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Perturbation theory for dual semigroups and its applications to age-dependent population dynamics

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In this paper we first summarize the perturbation theory in a sun-reflexive Banach space developed by CLÉMENT et al. (1987a, 1988). Next we shall introduce the concept of strong ergodicity for the evolutionary system and then we investigate conditions under which the evolutionary system generated by the perturbation of a C_0-semigroup in a sun-reflexive Banach space becomes strongly ergodic. Subsequently, we construct the evolutionary system corresponding to Lotka's renewal equation and apply it to prove strong ergodicity of the age-structured population with time-dependent vital rates. Finally, we study the controllability of the age-structured population controlled by changing its total fertility rate.

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1. INTRODUCTION

In age-dependent population dynamics, the basic population model is described by the Lotka-McKendrick-Von Foerster system:

$$(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})p(a,t) = -\mu(a)p(a,t), \quad t>0,\ 0<a<\omega, \tag{1.1a}$$

$$p(0,t) = \int_0^\omega m(a)p(a,t)da, \quad t>0, \tag{1.1b}$$

$$p(a,0) = \phi(a), \quad 0\leq a\leq\omega. \tag{1.1c}$$

In this system, $p(a,t)$ denotes the age-density at time t, that is, $\int_\alpha^\beta p(a,t)da$ is the number of individuals at time t between age α and age β. $\mu(a)$ and $m(a)$ are the per capita death rate and birth rate at age a, respectively. The integral boundary condition (1.1b) implies that newborns have age zero. The number ω is the life span of the members of the population or is the upper bound of the reproductive age, i.e. $m(a) = 0$ for $a>\omega$.

Although traditionally the system (1.1) has been solved by reducing it to a renewal integral equation, recently the semigroup approach to the above system has been widely developed by several authors and has shown its usefulness to study the asymptotic behavior of the population system (WEBB, 1985; METZ and DIEKMANN, 1986; INABA, 1988). Assume that $\phi\in X:= L^1(0,\omega)$, μ, $m\in L_+^\infty(0,\omega)$ and define the *population operator* A as

$$(A\phi)(a) = -\frac{d}{da}\phi(a)-\mu(a)\phi(a), \tag{1.2}$$

$$D(A) = \{\phi\in X\colon \phi\in AC[0,\omega],\ \phi(0) = \int_0^\omega m(a)\phi(a)da\},$$

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where $AC[0,\omega]$ is the set of absolutely continuous functions on $[0,\omega]$. Then the system (1.1) can be formulated as an abstract Cauchy problem in the Banach space X:

$$\frac{d}{dt}p(t) = Ap(t), \ p(0) = \phi\in X, \tag{1.3}$$

where $p(t)=p(.,t)\in X$. Since the population operator A generates a C_0-semigroup $T(t)$, $t\geq 0$, called the *population semigroup*, the evolution of the population in the state space X is given by

$$p(t) = T(t)\phi, \ t\geq 0. \tag{1.4}$$

However, if we want to deal with the time-inhomogeneous problem, i.e. the population with time-dependent vital rates, the semigroup approach has to be extended. In this case, since the domain of the population operator includes the time-dependent fertility rate $m(a,t)$, we have not so far had a general way to construct the evolutionary system $U(t,s)$, $0\leq s\leq t$, such that $U(t,s)\phi, \phi\in X$ gives the solution of the Cauchy problem

$$\frac{d}{dt}p(t) = A(t)p(t), \ p(s) = \phi\in X, \tag{1.5}$$

where

$$(A(t)\phi)(a) = -\frac{d}{da}\phi(a)-\mu(a,t)\phi(a), \tag{1.6}$$

$$D(A(t)) = \{\phi\in X{:}\phi\in AC[0,\omega],\ \phi(0) = \int_0^\omega m(a,t)\phi(a)da\}.$$

Recently Clément et al. (1987a, 1988) developed a systematic method to construct evolutionary systems by perturbing C_0-semigroups in sun-reflexive Banach spaces. Instead of the general system (1.3), consider the special simple case;

$$\frac{d}{dt}p(t) = A_0p(t), \ p(0) = \phi\in X, \tag{1.7}$$

where the operator A_0 is the closed operator in X given by

$$(A_0\phi)(a) = -\frac{d}{da}\phi(a)-\mu(a)\phi(a), \tag{1.8}$$

$$D(A_0) = \{\phi\in X{:}\ \phi\in AC[0,\omega],\ \phi(0) = 0\}.$$

Note that A_0 generates a C_0-semigroup $T_0(t)$, $t\geq 0$. Then, formally speaking, the system (1.1) can be seen as a perturbed system

$$\frac{d}{dt}p(t) = A_0p(t)+Bp(t), \ p(0) = \phi\in X. \tag{1.9a}$$

The operator B is defined by

$$(B\phi)(a) = \delta\int_0^\omega m(a)\phi(a)da, \tag{1.9b}$$

where δ denotes Dirac's delta function. If the operator B is a bounded linear operator from X to X, it is well known that A_0+B generates the C_0-semigroup $T(t)$, $t\geq 0$ defined as a solution of the variation-of-constants formula

$$T(t)\phi = T_0(t)\phi+\int_0^t T_0(t-\tau)BT(\tau)\phi d\tau. \tag{1.10}$$

However, since the operator given by (1.9b) maps out of the space X into some bigger space Y, we cannot apply this well known result to our problem (1.9). However Clément et al. (1987a) proved that in a sun-reflexive Banach space with respect to A_0, such a bigger space can be constructed systematically and the extended variation-of-constants formula defines the C_0-semigroup $T(t)$, $t \geqslant 0$. Moreover, they proved that the time-dependent perturbation $B(t): X \to Y$ defines the evolutionary system $U(t,s)$, $0 \leqslant s \leqslant t$ (CLÉMENT et al., 1988).

In this paper we first summarize the perturbation theory in a sun-reflexive Banach space developed by CLÉMENT et al. (1987a, 1988). Next we shall introduce the concept of strong ergodicity for the evolutionary system, which is a natural extension of the idea of asynchronous exponential growth for C_0-semigroups introduced by WEBB (1987). Then we investigate conditions under which the evolutionary system generated by the perturbation of a C_0-semigroup in a sun-reflexive space becomes strongly ergodic. We shall apply this perturbation approach to two demographic problems. First, we construct the evolutionary system corresponding to Lotka's renewal equation. We shall prove sufficient conditions for strong ergodicity of the age-structured population with time-dependent vital rates. Next we study the controllability of the age-structured population controlled by changing its total fertility rate.

2. DUAL SEMIGROUPS

In this section we summarize the results for dual semigroups which are needed for our purpose. For their proofs, the reader may refer to BUTZER and BERENS (1967), CLÉMENT, HEIJMANS, et al. (1987c), HILLE and PHILLIPS (1957) and YOSIDA (1980). Let X be a (non-reflexive) Banach space and let $T(t)$, $t \geqslant 0$ be a strongly continuous semigroup of bounded linear operators on X with infinitesimal generator A. Let $T^*(t)$, $t \geqslant 0$ denote the semigroup of adjoint operators acting on the dual space X^* and let A^* denote the adjoint of A.

PROPOSITION 2.1. *(1) For any $x^* \in X^*$ the map $t \to T^*(t)x^*$ from R^+ into X^* is weakly * continuous.* *(2) A^* is the weak * generator of $T^*(t)$, that is, x^* belongs to $D(A^*)$ if and only if $\frac{1}{t}(T^*(t)x^* - x^*)$ converges in the weak * topology as $t \downarrow 0$, and whenever there is convergence the limit is A^*x^*.*

The above proposition implies that $u^*(t) = T^*(t)x^*$ is a solution of the Cauchy problem

$$\frac{d}{dt}u^*(t) = A^*u^*(t), \; u^*(0) = x^* \in X^*, \tag{2.1}$$

whenever $x^* \in D(A^*)$ if differentiation is understood in the weak * sense. Generally the semigroup of bounded linear operators $T^*(t)$, $t \geqslant 0$ is not necessarily strongly continuous, although $T^*(t)$, $t \geqslant 0$ is a weak * continuous semigroup. Now we define a subspace $X^{\odot}$ of X^* by

$$X^{\odot} := \{x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0\}. \tag{2.2}$$

Then it is easily seen that the subspace $X^{\odot}$ is invariant under $T^*(t)$ and that $X^{\odot}$ is norm-closed. Let $T^{\odot}(t)$, $t \geqslant 0$ denote the restriction of $T^*(t)$ to $X^{\odot}$. Then $T^{\odot}(t)$, $t \geqslant 0$ is strongly continuous. Moreover the following holds:

PROPOSITION 2.2. *Let $A^{\odot}$ be the infinitesimal generator of $T^{\odot}(t)$, $t \geqslant 0$. Then the following holds;*
(1) $X^{\odot} = \overline{D(A^*)}$.
(2) *$A^{\odot}$ is the part of A^* in $X^{\odot}$, i.e. the largest restriction of A^* with both domain and range in $X^{\odot}$.*
(3) *$D(A^{\odot})$ is weak * dense in X^*.*

In what follows the elements of $X, X^*, X^{\odot}$ etc. are denoted by $x, x^*, x^{\odot}$ etc. We use $\langle x, x^* \rangle$ and $\langle x^*, x \rangle$ interchangeably to denote the value of x^* at x when $x \in X$ and $x^* \in X^*$. Moreover integrals of functions with values in a dual Banach space are regarded as weak * Riemann integrals. Hence if $t \to x^*(t)$ is continuous from $[a,b]$ to X^* equipped with its weak * topology, then $\int_a^b x^*(\tau)d\tau$ is defined as the unique element of X^* satisfying

$$<x, \int_a^b x^*(\tau)d\tau> = \int_a^b <x,x^*(\tau)>d\tau \text{ for all } x\in X.$$

The *prime norm* on X is defined as follows:

$$\|x\|' := \sup\{|<x,x^\odot>|: x^\odot\in X^\odot, \|x^\odot\|\leq 1\} \text{ for } x\in X.$$

Then it follows that

LEMMA. 2.3. *(1) The prime norm is equivalent with the original norm and when $T(t)$ is a contraction semigroup the two norms are actually the same.*
(2) If we equip X with the prime norm, the norm on $X^\odot$ remains unchanged, i.e.

$$\|x^\odot\| := \sup\{|<x,x^\odot>|: x\in X, \|x\|'\leq 1\} \text{ for } x^\odot\in X^\odot.$$

By taking the dual once more, we have again a weak * continuous semigroup $T^{\odot *}(t)$ with weak * generator $A^{\odot *}$ on $X^{\odot *}$. Every $x\in X$ defines a continuous linear functional on X^*, and hence can be considered as an element of $X^{\odot *}$. Since $X^\odot$ is weak * dense in X^* and X^* separates the points of X, $<x-y,x^\odot> = 0$ for all $x^\odot\in X^\odot$ implies that $x = y$ for $x,y\in X$. Therefore if we equip X with the prime norm there exists an isometric isomorphism of X onto a closed subspace of $X^{\odot *}$, that is, we can embed X into $X^{\odot *}$ by means of the natural mapping. In the following we shall identify X with its embedding into $X^{\odot *}$. By taking the restriction we introduce the subspace;

$$X^{\odot\odot} := \{x^{\odot *}\in X^{\odot *}: \lim_{t\downarrow 0}\|T^{\odot *}(t)x^{\odot *}-x^{\odot *}\| = 0\}.$$

In this case it can be proved that the prime norm on $X^\odot$ is the same as the original norm (see HILLE and PHILLIPS, 1957). It is clear that $X\subset X^{\odot\odot}$ since $T(t)$ is strongly continuous.

DEFINITION 2.4. X is called *sun-reflexive* ($\odot$-reflexive) with respect to A if and only if $X = X^{\odot\odot}$.

3. EVOLUTIONARY SYSTEMS AND THE VARIATION-OF-CONSTANTS FORMULA

In this section we state some results for the perturbation theory of dual semigroups in sun-reflexive Banach spaces without proofs. The reader may refer to CLÉMENT et al.(1987a, 1988) for their proofs.

Let X be a Banach space and let $T_0(t)$, $t\geq 0$ be a C_0-semigroup with generator A_0 and assume that X is sun-reflexive with respect to A_0. Let $T>0$ and let $B(t)$, $t\in[0,T]$ be a family of bounded linear operators from X to $X^{\odot *}$. We assume that $B(t)$, $t\in[0,T]$ is strongly continuous, i.e. for each $x\in X$ the mapping $t\rightarrow B(t)x$ is continuous from $[0,T]$ to $X^{\odot *}$.

DEFINITION 3.1. Let $T>0$ and let $\Delta := \{(t,s)\in R^2: 0\leq s\leq t\leq T\}$, $\Delta^* := \{(s,t)\in R^2: 0\leq s\leq t\leq T\}$. A two-parameter family $U(t,s)$, $(t,s)\in\Delta$ of bounded linear operators on a Banach space X is called a *forward evolutionary system* on X if the following two conditions are satisfied:
(1) $U(s,s)=I$ (the identity), $0\leq s\leq T$,
(2) $U(t,r)U(r,s)=U(t,s)$, $0\leq s\leq r\leq t\leq T$, (multiplicative property).
A two-parameter family $V(s,t),(s,t)\in\Delta^*$ of bounded linear operators on X is called a *backward evolutionary system* if
(1) $V(t,t)=I$, $0\leq t\leq T$,
(2) $V(s,r)V(r,t)=V(s,t)$, $0\leq s\leq r\leq t\leq T$.
An evolutionary system $U(t,s),(t,s)\in\Delta$ (or $V(s,t),(s,t)\in\Delta^*$) is said to be strongly continuous if for each $x\in X$ the mapping $(t,s)\rightarrow U(t,s)x$ ($(s,t)\rightarrow V(s,t)x$) is continuous from Δ (Δ^*) to X. Let $U(t,s)^*$ be the adjoint operator of $U(t,s)$. If we define the system $U^*(s,t)$, $(s,t)\in\Delta^*$ by $U^*(s,t) := U(t,s)^*$, $0\leq s\leq t\leq T$, then it is easily seen that $U^*(s,t)$, $(s,t)\in\Delta^*$ forms a backward evolutionary system. We

call it the *dual system* of $U(t,s)$, $(t,s)\in\Delta$.

LEMMA 3.2. *Let $f: \Delta \to X^{\odot *}$ be continuous. Then the function F defined by*

$$F(t,s) = \int_s^t T_0^{\odot *}(t-\tau) f(\tau,s)d\tau, \ (t,s)\in\Delta,$$

is continuous from Δ to X.

PROPOSITION 3.3. *The variation-of-constants formula*

$$U(t,s)x = T_0(t-s)x + \int_s^t T_0^{\odot *}(t-\tau)B(\tau)U(\tau,s)xd\tau, \ (t,s)\in\Delta, \ x\in X, \tag{3.1}$$

uniquely defines a strongly continuous forward evolutionary system $U(t,s)$, $(t,s)\in\Delta$ satisfying

$$\|U(t,s)\| \leqslant M\exp\{[\omega + MK(t,s)](t-s)\}, \tag{3.2}$$

where M and ω are such that $\|T_0(t)\| \leqslant M\exp(\omega t)$ and

$$K(t,s) := \sup_{s\leqslant\tau\leqslant t} \|B(\tau)\|.$$

In particular, if $t \to \|B(t)\|$ is measurable, we can sharpen the estimate (3.2) as follows:

$$\|U(t,s)\| \leqslant M\exp\{\int_s^t [\omega + M\|B(\tau)\|]d\tau\}. \tag{3.3}$$

The generation expansion

$$U(t,s) = \sum_{n\geqslant 0} U_n(t,s), \tag{3.4}$$

$$U_0(t,s) = T_0(t-s), \ U_n(t,s) = \int_s^t T_0^{\odot *}(t-\tau)B(\tau)U_{n-1}(\tau,s)xd\tau,$$

converges in the uniform operator topology uniformly on Δ.

In the case that the perturbation $B(t)$ is time-independent, the variation-of-constants formula (3.1) defines a C_0-semigroup $T(t)=U(t,0)$, $t\geqslant 0$ which is generated by the part of $A_0^{\odot *}+B$ in X (see CLÉMENT et al., 1987a, 1987c). Hence $T(t)\phi, \phi\in D(A)$ is the unique solution for the Cauchy problem

$$\frac{d}{dt}u(t) = Au(t), \ u(0) = \phi, \tag{3.5}$$

$$Ax = A_0^{\odot *}x + Bx, \ x\in D(A),$$

$$D(A) = \{x\in D(A_0^{\odot *}) : A_0^{\odot *}x + Bx \in X\}.$$

In the non-autonomous case we are confronted with a more complicated situation. Let $A(s)$ be the part of $A_0^{\odot *}+B(s)$ in X, i.e.

$$A(s)x = A_0^{\odot *}x + B(s)x, \tag{3.6}$$

$$x\in D(A(s)) := \{x\in D(A_0^{\odot *}) : A_0^{\odot *}x + B(s)x \in X\}.$$

Then it can be proved that:

PROPOSITION 3.4. *Let $x\in D(A(s))$. Then*

$$\frac{\partial^+}{\partial t} U(t,s)x|_{t=s} = A(s)x, \tag{3.7a}$$

$$\frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x, \tag{3.7b}$$

where the right derivative in (3.7a) and the derivative in (3.7b) are in the norm topology of X.

Since in general we cannot expect that $U(t,s)x$ is differentiable with respect to t in the norm topology of X, $U(t,s)x$ is not necessarily the solution of the Cauchy problem

$$\frac{d}{dt}u(t) = A(t)u(t),\ u(s) = x \in D(A(s)). \tag{3.8}$$

However, since X is embedded into $X^{\odot *}$, there are at least two other natural topologies in which $U(t,s)x$ could be differentiable. We can introduce some definitions:

DEFINITION 3.5. A function $f:[a,b] \to X^{\odot *}$ is *weak *- differentiable* with *weak *-derivative* g if for every $x^{\odot} \in X^{\odot}$, the real valued function $t \to \langle f(t), x^{\odot}\rangle$ is differentiable with derivative $\langle g(t), x^{\odot}\rangle$. A weak *-differentiable function f is continuously weak *-differentiable if in addition the function $t \to \langle g(t), x^{\odot}\rangle$ is continuous for all $x^{\odot} \in X^{\odot}$.

Now let us consider the (forward) Cauchy problem

$$\frac{d}{dt}u(t) = (A_0^{\odot *} + B(t))u(t),\ u(s) = x^{\odot *}. \tag{3.9}$$

DEFINITION 3.6. (a) A function $u:[s,T] \to X$ is called a *weak solution* to the Cauchy problem (3.9) with $x^{\odot *} = x \in X$ if $u(s) = x$ and if for every $x^{\odot} \in D(A_0^*)$ the real-valued function $t \to \langle u(t), x^{\odot}\rangle$ is continuously differentiable and

$$\frac{d}{dt}\langle u(t), x^{\odot}\rangle = \langle u(t), (A_0^* + B^*(t))x^{\odot}\rangle \text{ for } s < t \leqslant T. \tag{3.10}$$

(b) A function $u:[s,T] \to X^{\odot *}$ is called a *weak *-solution* to (3.9) if $u(s) = x^{\odot *}$, $u(t) \in D(A_0^{\odot *})$, $s < t \leqslant T$ and u is continuously weak *-differentiable on $(s,T]$ with weak *-derivative $(A_0^{\odot *} + B(t))u(t)$, i.e.

$$\frac{d}{dt}\langle u(t), x^{\odot}\rangle = \langle (A_0^{\odot *} + B(t))u(t), x^{\odot}\rangle \text{ for } s < t \leqslant T \text{ and all } x^{\odot} \in X^{\odot}. \tag{3.11}$$

PROPOSITION 3.7. (a) *For all* $x \in X, s \in [0,T), t \to U(t,s)x$ *is the unique weak solution to the forward problem* (3.9).
(b) *If* $U(t,s)D(A_0^{\odot *}) \subset D(A_0^{\odot *})$ *for every* $(t,s) \in \Delta$ *and* $\sup_{s \leqslant t \leqslant T} \|A_0^{\odot *}U(t,s)x\| < \infty$ *for all* $x \in D(A_0^{\odot *}), s \in [0,T)$, *then for every* $x \in D(A_0^{\odot *})$, $s \in [0,T)$, *the function* $t \to U(t,s)x$ *is the unique weak *-solution of the forward problem.*

REMARK 3.8. If $t \to B(t)$ is Lipschitz continuous from $[0,T]$ to $B(X, X^{\odot *})$ then the condition of the above proposition (b) is satisfied. (CLÉMENT et. al, 1988, Theorem 4.9). Therefore the Cauchy problem (3.9) has a unique weak *-solution under this condition.

Next we consider the backward problem. Here we assume that the mapping $t \to B(t)$ is continuous from $[0,T]$ to $B(X, X^{\odot *})$ equipped with the operator norm. Then the following can be proved:

PROPOSITION 3.9. *The subspace* $X^{\odot}$ *is invariant under* $U^*(s,t)$.

Since $X^{\odot}$ is invariant under $U^*(s,t), (s,t) \in \Delta^*$, we can define a backward system $U^{\odot}(s,t)$ on $X^{\odot}$ by restricting $U^*(s,t)$ on $X^{\odot}$, and $U^{\odot *}(t,s)$ forms a forward system on $X^{\odot *}$, which extends $U(t,s)$.

Then it follows that:

PROPOSITION 3.10. *The backward system* $U^{\odot}(s,t),(s,t)\in\Delta^*$ *is strongly continuous.*

PROPOSITION 3.11. *The function* $s\rightarrow U^{\odot *}(t,s)x^{\odot *}$ *is continuously weak *-differentiable on* $[0,t]$ *if and only if* $x^{\odot *} = x\in D(A_0^{\odot *})$ *and in that case the derivative equals* $-U^{\odot *}(t,s)(A_0^{\odot *}+B(s))x$.

REMARK 3.12. Although we only deal with linear problems here, it is easily seen that the perturbation theory for dual semigroups would provide a suitable framework to semilinear Cauchy problems. The reader may refer to CLÉMENT et al. (1987b) for such problems.

4. ASYMPTOTIC PROPERTIES OF EVOLUTIONARY SYSTEMS

In this section we consider asymptotic properties of the evolutionary system generated by a Lipschitz continuous perturbation of a C_0-semigroup. Of our interest here is to find out conditions under which the evolutionary system has a time-independent asymptotic structure. First we introduce a concept of *strong ergodicity* for the evolutionary system. This concept has been developed in the theory of Markov chains (MADSEN and CONN, 1973; SENETA, 1981) and originally stems from the theory of infinite products of matrices (THOMPSON, 1978; ARTZROUNI, 1986a).

DEFINITION 4.1. The evolutionary system $U(t,s),0\leqslant s\leqslant t<\infty$ is called *strongly ergodic* with *asymptotic growth rate r* if there exists a rank one operator P_s such that

$$\lim_{t\rightarrow\infty}\exp(-r(t-s))U(t,s) = P_s, \tag{4.1}$$

where the limit is in the operator norm topology.

From the definition, it follows immediately that

LEMMA 4.2. *Let* P_s *be the rank one operator defined by (4.1). Then there exists* $\phi_0\in X, f_t\in X^*$ *such that*

$$P_s\phi = \langle f_s,\phi\rangle\phi_0,\ U^*(s,t)f_t = \exp(r(t-s))f_s. \tag{4.2}$$

PROOF. From (4.1) and the equality

$$\exp(-r(u-t))U(u,t)\exp(-r(t-s))U(t,s) = \exp(-r(u-s))U(u,s),$$

it follows that

$$P_t\exp(-r(t-s))U(t,s) = P_s,\ t\geqslant s. \tag{4.3}$$

Hence we know that Range P_t = Range $P_s, t\geqslant s$ and that there exist $\phi_0\in X,\ f_t\in X^*$ such that $P_s\phi = \langle f_s,\phi\rangle\phi_0$. Moreover, from (4.3), we have

$$\langle f_t,\exp(-r(t-s))U(t,s)\phi\rangle = \langle f_s,\phi\rangle,$$

for each $\phi\in X$. Consequently $f_s = \exp(-r(t-s))U^*(s,t)f_t$. □

A C_0-semigroup of bounded linear operators $T(t),t\geqslant 0$ is an example of a strongly ergodic evolutionary system if $T(t),t\geqslant 0$ has *asynchronous exponential growth* with intrinsic growth constant r in the sense of WEBB (1987) and if the operator $P := \lim_{t\rightarrow\infty}\exp(-rt)T(t)$ is rank one. Another important example is derived from the theory of weakly ergodic multiplicative processes (BIRKHOFF, 1965, 1967; INABA, 1989). Assume that the Banach space X forms a Banach lattice with a natural positive cone K. If the evolutionary system $U(t,s),0\leqslant s\leqslant t$ is nonnegative, i.e. $U(t,s)K\subset K$, it forms a time-inhomogeneous *multiplicative process* in the sense of Birkhoff. Let $d(x,y),x,y\in K\setminus\{0\}$ be *Hilbert's projective metric.* That is,

$$d(x,y) := \log[\frac{\sup(x/y)}{\inf(x/y)}], \text{ for } x,y \in K \setminus \{0\},$$

where $\sup(x/y) := \inf\{\lambda : x \leqslant \lambda y\}$, $\inf(x/y) := \sup\{\mu : \mu x \leqslant y\}$ for $x \in X, y \in K \setminus \{0\}$ and we adopt the convention $\inf \phi = \infty$ and $\sup \phi = -\infty$. Then the multiplicative process $U(t,s), 0 \leqslant s \leqslant t$ is called *weakly ergodic* if

$$\lim_{t \to \infty} d(U(t,s)x, U(t,s)y) = 0 \text{ for all } x,y \in K \setminus \{0\}.$$

The definition of weak ergodicity implies that any two orbits $U(t,s)x$ and $U(t,s)y, x,y \in K \setminus \{0\}$ will be asymptotically proportional as time evolves. Then we can state that if $U(t,s)$ is a weakly ergodic process, then there exists a positive functional $v^*(s) \in K^*$ such that for $d(x,y) < \infty$

$$U(t,s)x = \langle v^*(s), x \rangle U(t,s)y + o(\| U(t,s)y \|), \tag{4.4}$$

where $o(\| U(t,s)y \|)/\| U(t,s)y \| \to 0$ as $t \to \infty$ and $v^*(s) = U^*(s,t)v^*(t)$ (INABA, 1989, Proposition 3.2). More precisely, the following estimate holds

$$\| U(t,s)x - \langle v^*(s), x \rangle U(t,s)y \| \leqslant \| U(t,s)y \| \operatorname{osc}(U(t,s)x / U(t,s)y), \tag{4.5}$$

where $\operatorname{osc}(x/y)$ is the *oscillation* of x and y defined by

$$\operatorname{osc}(x/y) = \sup(x/y) - \inf(x/y) \text{ for } x,y \in K \setminus \{0\}.$$

A time-inhomogeneous multiplicative process is called *uniformly primitive* for positive time when for some $\alpha > 0$, there exist for any $K > 0$ some t and s with $K < s < t$ such that $\Delta(U(t,s)) \leqslant \alpha$, where $\Delta(U(t,s))$ is the *projective diameter* defined by

$$\Delta(U(t,s)) := \sup\{d(U(t,s)x, U(t,s)y) : x,y \in K \setminus \{0\}\}$$

Then it can be proved that if $U(t,s)$, $0 \leqslant s \leqslant t$ is uniformly primitive, it is weakly ergodic (INABA, 1989, Proposition 3.3). Using these facts, we can prove that:

PROPOSITION 4.3. *Let X be a Banach lattice with positive cone K. Suppose that the evolutionary system $U(t,s), 0 \leqslant s \leqslant t$ is nonnegative and has an invariant element $\phi_0 \in K \setminus \{0\}$ such that $U(t,s)\phi_0 = \exp(r(t-s))\phi_0$. If $U(t,s), 0 \leqslant s \leqslant t$ is uniformly primitive, then $U(t,s), 0 \leqslant s \leqslant t$ is strongly ergodic with asymptotic growth rate r.*

PROOF. From uniform primitivity of $U(t,s)$, we can prove that for $x \in K \setminus \{0\}$

$$\| \exp(-r(t-s)) U(t,s)x - \langle v^*(s), x \rangle \phi_0 \| \leqslant \| \phi_0 \| \operatorname{osc}(U(t,s)x / U(t,s)\phi_0), \tag{4.6}$$

where the functional $v^*(s)$ is defined by

$$\langle v^*(s), x \rangle := \liminf_{t \to \infty} (U(t,s)x / U(t,s)\phi_0) = \limsup_{t \to \infty} (U(t,s)x / U(t,s)\phi_0),$$

(see INABA, 1989, Proposition 3.2). Originally the domain of the positive functional $v^*(s)$ is the cone K, but it can be extended to the whole space X by

$$\langle v^*(s), x \rangle := \langle v^*(s), x_+ \rangle - \langle v^*(s), x_- \rangle,$$

where $x = x_+ - x_- \in X, x_+ \in K, x_- \in K$ are the positive part and the negative part of x respectively. Then if we define a rank one operator P_s as

$$P_s x := \langle v^*(s), x \rangle \phi_0, \; x \in X,$$

then from (4.6) we obtain that for $x = x_+ - x_- \in X$,

$$\| \exp(-r(t-s)) U(t,s)x - P_s x \| \leqslant \| \phi_0 \| [\operatorname{osc}(U(t,s)x_+ / U(t,s)\phi_0) + \operatorname{osc}(U(t,s)x_- / U(t,s)\phi_0)], \tag{4.7}$$

where we adopt the convention $\operatorname{osc}(0/y) = 0$ for $y \in K \setminus \{0\}$. Let $N(A)$ be the *oscillation ratio* of a positive operator A defined by

$$N(A) := \inf\{\lambda : \operatorname{osc}(Ax/Ay) \leqslant \lambda \operatorname{osc}(x/y), x,y \in K \setminus \{0\}\}.$$

Then it follows from Birkhoff's theorem and Ostrowski's theorem (see BUSHELL, 1973) that

$$N(A) \leqslant \tanh(\frac{\Delta(A)}{4}),$$

from which we have $N(A) < 1$ as long as $\Delta(A)$ is finite. On the other hand, from uniform primitivity of the multiplicative process $U(t,s)$, there exist a number $\alpha > 0$ and an infinite sequence of positive numbers $s = t_0 < t_1 < \ldots$ tending to ∞ such that $\Delta(U(t_{2n+1}, t_{2n})) \leqslant \alpha$. Then it is easily seen that for $x \in K \setminus \{0\}, t \geqslant t_{2n+1}$,

$$\operatorname{osc}(U(t,s)x / U(t,s)\phi_0) \leqslant (\tanh(\frac{\alpha}{4}))^n \operatorname{osc}(U(t_1,s)x / U(t_1,s)\phi_0). \tag{4.8}$$

Thus it follows that for $x \in K \setminus \{0\}$

$$\begin{aligned} &\operatorname{osc}(U(t_1,s)x / U(t_1,s)\phi_0) \leqslant (e^{\alpha} - 1)\inf(U(t_1,s)x / U(t_1,s)\phi_0) \\ &\leqslant (e^{\alpha} - 1)\frac{\|U(t_1,s)x\|}{\|U(t_1,s)\phi_0\|} \leqslant (e^{\alpha} - 1)e^{-r(t_1 - s)}\frac{\|U(t_1,s)\|}{\|\phi_0\|}\|x\|, \end{aligned} \tag{4.9}$$

where we use the monotonicity of $\|x\|$ as a function of $|x|$. Therefore we conclude that

$$\|\exp(-r(t-s))U(t,s)x - P_s x\| \leqslant 2\|\phi_0\|(\tanh(\frac{\alpha}{4}))^n(e^{\alpha} - 1)e^{-r(t_1 - s)}\frac{\|U(t_1,s)\|}{\|\phi_0\|}\|x\|, \tag{4.10}$$

for all $x \in X$. Then we have

$$\lim_{t \to \infty} \|\exp(-r(t-s))U(t,s) - P_s\| = 0.$$

This completes our proof. □

Next we consider the asymptotic behavior of the evolutionary systems generated by a Lipschitz continuous perturbation of a C_0- semigroup in a sun-reflexive Banach space. Of our concern here is to show conditions such that a perturbed strongly ergodic evolutionary system is again strongly ergodic. Let A_0 be the infinitesimal generator of the C_0-semigroup $T_0(t), t \geqslant 0$, let X be sun-reflexive with respect to A_0 and let Ω be the set of Lipschitz continuous perturbations defined by $\Omega := \{B(t) : t \to B(t)$ is Lipschitz continuous from R^+ to $B(X, X^{\odot *})\}$, where $B(X, X^{\odot *})$ is the set of bounded linear operators from X to $X^{\odot *}$. Let $U_B(t,s)$, $B \in \Omega$ be the evolutionary system defined by the variation-of-constants formula

$$U_B(t,s)x = T_0(t-s)x + \int_s^t T_0^{\odot *}(t-\tau)B(\tau)U_B(\tau,s)x d\tau. \tag{4.11}$$

If $U_B(t,s)$ is strongly ergodic with asymptotic growth rate r, we define the rank one operator $P_s(B)$ by

$$\lim_{t \to \infty} \exp(-r(t-s))U_B(t,s) = P_s(B), \tag{4.12}$$

We define a subset

$$\Omega(B_0) := \{B \in \Omega : \int_0^{\infty} \|B(\tau) - B_0(\tau)\| d\tau < \infty\},\ B_0 \in \Omega.$$

LEMMA 4.4. *If $B(t), C(t) \in \Omega$, then the following variation-of- constants formula holds*

$$U_B(t,s)x = U_C(t,s)x + \int_s^t U_C^{\odot *}(t,\tau)(B(\tau) - C(\tau))U_B(\tau,s)x d\tau,\ x \in X. \tag{4.13}$$

PROOF. Note that under our assumption, $U_B(t,s)D(A_0^{\odot *})\subset D(A_0^{\odot *})$ (CLÉMENT et al., 1988, Theorem 4.9), and for every $x\in D(A_0^{\odot *})$, the function $\sigma\rightarrow U_B(\sigma,s)x$, $s\leqslant\sigma$ has the weak *-derivative $(A_0^{\odot *}+B(\sigma))U(\sigma,s)x$ and the function $\sigma\rightarrow U_C(t,\sigma)x$ has the weak *- derivative $-U_C^{\odot *}(t,\sigma)(A_0^{\odot *}+C(\sigma))x$. Then for $x^{\odot}\in X^{\odot},x\in D(A_0^{\odot *})$ we obtain

$$\frac{\partial}{\partial\sigma}<U_C(t,\sigma)U_B(\sigma,s)x,x^{\odot}>$$
$$= <U_C^{\odot *}(t,\sigma)(A_0^{\odot *}+B(\sigma))U_B(\sigma,s)x,x^{\odot}>-<U_C^{\odot *}(t,\sigma)(A_0^{\odot *}+C(\sigma))U_B(\sigma,s)x,x^{\odot}>$$
$$= <U_C^{\odot *}(t,\sigma)(B(\sigma)-C(\sigma))U_B(\sigma,s)x,x^{\odot}>.$$

Hence it follows that

$$<\int_s^t U_C^{\odot *}(t,\sigma)(B(\sigma)-C(\sigma))U_B(\sigma,s)x d\sigma,x^{\odot}>= \int_s^t <U_C^{\odot *}(t,\sigma)(B(\sigma)-C(\sigma))U_B(\sigma,s)x,x^{\odot}>d\sigma$$
$$= \int_s^t \frac{\partial}{\partial\sigma}<U_C(t,\sigma)U_B(\sigma,s)x,x^{\odot}>d\sigma= <U_B(t,s)x,x^{\odot}>-<U_C(t,s)x,x^{\odot}>.$$

Since $D(A_0^{\odot *})$ is dense in X, we arrive at the variation-of- constants formula (4.13). □

PROPOSITION 4.5. *Suppose that $U_B(t,s)$ is strongly ergodic with asymptotic growth rate r at $B = B_0\in\Omega$. Then for every $B\in\Omega(B_0)$, $U_B(t,s)$ is strongly ergodic with asymptotic growth rate r, and*

$$P_s(B)x = \lim_{t\to\infty}<f_t,\exp(-r(t-s))U_B(t,s)x>\phi_0, \tag{4.14}$$

where $f_t\in X^*$ and $\phi_0\in X$ are such that $P_s(B_0)x = <f_s,x>\phi_0$.

PROOF. From Lemma 4.4, for $B\in\Omega(B_0)$, we have an expression for $U_B(t,s)$ by the variation-of-constants formula

$$U_B(t,s)x = U_{B_0}(t,s)x+\int_s^t U_{B_0}^{\odot *}(t,\tau)(B(\tau) - B_0(\tau))U_B(\tau,s)xd\tau,\ \ x\in X, \tag{4.15}$$

where we assume that

$$\lim_{t\to\infty}\exp(-r(t-s))U_{B_0}(t,s) = P_s(B_0).$$

For simplicity, we define the evolutionary system $V_B(t,s)$, $0\leqslant s\leqslant t$ by $V_B(t,s):=\exp(-r(t-s))U_B(t,s)$. Observe that for $s\leqslant t_0<t<t'$,

$$\|V_B(t',s)x-V_B(t,s)x\|\leqslant\|V_B(t',s)x-V_{B_0}(t',t_0)V_B(t_0,s)x\|$$
$$+\|V_{B_0}(t',t_0)V_B(t_0,s)x-V_{B_0}(t,t_0)V_B(t_0,s)x\| + \|V_B(t,s)x-V_{B_0}(t,t_0)V_B(t_0,s)x\| := I+J+K.$$

Then it follows that

$$I := \|V_B(t',s)x-V_{B_0}(t',t_0)V_B(t_0,s)x\|\leqslant\|V_B(t',t_0)-V_{B_0}(t',t_0)\|\|V_B(t_0,s)\|\|x\|. \tag{4.16}$$

From (4.15), we obtain

$$\|V_B(t',t_0)-V_{B_0}(t',t_0)\|\leqslant\int_{t_0}^{t'}\|V_{B_0}^{\odot *}(t,\tau)\|\|B(\tau)-B_0(\tau)\|\|V_B(\tau,t_0)\|d\tau.$$

Since $\lim_{t\to\infty}V_{B_0}(t,s) = P_s(B_0)$, then there exists a number $M(s,B_0)$ such that $\|V_{B_0}^{\odot *}(t,s)\|\leqslant\|V_{B_0}(t,s)\|\leqslant M(s,B_0)$ for all $t\geqslant s$. From (4.15) the following inequality holds

$$\|V_B(t,s)\|\leqslant M(s,B_0)+M(s,B_0)\int_s^t\|B(\tau)-B_0(\tau)\|\|V_B(\tau,s)\|d\tau,$$

from which we have

$$\|V_B(t,s)\| \leq M(s,B_0)\exp(M(s,B_0)\int_s^t \|B(\tau)-B_0(\tau)\|d\tau) \leq M(s,B)$$

for all $t \geq s$, where

$$M(s,B) := M(s,B_0)\exp(M(s,B_0)\int_0^\infty \|B(\tau)-B_0(\tau)\|d\tau).$$

Therefore we arrive at the inequality

$$I \leq M(s,B_0)M(s,B)^2 \int_{t_0}^{t'} \|B(\tau)-B_0(\tau)\|d\tau\|x\|.$$

Then for any $\epsilon>0$ we can choose t_0 so large that $I \leq \frac{\epsilon}{3}\|x\|$. In exactly the same way, we can show that $K \leq \frac{\epsilon}{3}\|x\|$ for large $t_0>s$. Next observe that

$$J := \|V_{B_0}(t',t_0)V_B(t_0,s)x - V_{B_0}(t,t_0)V_B(t_0,s)x\| \leq \|V_{B_0}(t',t_0)-V_{B_0}(t,t_0)\|M(s,B)\|x\|.$$

Since $\lim_{t\to\infty} V_{B_0}(t,s) = P_s(B_0)$, for any $\epsilon>0$ we can choose a $T_0>t_0$ such that for $t',t>T_0$, $J \leq \frac{\epsilon}{3}\|x\|$. Then for any $\epsilon>0$ if t,t', t_0 are sufficiently large, we have $I+J+K \leq \epsilon\|x\|$, which shows that there exists an operator $P_s(B)$ such that

$$\lim_{t\to\infty} V_B(t,s) = P_s(B).$$

From (4.16), we obtain

$$\|P_s(B)-P_{t_0}(B_0)V_B(t_0,s)\| \to 0 \text{ as } t_0\to\infty.$$

Then $P_s(B)$ is a rank one operator and

$$P_s(B)x = \lim_{t\to\infty} \langle f_t, V_B(t,s)x\rangle \phi_0.$$

This completes the proof. □

COROLLARY 4.6. *Suppose that $T(t)$, $t\geq 0$ is a C_0-semigroup generated by a constant perturbation $B_0\in\Omega$, $T(t)$ has asynchronous exponential growth with intrinsic growth constant r and the operator*

$$P = \lim_{t\to\infty}\exp(-rt)T(t),$$

is rank one. If

$$\int_0^\infty \|B(\tau)-B_0\|d\tau < \infty,$$

then the evolutionary system $U_B(t,s)$ is strongly ergodic with asymptotic growth rate r and

$$P_0(B) = \lim_{t\to\infty}\exp(-rt)PU_B(t,0).$$

REMARK 4.7. The case that $B(t)\in\Omega$ has a period $\omega>0$ is another interesting case for which we can expect that there exists a time-invariant asymptotic structure for the evolutionary system. First, using the variation-of-constants formula, it is easily seen that the following holds, though we omit the proof:

$$U_B(t+\omega,s+\omega) = U_B(t,s) \text{ for all } 0\leq s\leq t. \tag{4.17}$$

Suppose that the evolutionary system $U(t,s)$, $0 \leqslant s \leqslant t$ forms a weakly ergodic multiplicative process on the Banach lattice (X,K). We assume that $U(t,s)$ satisfies the property (4.17) and the positive operator $U(s+\omega,s)$ has a positive eigenvector $x_0 \in K \setminus \{0\}$ associated with the eigenvalue $\lambda(s)>0$. Then we have

$$U(t+\omega,s)x_0 = U(t+\omega,s+\omega)U(s+\omega,s)x_0 = \lambda(s)U(t,s)x_0. \tag{4.18}$$

Defining a continuous function $t \to g(t)$ from $[s,\infty)$ to K by

$$g(t) = \exp(-rt)U(t,s)x_0,\ r = \frac{\log\lambda(s)}{\omega}. \tag{4.19}$$

Then, by using (4.18), it is easy to check that $g(t)$ has the period ω. From (4.4), there exists a positive functional $v^*(s)$ as

$$\exp(-rt)U(t,s)x = \langle v^*(s),x\rangle g(t)+o(\|g(t)\|)$$

for any x such that $d(x,x_0)<\infty$. Since $\|g(t)\|$ is bounded above, it follows that for $d(x,x_0)<\infty$

$$\lim_{t\to\infty} \|\exp(-rt)U(t,s)x - \langle v^*(s),x\rangle g(t)\| = 0. \tag{4.20}$$

This can be seen as an infinite dimensional analogue of the Floquet theory for systems of ordinary differential equation.

5. Generalized stable population theory

In this section we consider a demographic application of the results obtained in the previous section.

Classical stable population theory states that a closed population subject to constant mortality and fertility schedule will asymptotically grow exponentially while the age distribution converges to a stable distribution. This phenomenon is generally called strong ergodicity of the population process in demography. Precisely speaking, the strong ergodicity means that if $p(a,t)$ is the age-density of the population at time t, there exist a number r, a positive functional $C(\phi)$ and an age-density function $u(a)$ such that

$$\lim_{t\to\infty} \exp(-rt)p(a,t) = C(\phi)u(a),$$

for the initial data $\phi = p(a,0)$ in L^1-convergence. Then the classical stable population theory implies that the constant mortality and fertility schedule is a sufficient condition for strong ergodicity of a population described by the Lotka- McKendrick-Von Foerster model.

In the real world, mortality and fertility rates are never constant. Instead they adapt to the changing technological and social environment. Once we remove the restriction that the vital rates are time-independent, we cannot generally expect that there exists a time-invariant stable age-structure. The most general result for the population process with time-dependent vital rates is known as the weak ergodicity theorem, and states that the age distribution will be asymptotically independent of the initial population (Lopez, 1961; Kim, 1987; Inaba, 1989). However, there remains a gap between weak and strong ergodicity and it is a problem to decide whether or not the population process with time-dependent vital rates converges to a fixed age distribution.

The purpose of the generalized stable population theory introduced by M. Artzrouni (1985) is to derive necessary and sufficient conditions under which a population subject to time-dependent vital rates is strongly ergodic. In the linear discrete-time model (Leslie model), Golubitsky et al. (1975), Thompson (1978), Artzrouni (1985) have already provided some conditions which guarantee the strong ergodicity of the age-dependent population with time-dependent vital rates. Moreover Artzrouni (1985) conjectured the necessary and the sufficient conditions for the strong ergodicity of the continuous-time model. In the following, we shall prove sufficient conditions, although they are slightly different from the conditions conjectured by Artzrouni.

The continuous-time population model (Lotka model) is formulated by system (1.1) or by the renewal integral equation. Here it is more convenient to start from the renewal integral equation: Let

$B(t)$ be the density of newborn children at time t, $m(a,t)$ be the fertility rate at age a and time t, $\mu(a,t)$ be the death rate at age a and time t and $p(a,t)$ be the age-density of the population at time t. Let $\Pi(a,t)$ be the survival function given by

$$\Pi(a,t) := \exp(-\int_0^a \mu(\rho, t-a+\rho)d\rho). \tag{5.1}$$

Then $\Pi(a,t)$ denotes the proportion of individuals born at time $t-a$ which survive to age a at time t. The dynamics of the population is described by the renewal equation (LANGHAAR, 1972);

$$B(t) = \int_0^\beta \phi(a,t)B(t-a)da, \quad t>0, \tag{5.2}$$

$$B(-a) = x(a) \in L^1(0,\beta),$$

where β is the upper bound of the reproductive age, i.e. $m(a,t)=0$ for $a \geq \beta$, $t \geq 0$, $\phi(a,t)$ is the *net maternity function* defined by $\phi(a,t) = m(a,t)\Pi(a,t)$ and $x(a)$ is the initial data (the *starting function*). In the following we assume that $m(\cdot,t), \mu(\cdot,t) \in L_+^\infty(0,\beta)$. Once $B(t)$ is determined by (5.2), the age-density function is given by

$$p(a,t) = \Pi(a,t)B(t-a). \tag{5.3}$$

At first, we shall clear the meaning of the generalized stable population.

DEFINITION 5.1. The population governed by (5.1)-(5.3) is called a *generalized stable population* (or strongly ergodic) if there exists a number r, a positive functional $C(\phi), \phi \in L^1$ and an age-density function $u(a) \in L^1(0,\beta)$ such that

$$\lim_{t\to\infty} \int_0^\beta |\exp(-rt)p(a,t) - C(\phi)u(a)|da = 0, \tag{5.4}$$

for the initial condition $p(a,0) = \phi \in L^1(0,\beta)$.

Then the following lemma follows immediately.

LEMMA 5.2. *A population $p(a,t)$ represented by (5.1)-(5.3) is a generalized stable population if (1) there exist a constant $K = K(x)$ and a constant r such that*

$$\lim_{t\to\infty} \int_0^\beta |\exp(-rt)B(t-a) - K\exp(-ra)|da = 0, \tag{5.5}$$

(2) there exists a time-independent function $\Pi(a)$ such that

$$\lim_{t\to\infty} \int_0^\beta |\Pi(a,t) - \Pi(a)|da = 0. \tag{5.6}$$

Under the above conditions, it follows that

$$\lim_{t\to\infty} \int_0^\beta |\exp(-rt)p(a,t) - K\exp(-ra)\Pi(a)|da = 0. \tag{5.7}$$

In the following we consider sufficient conditions which guarantee (5.5). If we define a function $t \to u(\cdot,t)$ from R^+ to $X = L^1(0,\beta)$ by $u(a,t) = B(t-a)$, we can rewrite the renewal equation (5.2) as the initial-boundary value problem

$$(\frac{\partial}{\partial a} + \frac{\partial}{\partial t})u(a,t) = 0, \quad t>0, \ 0<a<\beta, \tag{5.8a}$$

$$u(0,t) = \int_0^\beta \phi(a,t)u(a,t)da, \tag{5.8b}$$

$$u(a,0) = x(a). \tag{5.8c}$$

Now we define an unperturbed operator A_0 on X by

$$A_0x = -x',\ D(A_0) = \{x\in X{:}x(0) = 0,\ x\in AC[0,\beta]\}, \tag{5.9}$$

where $AC[0,\beta]$ denotes the set of absolutely continuous functions on $[0,\beta]$. Then it is easily seen that the unperturbed C_0- semigroup $T_0(t), t\geqslant 0$ generated by A_0 is given by

$$(T_0(t)x)(a) = \begin{cases} x(a-t),\ a-t>0, \\ \quad & x\in X. \\ 0,\ t-a>0, \end{cases} \tag{5.10}$$

Moreover $X^* = L^\infty(0,\beta)$ and the weak *-continuous semigroup $T_0^*(t)$, $t\geqslant 0$ on X^* is given by

$$(T_0^*(t)\psi)(a) = \begin{cases} \psi(a+t),\ a+t\leqslant\beta, \\ \quad & \psi\in X^*. \\ 0,\ a+t>\beta, \end{cases} \tag{5.11}$$

Its weak *-generator A_0^* is given by

$$A_0^*\psi = \psi',\ D(A_0^*) = \{\psi\in AC[0,\beta]{:}\ \psi'\in X^*,\ \psi(\beta) = 0\}. \tag{5.12}$$

Hence it follows that

$$X^\odot = C_0[0,\beta] = \overline{D(A_0^*)} = \{\psi\in C[0,\beta]{:}\ \psi(\beta) = 0\}, \tag{5.13}$$

and the action of $T_0^\odot(t)$ is the same as the action of $T_0^*(t)$. Let $A_0^\odot$ be the part of A_0^* in $X^\odot$. Then

$$A_0^\odot\psi = \psi',\ D(A_0^\odot) = \{\psi\in C^1[0,\beta]{:}\ \psi(\beta) = \psi'(\beta) = 0\}. \tag{5.14}$$

In general, $C[0,\beta]^*$ can be identified with $NBV[0,\beta]$, i.e. the space of functions of bounded variation which vanish on $(-\infty,0]$, are right continuous on $(0,\beta)$ and constant on $[\beta,\infty)$, with duality pairing

$$<\psi,\rho> = \int_0^\beta \psi(a)d\rho(a),\ \psi\in C[0,\beta],\ \rho\in NBV[0,\beta]. \tag{5.15}$$

Let $H(a)\in NBV[0,\beta]$ be the Heaviside function and let $H_\beta = H(a-\beta)$. Then it is easily seen that $C_0[0,\beta]^* = NBV[0,\beta]/\{\alpha H_\beta{:}\alpha\in R\}$, that is, $X^{\odot *}$ can be identified with the set of equivalence classes $\{\rho\} = \{\rho+\alpha H_\beta{:}\alpha\in R, \rho\in NBV[0,\beta]\}$. The weak *-continuous semigroup $T_0^{\odot *}(t), t\geqslant 0$ is given by

$$(T_0^{\odot *}(t)\rho)(a) = \begin{cases} \rho(a-t),\ a-t>0, \\ \quad & \rho\in X^{\odot *}, \\ 0,\ a-t<0, \end{cases} \tag{5.16}$$

where ρ denotes the equivalence class $\{\rho\}$. The infinitesimal generator is given by

$$A_0^{\odot *}\rho = -\rho', D(A_0^{\odot *}) = \{\rho\in X^{\odot *}{:}\rho\in AC[0,\beta], \rho'\in NBV[0,\beta]\}. \tag{5.17}$$

Therefore it follows that

$$X^{\odot\odot} = \{\rho\in X^{\odot *}{:}\rho\in AC[0,\beta]\}. \tag{5.18}$$

The space of all absolutely continuous functions $X^{\odot\odot}$ is a closed linear subspace of $NBV[0,\beta]$ and there is an isometric isomorphism between $AC[0,\beta]$ and $L^1(0,\beta)$ given by correspondence $\rho'\leftrightarrow f$ with

$\|\rho\|_{NBV} = \|f\|_{L^1}, \|\rho\|_{NBV} := \int_{-\infty}^{\infty} |d\rho(a)|$. Then we can identify $X^{\odot\odot}$ with $X = L^1(0,\beta)$, that is, the Banach space X is sun-reflexive.

Now the system (5.8) can be formulated as an abstract Cauchy problem

$$\frac{d}{dt}u(t) = (A_0^{\odot *} + B(t))u(t),\ u(0) = x \in X, \tag{5.19}$$

where the perturbation term $B(t)$: $X \to X^{\odot *}$ is given by

$$(B(t)x)(a) = H(a)\int_0^\beta \phi(a,t)x(a)da,\ x \in X. \tag{5.20}$$

If the net maternity function $\phi(a,t)$ is Lipschitz continuous in the time variable uniformly in the age variable, then $t \to B(t)$ is Lipschitz continuous from R^+ to $B(X, X^{\odot *})$. Then the weak *- solution for $x \in D(A_0^{\odot *})$ is given by

$$u(t) = U(t,0)x,\ t \geq 0, \tag{5.21}$$

where the evolutionary system $U(t,s), 0 \leq s \leq t$ is defined by the variation-of-constants formula

$$U(t,s)x = T_0(t-s)x + \int_s^t T_0^{\odot *}(t-\tau)B(\tau)U(\tau,s)xd\tau,\ x \in X. \tag{5.22}$$

Next we shall check that the weak *-solution $U(t,s)$ is indeed the evolutionary system corresponding to the system (5.8). First observe that

$$<\int_0^t T_0^{\odot *}(\tau)Hd\tau, \phi(\cdot,t)> = \int_0^t \phi(a,t)da. \tag{5.23}$$

If we define $y(t,s;x), f(t,s;x)$ by

$$y(t,s;x) := <U(t,s)x, \phi(\cdot,t)> = \int_0^\beta \phi(a,t)u(a,t)da,$$

$$f(t,s;x) := <T_0(t-s)x, \phi(\cdot,t)> = \int_{t-s}^\beta \phi(a,t)x(a-t+s)da,$$

then, from (5.22), we have

$$y(t,s;x) = f(t,s;x) + <\int_s^t T_0^{\odot *}(t-\tau)Hy(\tau,s;x)d\tau, \phi(\cdot,t)>. \tag{5.24}$$

By using Lemma 5.1 in Clément et al. (1987a), it follows that

$$<\int_s^t T_0^{\odot *}(t-\tau)Hy(\tau,s;x)d\tau, \phi(\cdot,t)> = \int_s^t \phi(t-\tau,t)y(\tau,s;x)d\tau. \tag{5.25}$$

By substituting (5.25) into (5.24), we obtain the Volterra integral equation

$$y(t,s;x) = f(t,s;x) + \int_s^t \phi(t-\tau,t)y(\tau,s;x)d\tau. \tag{5.26}$$

Once $y(t,s;x)$ is solved from (5.26), the evolutionary system $U(t,s)$ is obtained as

$$U(t,s)x = T_0(t-s)x + \int_s^t T_0^{\odot *}(t-\tau)Hy(\tau,s;x)d\tau, \tag{5.27}$$

from which we have the direct representation

$$(U(t,s)x)(a) = \begin{cases} x(a-t+s), & a>t-s, \\ y(t-a,s;x), & t-s>a. \end{cases} \tag{5.28}$$

This shows that the weak *-solution $U(t,s)x$, $0\leqslant s\leqslant t$ indeed gives the evolutionary system corresponding to the system (5.8) or the renewal equation (5.2).

Now we set up the following assumption:

Assumption 5.3. (a) There exist an interval $[\gamma_1,\gamma_2]\subset(0,\beta)$ and a positive number $\epsilon>0$ such that $\phi(a,t)\geqslant\epsilon$ for all $(a,t)\in[\gamma_1,\gamma_2]\times[0,\infty)$.
(b) There exists a small number $\eta>0$ such that $m(a,t)>0$ for almost all $a\in(\beta-\eta,\beta)$ and all $t\geqslant 0$.

Under the above assumption, it can be shown that the evolutionary system $U(t,s)$ given by (5.28) is uniformly primitive (INABA, 1989). Therefore, $U(t,s)$ is weakly ergodic. Next we introduce another assumption:

Assumption 5.4. There exists a number $r\in R$ such that

$$\int_0^\infty \left| \int_0^\beta \phi(a,t)\exp(-ra)da - 1 \right| dt<\infty. \tag{5.29}$$

REMARK 5.5. The relation (5.29) implies that the function $t\to\int_0^\beta \phi(a,t)\exp(-ra)da$ *rapidly* converges to unity as $t\to\infty$ in the sense of ARTZROUNI (1985). An important case that (5.29) is satisfied is the case that the kernel $\phi(a,t)$ rapidly converges to a time-independent positive kernel $\phi(a)$ in the following sense

$$\int_0^\infty\int_0^\beta |\phi(a,t)-\phi(a)|\,dadt<\infty. \tag{5.30}$$

In fact, in this case it is easy to see that there exists a unique real value r such that

$$\int_0^\beta \phi(a)\exp(-ra)da = 1,$$

and then it follows that

$$\int_0^\infty \left| \int_0^\beta \phi(a,t)\exp(-ra)da - 1 \right| dt<\infty.$$

So Assumption 5.4 is satisfied. Moreover note that Assumption 5.4 is different from the condition conjectured by Artzrouni (1985, Conjecture 6.1, A1). His condition is that there exists a number r such that

$$\int_0^\infty |r(t)-r|\,dt<\infty, \tag{5.31}$$

where $r(t)$ is a unique real root of the characteristic equation

$$\int_0^\beta \exp(-r(t)a)\phi(a,t)da = 1. \tag{5.32}$$

However if $r(t)$ converges to r in the sense that

$$\int_0^\infty\int_0^\beta |\exp(-r(t)a)-\exp(-ra)|\,dadt<\infty,$$

then it is easily seen that Assumption 5.4 is satisfied.

Under Assumption 5.4, we can decompose the net maternity function $\phi(a,t)$ as

$$\phi(a,t) = \phi_0(a,t)+(h(t)-1)\phi_0(a,t), \tag{5.33}$$

$$\phi_0(a,t) := \frac{\phi(a,t)}{h(t)},\ h(t) := \int_0^\beta e^{-ra}\phi(a,t)da.$$

Then it follows necessarily that

$$\int_0^\beta \exp(-ra)\phi_0(a,t)da = 1, \tag{5.34}$$

and $\phi_0(a,t)$ satisfies Assumption 5.3. Let $U_0(t,s)$, $0 \leqslant s \leqslant t$ be the evolutionary system defined by

$$U_0(t,s)x = T_0(t-s)x + \int_s^t T_0^{\odot *}(t-\tau)B_0(\tau)U_0(\tau,s)xd\tau,/x \in X, \tag{5.35}$$

where the perturbation term $B_0(t)$ is given by

$$(B_0(t)x)(a) = H(a)\int_0^\beta \phi_0(a,t)x(a)da. \tag{5.36}$$

It is easily seen that $U_0(t,s)$ has an invariant element $\psi_0(a) := \exp(-ra)$ such that

$$(U_0(t,s)\psi_0)(a) = \exp(r(t-s))\psi_0(a). \tag{5.37}$$

Since $U_0(t,s)$ is uniformly primitive by Assumption 5.3, it follows from Proposition 4.3 that $U_0(t,s)$ is strongly ergodic, i.e. there exists a rank one operator $P_s(B_0)$ such that

$$\lim_{t\to\infty} \exp(-r(t-s))U_0(t,s) = P_s(B_0). \tag{5.38}$$

Note that from Assumption 5.4, we have

$$\int_0^\infty \|B(\tau)-B_0(\tau)\|d\tau \leqslant \|H\|\sup|\phi_0|\int_0^\infty |h(t)-1|dt < \infty. \tag{5.39}$$

Therefore, from Proposition 4.5, we know that the evolutionary system $U(t,s)$, $0 \leqslant s \leqslant t$ is also strongly ergodic with asymptotic growth rate r. Moreover if we define a rank one operator $P_s(B)$ by

$$\lim_{t\to\infty} \exp(-r(t-s))U(t,s) = P_s(B), \tag{5.40}$$

then

$$P_s(B)x = \lim_{t\to\infty} \langle f_t, \exp(-r(t-s))U(t,s)x \rangle \psi_0, \tag{5.41}$$

where $f_t \in X^*$ is defined such that $P_s(B_0)x = \langle f_s, x\rangle \psi_0$. This implies that there exists a number $K(x) = \lim_{t\to\infty} \langle f_t, \exp(-rt)U(t,0)x\rangle$ such that

$$\|\exp(-rt)U(t,0)x - K\psi_0\| \to 0 \text{ as } t \to \infty. \tag{5.42}$$

This shows exactly that (5.5) holds. From Lemma 5.2, we arrive at the following conclusion:

PROPOSITION 5.5. *Suppose that the condition (5.6), Assumption 5.3 and 5.4 hold and the net maternity function $\phi(a,t)$ is Lipschitz continuous in the time variable t uniformly in the age variable a. Then the population $p(a,t)$ governed by (5.1)-(5.3) is a generalized stable population.*

REMARK 5.6. By using the evolutionary system $U(t,s)$, the age distribution $p(a,t)$ of the system (5.1)-(5.3) is expressed as

$$p(a,t) = (L(t)U(t,s)L^{-1}(s)\phi)(a),$$

where $\phi(a) = p(a,s)$ and the operator $L(t)$ in L^1 is defined by

$$(L(t)\psi)(a) = \Pi(a,t)\psi(a), \quad \psi \in L^1(0,\beta).$$

Then $V(t,s) := L(t)U(t,s)L^{-1}(s)$ forms an evolutionary system on the state space for the age distributions.

6. TFR-CONTROLLABILITY OF THE POPULATION SYSTEM

It is well known that many problems of infinite-dimensional control systems can be formulated in the canonical form

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t),\ x(0) = x_0 \in X, \tag{6.1}$$

where X is a Banach space (the state space), $x(t)$ is the state vector of the system, x_0 is the initial state, A is the infinitesimal generator of the C_0-semigroup $T(t), t \geq 0$ on X, B is the bounded linear operator from another Banach space U (the control space) to X and $u(t)$ is the control term. The mild solution of (6.1) is given by the variation-of-constants formula

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds. \tag{6.2}$$

However it is often too restrictive for many applications to assume that the operator B maps U into X. For example, in the boundary control system, B maps U into some space Y bigger than X. So it is clear that the perturbation theory for dual semigroups in sun-reflexive Banach spaces could give a suitable framework to generalize the classical control theory. HEIJMANS (1986) first gave controllability and observability results for the canonical control system in sun-reflexive Banach spaces.

In this section we consider a control problem in demography which cannot be formulated by the canonical form as (6.1). Consider a one-sex closed population system described by the equation

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right)p(a,t) = -\mu(a)p(a,t), \tag{6.3a}$$

$$p(0,t) = \beta(t)\int_0^\omega h(a)p(a,t)da, \tag{6.3b}$$

$$p(a,0) = x_0(a). \tag{6.3c}$$

where $p(a,t)$ is the age-density at time t, $x_0(a)$ is the initial data, $h(a)$ is the normalized fertility function such that

$$\int_0^\omega h(a)\ell(a)da = 1, \tag{6.4}$$

where ω is a fixed number larger than the upper bound of reproductive age and $\ell(a)$ is the survival rate function given by

$$\ell(a) = \exp\left(-\int_0^a \mu(\rho)d\rho\right),$$

and $\beta(t)$ is the control variable. Demographically $\beta(t)$ is the *total fertility rate* (TFR) of the population at time t, i.e. the average number of childbirths per female during her reproductive period and $h(a)$ is the age-pattern of the fertility rate. We call (6.3) the *TFR-control system*. The TFR-control system corresponds to the situation that the controller wants to control the age-structured population only through changing its TFR without changing the age-pattern of the fertility rate. This problem has been first investigated by Chinese scientists in order to provide a mathematical foundation for the one child policy in the People's Republic of China (SONG et al., 1985; YU et al., 1987a, 1987b; SONG and YU, 1988). Though they have already investigated the above problem by using the discrete or the semi-discrete model, we here deal with the full continuous model.

Let $X = L^1(0,\omega)$ be the state space of the population and assume that $h(a)$, $\mu(a) \in L^\infty_+(0,\omega)$ and $h(a)$ has a compact support in $[0,\omega]$. Then the feedback control system (6.3) can be formulated as

$$\frac{d}{dt}p(t) = A_0 p(t) + \beta(t) B p(t),\ p(0) = x \in X, \tag{6.5}$$

where the operator A_0 is defined by

$$(A_0 x)(a) = -\frac{d}{da}x(a) - \mu(a)x(a), \tag{6.6}$$

$$D(A_0) = \{x \in X\colon x \in AC[0,\omega],\ x(0) = 0\}.$$

Then it can be proved that X is sun-reflexive with respect to A_0 and the space $X^{\odot *}$ can be identified with $C_0[0,\omega]^*$ given in the previous section 5 (also see CLÉMENT, HEIJMANS, et al., 1987c, Chap.10). The perturbation $B: X \to X^{\odot *}$ is defined by

$$(Bx)(a) = H(a)\int_0^\omega h(a)x(a)da, \tag{6.7}$$

where $H \in X^{\odot *}$ is the Heaviside function. If $\beta(t) = \gamma = $ const., the Cauchy problem (6.5) has the unique mild solution $T_\gamma(t)x$, $t \geq 0$, $x \in X$ where $T_\gamma(t)$ is the C_0-semigroup generated by the operator $A_\gamma = A_0^{\odot *} + \gamma B$, $D(A_\gamma) = \{x \in D(A_0^{\odot *})\colon A_0^{\odot *}x + \gamma Bx \in X\}$. Furthermore, if we identify $X^{\odot\odot}$ with $X = L^1(0,\omega)$, it follows that

$$(A_\gamma x)(a) = -\frac{d}{da}x(a) - \mu(a)x(a), \tag{6.8}$$

$$D(A_\gamma) = \{x \in X\colon x \in AC[0,\omega], x(0) = \gamma\int_0^\omega h(a)x(a)da\}.$$

Then the operator A_γ has a strictly dominant real eigenvalue r_γ such that

$$\gamma\int_0^\omega h(a)\ell(a)\exp(-r_\gamma a)da = 1. \tag{6.9}$$

In particular, $r_\gamma = 0$ if $\gamma = 1$; $r_\gamma > 0$ if $\gamma > 1$; $r_\gamma < 0$ if $\gamma < 1$. The eigenvector v_γ of A_γ^* corresponding to r_γ is called the *reproductive value* vector. By the strong ergodic theorem, the following holds (INABA, 1988)

$$\lim_{t\to\infty}\exp(-r_\gamma t)T_\gamma(t)x = \langle v_\gamma, x\rangle \exp(-r_\gamma a)\ell(a), \tag{6.10}$$

where v_γ is normalized as $\langle v_\gamma, \psi_\gamma\rangle = 1$, $\psi_\gamma(a) := \exp(-r_\gamma a)\ell(a)$. Moreover it follows that

$$\langle v_\gamma, T_\gamma(t)x\rangle = \langle v_\gamma, x\rangle \exp(r_\gamma t),\quad t \geq 0. \tag{6.11}$$

The most important question for the control system (6.5) is whether the desired state can be attained from a given initial state by changing the control variable. From the perturbation theory for dual semigroups in sun-reflexive Banach spaces, we know that if $\beta(t) \in PC[0,\infty)$ (i.e. the set of piece-wise continuous function on R^+), then the system (6.5) defines the evolutionary system $U_\beta(t,s)$ by

$$U_\beta(t,s)x = T_0(t-s)x + \int_s^t T_0^{\odot *}(t-\tau)\beta(\tau)BU_\beta(\tau,s)xd\tau,\ x \in X, \tag{6.12}$$

where $T_0(t)$ is the C_0-semigroup generated by A_0. Now we introduce a definition:

DEFINITION 6.1. Let $\Omega_t(x_0;\Lambda) := \{U_\beta(t, 0)x_0 : \beta \in \Lambda\}$ where Λ is a subset of $PC[0,\infty)$ and let

$$\Omega(x_0;\Lambda) = \bigcup_{t>0} \Omega_t(x_0;\Lambda). \tag{6.13}$$

Then $\Omega(x_0;\Lambda)$ is called the *controllability space* with respect to $x_0 \in X$ under the *admissible control* Λ and $\overline{\Omega}(x_0;\Lambda)$ is called the *approximately controllability space* with respect to x_0 under the admissible control Λ.

Then it is easy to see that $x \in \Omega(x_0;\Lambda)$ if and only if there exist $t_0>0$ and $\beta(t)\in\Lambda$ such that $U_\beta(t_0,0)x_0 = x$, and $x\in\overline{\Omega}(x_0;\Lambda)$ if and only if for any $\epsilon>0$ there exist $t_0>0, \beta(t)\in\Lambda$ such that $\|U_\beta(t_0,0)x_0 - x\|<\epsilon$. In the TFR-control problem, the desired state (the target population) is a stationary population, i.e. the target population has a form such that $b\ell(a), b\in R^+$, where b is given by

$$b = \frac{N}{\int_0^\omega \ell(a)da},$$

and N is the total size of the target population. Since $\beta(t)$ is TFR of the population, the largest admissible control Λ_0 is given by $\Lambda_0 = \{\beta(t)\in PC[0,\infty): 0\leqslant\beta(t)\leqslant M\}$ where the number $M<\infty$ is the upper bound of TFR. However, in practice, the admissible control should be more restricted than Λ_0, because there exist a lot of socio-economic, ethical and psychological constraints for reproductive behavior in real human societies. Here we choose the set

$$\Lambda = \{\beta(t)\in C^\infty[0,\infty): 0<1-\delta\leqslant\beta(t)\leqslant 1+\delta<M\}, \tag{6.14}$$

as the admissible control, where δ is a given small number. Our main purpose here is to prove the following:

PROPOSITION 6.2. *Let X_+ be the natural positive cone of X and let $\Pi := \{b\ell(a): b\in R^+\}$ be the set of target populations. If $x\in K\setminus\{0\}, <v_1,x>\neq 0$, then $\Pi\subset\overline{\Omega}(x;\Lambda)$, that is, the TFR-control system is approximately controllable with respect to the initial population whose total reproductive value is not zero.*

PROOF. Let $b^*\ell(a)\in\Pi$ be a target population. Suppose that the reproductive value vector v_1 is normalized such that $<v_1,\psi_1> = 1$ where $\psi_1 = \ell(a)$ is the eigenvector of the operator A_1 corresponding to the eigenvalue $r_1 = 0$. Note that the condition $<v_1,x>\neq 0$ for $x\in K\setminus\{0\}$ implies that $<v_\gamma,x>\neq 0$ for all $\gamma>0$, because v_γ has the form

$$v_\gamma(a) = C\int_a^\omega \exp(-r_\gamma(\zeta-a))\frac{\gamma h(\zeta)\ell(\zeta)}{\ell(a)}d\zeta,$$

where a constant C is given as $<v_\gamma,\psi_\gamma> = 1$. From (6.10) it follows that if $<v_1,x> = b^*$, then

$$\lim_{t\to\infty} T_1(t)x = \lim_{t\to\infty} U_\beta(t, 0)x = b^*\ell(a), \tag{6.15}$$

where $\beta(t) = 1$ for all $t\geqslant 0$. Next assume that $<v_1,x><b^*$. For any small number $\delta>0$, it follows that $r_{1+\delta}>0$ and $<v_1,T_{1+\delta}(t)x>$ goes to infinity as $t\to\infty$, because

$$\lim_{t\to\infty} <v_1,\exp(-r_{1+\delta}t)T_{1+\delta}(t)x> = <v_{1+\delta},x><v_1,\psi_{1+\delta}>,$$

where $\psi_{1+\delta}$ is the eigenvector of $A_{1+\delta}$ corresponding to the eigenvalue $r_{1+\delta}$. Then we can choose a time $t_0>0$ such that

$$<v_1,T_{1+\delta}(t_0)x> = b^*. \tag{6.16}$$

Now we define $\beta_0(t) \in PC[0,\infty)$ such that

$$\beta_0(t) = \begin{cases} 1+\delta, & 0 \leqslant t \leqslant t_0, \\ 1, & t_0 < t. \end{cases} \tag{6.17}$$

Then we obtain the representation

$$U_{\beta_0}(t,0)x = \begin{cases} T_{1+\delta}(t)x, & 0 \leqslant t \leqslant t_0, \\ T_1(t-t_0)T_{1+\delta}(t_0)x, & t_0 < t. \end{cases} \tag{6.18}$$

From (6.10) and (6.16), it follows that

$$\lim_{t\to\infty} U_{\beta_0}(t,0)x = \lim_{t\to\infty} T_1(t-t_0)T_{1+\delta}(t_0)x = <v_1, T_{1+\delta}(t_0)x>\ell(a) = b^*\ell(a).$$

Similarly if $<v_1,x>>b^*$, we can prove that there exists $\beta_0(t) \in PC[0,\infty)$ such that $0<1-\delta \leqslant \beta_0(t) \leqslant 1$ and

$$\lim_{t\to\infty} U_{\beta_0}(t,0)x = b^*\ell(a). \tag{6.19}$$

Therefore we conclude that Π is included in the approximately controllability space under the admissible control $\Lambda_1 := \{\beta(t) \in PC[0,\infty): 0<1-\delta \leqslant \beta(t) \leqslant 1+\delta\}$ with respect to $x \in X_+ \setminus \{0\}$ such that $<v_1,x>>0$. Let $\beta_0(t) \in \Lambda_1$ satisfy (6.19). Then we can choose $\beta_n(t) \in C^\infty[0,\infty)$ such that

$$|\beta_n(t)-1| \leqslant \delta, \quad \lim_{n\to\infty} \int_0^\infty |\beta_n(\tau)-\beta_0(\tau)|\,d\tau = 0. \tag{6.20}$$

Then we have the following estimation from (6.12)

$$\|U_{\beta_n}(t,0)x\| \leqslant M\|x\|\exp\{M(1+\delta)\|B\|t\}, \tag{6.21}$$

where the constant M is given such that $\|T_0(t)\| \leqslant M$. By using (6.12), (6.21) and Gronwall's inequality, it is easily seen that the following holds

$$\|U_{\beta_n}(t,0)x - U_{\beta_0}(t,0)x\| \leqslant M^2\|x\|\|B\|\exp\{2M\|B\|(1+\delta)t\} \int_0^\infty |\beta_n(\tau)-\beta_0(\tau)|\,d\tau. \tag{6.22}$$

By the definition of β_0, for any $\epsilon>0$ we can take a large $T>0$ such that

$$\|U_{\beta_0}(T,0)x - b^*\ell(\cdot)\| < \frac{\epsilon}{2},$$

and from (6.20) and (6.22) there exists a sufficiently large $n = n(T)$ such that

$$\|U_{\beta_n}(T,0)x - U_{\beta_0}(T,0)x\| < \frac{\epsilon}{2}.$$

Then we arrive at

$$\|U_{\beta_n}(T,0)x - b^*\ell(\cdot)\| < \epsilon,$$

which shows that $\Pi \subset \overline{\Omega(x;\Lambda)}$ when $<v_1,x>\neq 0$ for $x \in X_+ \setminus \{0\}$. This completes the proof. □

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