in (2.1) can be rewritten as

$$\frac{\mathbf{z}^*_{1,n}}{n(n-1)} = \mathbf{U}_n + \frac{\Sigma_{j=1}^n \mathbf{w}_j \mathbf{p}_j}{n(n-1)}. \tag{2.3}$$

We observe that $\mathbf{z}^*_{1,n}$ normalized by $n(n-1)$ is a U-statistic perturbed by a normalized sum of i.i.d. random variables. The influence of the latter term on the asymptotic behaviour of $\mathbf{z}^*_{1,n}$ will turn out to be negligible.

2.2. Convergence properties of $\mathbf{z}^*_{1,n}$

As a prerequisite for convergence theorems on U-statistics that we shall use, the variance of the conditional expectation $E(h((\mathbf{w},\mathbf{p}), (\mathbf{w}', \mathbf{p}'))|(\mathbf{w},\mathbf{p}))$ should not be equal to zero. Let us denote this conditional expectation by $h_1(\mathbf{w},\mathbf{p})$. Under this condition the theory of U-statistics resembles strongly that of sums of i.i.d. random variables. The condition is easily verified in our case if we assume that $(\mathbf{w},\mathbf{p})$ has a distribution whose support contains a two dimensional convex set.

LEMMA 2.1. *Under the above assumption and* $E\mathbf{w}_1 < \infty$, $E\mathbf{p}_1 < \infty$, *we have* $\mathrm{Var}(h_1(\mathbf{w},\mathbf{p})) > 0$.

PROOF. If the lemma were not true, then there exists a constant c such that $h_1(\mathbf{w},\mathbf{p}) = c$ $(wp\,1)$. Now take any (w,p) and $(\alpha w, \alpha p)$ from the convex set in the support of the distribution of $(\mathbf{w},\mathbf{p})$, with $\alpha > 0$, $\alpha \neq 1$. Thus, we have $h_1(w,p) = h_1(\alpha w, \alpha p) = c$. However, from definition (2.2) of $h((w,p), (w',p'))$ it is easy to see that $h_1(\alpha w, \alpha p) = \alpha h_1(w,p)$ a contradiction if $0 < c < \infty$. But $h_1(\mathbf{w},\mathbf{p}) = 0$ $(wp\,1)$ can only occur if the distribution of $\mathbf{p}/\mathbf{w}$ is concentrated on only one point, which we excluded by our assumptions. Obviously, the assumptions $E\mathbf{w} < \infty$, $E\mathbf{p} < \infty$ imply $h_1(\mathbf{w},\mathbf{p}) < \infty$ $(wp\,1)$. □

Now we are ready to apply directly theorems from [SERFLING 1980, Chapter 5] to establish, respectively, a law of large numbers, a law of the iterated logarithm, and a central limit theorem for $\mathbf{z}^*_{1,n}/(n(n-1))$. Let $\theta = Eh((\mathbf{w},\mathbf{p}), (\mathbf{w}',\mathbf{p}'))$.

THEOREM 2.2. *Law of large numbers. If* $E\mathbf{w}_1 < \infty$, $E\mathbf{p}_1 < \infty$ *and* $E\mathbf{w}_1\mathbf{p}_1 < \infty$, *then*

$$\lim_{n\to\infty} \frac{\mathbf{z}^*_{1,n}}{n(n-1)} = \theta \quad (wp\,1).$$

PROOF. Recall formula (2.3). Using the independence of $(\mathbf{w},\mathbf{p})$ and $(\mathbf{w}',\mathbf{p}')$ we have

$$\theta = E(h((\mathbf{w},\mathbf{p}), (\mathbf{w}',\mathbf{p}')) \leqslant \tfrac{1}{2}E(\mathbf{w}\mathbf{p}' + \mathbf{w}'\mathbf{p})$$
$$\leqslant \tfrac{1}{2}E\mathbf{w}E\mathbf{p}' + \tfrac{1}{2}E\mathbf{w}'E\mathbf{p} < \infty$$

by the assumptions. Hence Theorem A from [SERFLING 1980, p. 190] can be applied to prove that

$$\lim_{n\to\infty} \mathbf{U}_n = \theta \quad (wp\,1). \tag{2.4}$$

Since $E\mathbf{w}_1\mathbf{p}_1 < \infty$, the usual strong law of large numbers implies that

$$\lim_{n\to\infty} \frac{\Sigma_{j=1}^n \mathbf{w}_j \mathbf{p}_j}{n(n-1)} = 0 \quad (wp\,1). \tag{2.5}$$

Together (2.4) and (2.5) imply the theorem . □

THEOREM 2.3. *Law of the iterated logarithm. Let* $\sigma^2 = \mathrm{Var}(h_1(\mathbf{w},\mathbf{p}))$. *Assume that* $\sigma^2 > 0$, $E\mathbf{w}_1^2 < \infty$ *and* $E\mathbf{p}_1^2 < \infty$. *Then*

$$\limsup_{n\to\infty} \frac{\sqrt{n}(\mathbf{z}^*_{1,n}/(n(n-1)) - \theta)}{2\sigma(2\log\log n)^{1/2}} = 1 \quad (wp\,1).$$

PROOF. Recall formula (2.3). By independence of $(\mathbf{w},\mathbf{p})$ and $(\mathbf{w}',\mathbf{p}')$ we have

$$E(h^2(\mathbf{w},\mathbf{p}),\ (\mathbf{w}'\mathbf{p}')) \leqslant \frac{1}{4}E(\mathbf{w}^2\mathbf{p}'^2 + \mathbf{w}'^2\mathbf{p}^2) < \infty. \tag{2.6}$$

Therefore, we may apply Theorem C from [SERFLING 1980, p. 191] to obtain

$$\limsup_{n\to\infty} \frac{\sqrt{n}(\mathbf{U}_n - \theta)}{2\sigma(2\log\log n)^{1/2}} = 1 \quad (wp\ 1). \tag{2.7}$$

By Cauchy-Schwartz' inequality we have

$$E\mathbf{w}_1\mathbf{p}_1 \leqslant (E\mathbf{w}_1^2 E\mathbf{p}_1^2)^{1/2} < \infty,$$

so that the strong law of large numbers implies that for every $\delta > 0$

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \mathbf{w}_j\mathbf{p}_j}{n^{1+\delta}} = 0 \quad (wp\ 1). \tag{2.8}$$

Together (2.7) and (2.8) imply the theorem. □

THEOREM 2.4. *Central limit theorem. Let* $\sigma^2 = \mathrm{Var}(h_1(\mathbf{w},\mathbf{p}))$. *Assume that* $\sigma^2 > 0$, $E\mathbf{w}_1^2 < \infty$, $E\mathbf{p}_1^2 < \infty$. *Then*

$$\frac{\sqrt{n}}{2\sigma}(\mathbf{z}^*_{1,n}/(n(n-1)) - \theta) \rightsquigarrow N(0,1).$$

PROOF. Since we have the same assumptions here as in Theorem 2.3, (2.6) holds. Therefore we may apply Theorem A from [SERFLING 1980 p. 192] to obtain

$$\frac{\sqrt{n}}{2\sigma}(\mathbf{U}_n - \theta) \rightsquigarrow N(0,1). \tag{2.9}$$

This together with (2.8), which holds here too, implies the theorem. □

3. THE *m*-MACHINE PROBLEM

The stochastic assumptions and the notation of the previous section are maintained here. In addition, we assume that the number of machines is fixed.

3.1. Upper and lower bounds on $\mathbf{z}^*_{m,n}$

For a lower bound on $\mathbf{z}^*_{m,n}$ we refer to [EASTMAN et al. 1964]: for any schedule of the jobs, such that on each machine the jobs assigned to it are scheduled in order of increasing $\mathbf{p}/\mathbf{w}$-ratio, the sum of the weighted completion times is bounded from below by

$$\frac{m-1}{m}\sum_{j=1}^{n} \mathbf{w}_j\mathbf{p}_j + \frac{1}{m}\mathbf{z}^*_{1,n}.$$

Simple exchange arguments show that any optimal schedule must satisfy this property. Hence,

$$\mathbf{z}^*_{m,n} \geqslant \frac{m-1}{m}\sum_{j=1}^{n} \mathbf{w}_j\mathbf{p}_j + \frac{1}{m}\mathbf{z}^*_{1,n} \tag{3.1}$$

For an upper bound we consider the before mentioned SWPT-rule as an approximation method for the solution of the m-machine problem. Every next job to be scheduled is assigned to the machine that is first available. Switching for once again to order statistics (see Section 2), we notice that the j-th job to be scheduled will start no later than $(1/m)\sum_{k=1}^{j-1} \mathbf{p}_{(k)}$, $j = 1,\ldots,n$. Hence, its weighted completion time is bounded from above by

$$(1/m)\mathbf{w}_{(j)}\sum_{k=1}^{j-1} \mathbf{p}_{(k)} + \mathbf{w}_{(j)}\mathbf{p}_{(j)},\quad j = 1,\ldots,n.$$

Summation over j yields the following upper bound on the value of the solution produced by the

SWPT-rule, denoted by $z_{m,n}^{SWPT}$, which is on its turn an upper bound on $z_{m,n}^{*}$.

$$z_{m,n}^{*} \leqslant z_{m,n}^{SWPT} \leqslant \frac{1}{m}\Sigma_{j=1}^{n} w_{(j)}\Sigma_{k=1}^{j-1} p_{(k)} + \Sigma_{j=1}^{n} w_{(j)} p_{(j)}$$

$$= \frac{1}{m}\Sigma_{j=1}^{n} w_{(j)}\Sigma_{k=1}^{j} p_{(k)} + \frac{m-1}{m}\Sigma_{j=1}^{n} w_{(j)} p_{(j)}.$$

Since the first term on the right hand side is equal to $(1/m)z_{1,n}^{*}$ (see (2.1)), we may rewrite the above inequality as

$$z_{m,n}^{*} \leqslant z_{m,n}^{SWPT} \leqslant \frac{1}{m} z_{1,n}^{*} + \frac{m-1}{m}\Sigma_{j=1}^{n} w_j p_j. \tag{3.2}$$

We see that both the upper and the lower bound consist of a term $(1/m)z_{1,n}^{*}$ and a term including $\Sigma_{j=1}^{n} w_j p_j$, which in Section 2.2 turned out to become negligible asymptotically.

3.2. Convergence properties of $z_{m,n}^{*}$

From the upper and lower bound (3.2) and (3.1) it is clear that the asymptotics of $z_{m,n}^{*}$ follow those of $(1/m)z_{1,n}^{*}$. Therefore we give the following convergence theorems for $z_{m,n}^{*}$ without proof.

THEOREM 3.1. *Law of large numbers. If* $Ew_1 < \infty, Ep_1 < \infty$ *and* $Ew_1 p_1 < \infty$, *then*

$$\lim_{n\to\infty} \frac{z_{m,n}^{*}}{n(n-1)} = \frac{\theta}{m} \quad (wp\ 1). \square$$

THEOREM 3.2. *Law of the iterated logarithm. Let* $\sigma^2 = \mathrm{Var}(h_1(\mathbf{w},\mathbf{p}))$. *If* $Ew_1^2 < \infty$, $Ep_1^2 < \infty$ *and* $\sigma^2 > 0$, *then*

$$\limsup_{n\to\infty} \frac{m\sqrt{n}(z_{m,n}^{*}/(n(n-1)) - \theta/m)}{2\sigma(2\log\log n)^{1/2}} = 1 \quad (wp\ 1). \square$$

THEOREM 3.3. *Central limit theorem. Let* $\sigma^2 = \mathrm{Var}(h_1(\mathbf{w},\mathbf{p}))$. *If* $Ew_1^2 < \infty$, $Ep_1^2 < \infty$ *and* $\sigma^2 > 0$, *then*

$$\frac{m\sqrt{n}}{2\sigma}(z_{m,n}^{*}/(n(n-1)) - \theta/m) \rightsquigarrow N(0,1). \square$$

All of the above holds if we substitute $z_{m,n}^{SWPT}$ for $z_{m,n}^{*}$. This implies among other things asymptotic optimality (*wp* 1) of the SWPT-rule for the minimum weighted flowtime scheduling problem:

$$\lim_{n\to\infty} \frac{z_{m,n}^{SWPT} - z_{m,n}^{*}}{z_{m,n}^{*}} = 0 \quad (wp\ 1).$$

4. Concluding remarks

The results presented in this paper generalize and extend results on the unweighted minimum flowtime problem in [Frenk et al. 1984]. We notice that it is also possible to obtain the results using the entropy approach to empirical process theory (cf. (cf. [Pollard 1984]). The results in Section 3 indicate nice behaviour of the SWPT-rule as an approximation method for the m-machine problem. It would be interesting to study the worst-case behaviour of this method.

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