Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

J.W. van der Woude

On the structure at infinity of a structured system

Department of Operations Research, Statistics, and System Theory Report BS-R8918 October

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).

On the Structure at Infinity of a Structured System

J.W. van der Woude
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

In this paper we develop a graph theoretic characterization of the generic structure at infinity of the transfer matrix of a structured system. We show that the generic structure at infinity can be determined by means of algorithms from combinatorial optimization based on the max-flow min-cut theorem, and on results concerning minimal cost flows. As an application of the obtained characterization, we propose a structural version of two well-known disturbance decoupling problems and we derive graph theoretic necessary and sufficient conditions for the solvability of each of the two problems.

1980 Mathematics Subject Classification (1985): 93C05, 93A99, 94C15.

Key Words and Phrases: Structured systems, generic rank, generic orders of zeroes at infinity, transfer matrix, graph theory, max-flow min-cut theorem, minimal cost flow, disturbance decoupling problem.

1. Introduction

In the present paper we introduce the generic rank and the generic orders of the zeroes at infinity, together forming the generic structure at infinity, of transfer matrices of a general class of structured linear systems. We represent structured systems by means of directed graphs, and we develop graph theoretic characterizations of the generic rank and the generic orders of the zeroes at infinity of the corresponding transfer matrix. We show that the obtained characterizations can be checked by means of well-known and efficient algorithms from combinatorial optimization. As an application of the obtained characterization, we propose a structural version of two well-known disturbance decoupling problems, and we derive graph theoretic necessary and sufficient conditions for the solvability of each of the two problems.

Having briefly sketched the contents of the paper, we now want to make clear why the study of structured systems is useful. To do this we may consider any well-established control problem, formulated for an appropriate linear system. For instance, we may think of the pole assignment problem, the disturbance decoupling problem or the problem of non interacting control (cf. Wonham [20]).

One of the main ideas behind the present paper now is that, before applying algorithms that check the solvability of the control problem and that compute the corresponding feedback control laws, it may be worthwhile to investigate if the system has any structure. And, if so, it may then be useful to try to determine from this structure, whether or not in some structural sense the control problem is solvable. Of course, it is therefore required that we have a characterization of the structural solvability of the control problem in terms of the structure of the system. Furthermore, it is clear that it might be useful to have an algorithm by which we can verify the characterization in an efficient way. Finally, it may be clear that when we can derive such a characterization and algorithm, we obtain a powerful tool which exploits the structure present in the system and which, in addition to the existing algorithms, helps us to the decide about the solvability of the control problem.

In the present paper we are led by the disturbance decoupling problem by state feedback, well-known from the geometric approach towards control theory (cf. Wonham [20]). We recall that the solvability

Report BS-R8918
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

of the disturbance decoupling problem is equivalent to the fact that certain elementary transfer matrices have the same rank and have zeroes at infinity of the same orders. Since we represent structured systems by means of graphs, it is therefore clear that our first interest lies in the development of a graph theoretic characterization of the rank and the orders of the zeroes at infinity of the transfer matrix of a structured system.

The outline of the present paper is as follows. In section 2 we introduce structured systems and we describe a way in which structured systems can be parametrized. In section 3 we introduce the rank and the orders of the zeroes at infinity of proper rational matrices. Furthermore, we state a result on the solvability of proper rational matrix equations over the proper rational matrices. In section 4 we introduce the generic rank and the generic orders of the zeroes at infinity of the transfer matrix of a structured system. In section 5 we introduce the graphs corresponding to the structured systems, and we recall some important notions and results from graph theory. In section 6 we state our main results. We first prove that the generic rank of the transfer matrix of a structured system is equal to the maximal number of disjoint paths from the set of input vertices to the set of output vertices in the graph corresponding to the structured system. If this rank equals r, we next prove that the generic orders of the zeroes at infinity can be determined by computing for i from 1 to r, the minimal number of state vertices appearing in any i-tuple of disjoint paths from the set of input vertices to the set of output vertices. In section 7 we discuss some of the computational aspects of the main results. We indicate that for a given structured system the generic rank and the generic orders of the zeroes at infinity can be computed using algorithms from combinatorial optimization based on the max-flow min-cut theorem and on results on minimal cost flows. In section 8 we propose a structural version of the disturbance decoupling problem and the so-called modified disturbance decoupling problem, and we apply our main results to obtain a graph theoretic characterization for the solvability of each of the two problems. In section 9 we offer some remarks and comments.

2. Structured systems

In this section we introduce structured systems, and we describe how these systems can be parametrized. Therefore, we consider the finite-dimensional linear time-invariant system

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{1.1.a}$$

$$y(t)=Cx(t), \tag{1.1.b}$$

with state $x(t)\in\mathbb{R}^n$, input $u(t)\in\mathbb{R}^m$, and output $y(t)\in\mathbb{R}^p$, and with A,B and C real matrices of dimensions $n\times n$, $n\times m$ and $p\times n$, respectively. To give an indication of what we mean by structured systems, we assume that the system (1.1) is a series interconnection of the following two subsystems

$$\dot{x}_1(t)=A_1x_1(t)+B_1u_1(t), \qquad \dot{x}_2(t)=A_2x_2(t)+B_2u_2(t),$$

$$y_1(t)=C_1x_1(t), \qquad y_2(t)=C_2x_2(t),$$

where $u_2(t)=y_1(t), u(t)=u_1(t)$ and $y(t)=y_2(t)$, and all vectors and matrices have appropriate dimensions. After interconnection of the two subsystems it follows that

$$A=\begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, B=\begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \text{ and } C=\begin{bmatrix} 0 & C_2 \end{bmatrix}.$$

The zeroes in the above representation of A,B and C are matrices with entries that are fixed zeroes. This means that these entries are always zero, no matter what the entries are in the matrices A_1, B_1, C_1, A_2, B_2 and C_2. In this paper we call such fixed zeroes in A, B and C *structural zeroes*. Entries in A, B and C that are not structural zeroes, we call *structural unknowns*, and we assume that the values of these entries are unknown and are independent of each other.

In this paper we say that a matrix is *structured* if its entries are either a structural zero or a structural unknown, and we call a system of type (1.1) a *structured system* if the matrices A, B and C are structured.

Given a structured system of type (1.1), we denote the number of structural unknowns in A, B and C by k, and we parametrize the set of all nominal systems that correspond to the same structured system by a parameter $\lambda \in \mathbf{R}^k$. To do this, we number the structural unknowns in A, B and C from 1 up to k, and we write λ_i at the i-th structural unknown. We denote the nominal values of A,B and C at the parameter value $\lambda \in \mathbf{R}^k$ by A_λ, B_λ and C_λ. Below we give an example of a structured system of type (1.1), together with a possible parametrization.

EXAMPLE
$k=9, n=3, m=2, p=2$ and 0 denotes a structural zero, and x a structural unknown.

$$A = \begin{bmatrix} 0 & x & 0 \\ x & 0 & 0 \\ 0 & x & x \end{bmatrix}, B = \begin{bmatrix} x & 0 \\ 0 & x \\ x & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}.$$

$$A_\lambda = \begin{bmatrix} 0 & \lambda_2 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & \lambda_4 \end{bmatrix}, B_\lambda = \begin{bmatrix} \lambda_5 & 0 \\ 0 & \lambda_7 \\ \lambda_6 & 0 \end{bmatrix}, C_\lambda = \begin{bmatrix} 0 & \lambda_8 & 0 \\ 0 & 0 & \lambda_9 \end{bmatrix}.$$

3. STRUCTURE AT INFINITY

In the present section we introduce the rank and the orders of the zeroes at infinity of proper rational matrices. However, we start with a brief introduction on rational functions.

We call a function a *rational function* if the function can be written as the quotient of two polynomials with real coefficients. Given such a representation and using the usual notion of degree for nonzero polynomials, we define the *degree* of a nonzero rational function to be the degree of the numerator polynomial minus the degree of the denominator polynomial. For rational functions identically equal to *zero*, we define the degree to be $-\infty$. Note that polynomials are rational functions and that for polynomials this new notion of degree coincides with the usual notion of degree. We call a rational function *proper* if its degree is negative or zero, and *strictly proper* if its degree is negative. This means that, if written as the quotient of two polynomials, a rational function is proper if the degree of the nominator polynomial is not larger than the degree of the denominator polynomial, and strictly proper if the degree of the nominator polynomial is less the degree of the denominator polynomial.

We call a matrix a *rational matrix* if its entries are rational functions, a *proper rational matrix* if its entries are proper rational functions and a *strictly proper rational matrix* if its entries are strictly proper rational functions. We say that a rational matrix has *rank r* if there is an r-th order minor of the matrix that is *unequal* to zero, while every $r+1$-th order minor of the matrix is *identically equal* to zero. We say that a square proper rational matrix is a *bicausal rational matrix* if the matrix is invertible and if its inverse is a proper rational matrix (cf. Hautus and Heymann [9]). Bicausal rational $t \times t$ matrices are the units in the ring of proper rational $t \times t$ matrices. It can be shown that a proper rational matrix is bicausal if and only if the determinant of its value at infinity is unequal to zero. Using bicausal rational matrices, we can state the following theorem concerning a factorization of proper rational matrices (cf. Descusse and Dion [3], Hautus [7], also compare with the Smith form for polynomial matrices).

THEOREM 3.1. *Given a proper rational matrix $T(s)$, there exists a factorization*

$$T(s) = V(s) \begin{bmatrix} \Gamma(s) & 0 \\ 0 & 0 \end{bmatrix} U(s),$$

with $U(s)$ and $V(s)$ bicausal rational matrices of suitable dimensions and $\Gamma(s) = diag\,(s^{-t_1}, s^{-t_2}, \ldots, s^{-t_r})$, where $r = rank\ T(s)$ and $t_1, t_2, \ldots, t_r$ are integers that satisfy $0 \leq t_1 \leq t_2 \leq \ldots \leq t_r$.

The integers $t_1, t_2, \ldots, t_r$ are known as the *orders of the zeroes at infinity* of $T(s)$, and are uniquely determined. We say that two proper rational matrices with the same rank, also have zeroes at infinity of the same orders, if the list of the orders of the zeroes at infinity for both matrices is the same. Using the Cauchy-Binet formula we can prove the following characterization of the orders of the zeroes at infinity of a proper rational matrix, where we denote $m_i = \sum_{j=1}^{i} t_i$ for $i = 1,2,...,r$. (cf. Gantmacher [5], Chapter 6.3, where a similar result for polynomial matrices is proved).

LEMMA 3.2. *Let $T(s)$ be a proper rational matrix of rank r with a factorization as given in theorem 3.1. Then for any $i = 1,2,...,r$, every i-th order minor of $T(s)$ is a proper rational function with a degree $\delta \leq -m_i$, and there exists at least one i-th order minor of $T(s)$ with a degree δ such that the equality holds, i.e. $\delta = -m_i$.*

Lemma 3.2 implies that the number m_i equals the exponent of the greatest power of s by which any i-th order minor of the proper rational matrix $T(s)$ can be multiplied such that the product remains proper. Clearly, we could have used the latter characterization to give an alternative definition of the orders of the zeroes at infinity in which the use of a factorization of a proper rational matrix is avoided. In fact, in the next section, we more or less use this alternative approach to introduce the *generic* orders of the zeroes at infinity of the transfer matrix of a structured system.

We now state a theorem that we need in section 8 to obtain suitable conditions for the solvability of the disturbance decoupling problem and the modified disturbance decoupling problem (cf. Emre and Hautus [4], Newman [15]).

THEOREM 3.3. *Let $T(s)$ and $S(s)$ be proper rational matrices that have the same number of rows. Then there exists a proper rational matrix $X(s)$ of suitable dimensions such that $T(s)X(s) = S(s)$ if and only if the rank and the orders of the zeroes at infinity of the matrix $T(s)$ and of the compound matrix $[T(s)\, S(s)]$ are the same.*

PROOF. The necessity of the conditions is immediate by the fact that $[T(s)\, S(s)]\,\tilde{U}(s) = [T(s)\; 0]$, where

$$\tilde{U}(s) = \begin{bmatrix} I & -X(s) \\ 0 & I \end{bmatrix}$$

is a bicausal rational matrix in which 0 and I denote zero matrices and identity matrices of suitable dimensions. Then, by theorem 3.1, $T(s)$, $[T(s)\; 0]$ and $[T(s)\, S(s)]$ have the same rank and have zeroes at infinity of the same orders.

To prove the sufficiency of the conditions, it is easy to see that without loss of generality we can assume that

$$T(s) = \begin{bmatrix} \Gamma(s) & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Gamma(s)$ is an $r \times r$ matrix as described in theorem 3.1, and that the matrix $S(s)$ consists of l columns and is partitioned as

$$S(s) = \begin{bmatrix} \tilde{S}(s) \\ \bar{S}(s) \end{bmatrix}$$

where $\tilde{S}(s)$ is an $r \times l$ matrix, and $\bar{S}(s)$ is a matrix of l columns and a suitable number of rows.

Because rank $T(s)$ = rank $[T(s)\, S(s)]$, we have that the matrix $\bar{S}(s)$ is identically equal to zero, i.e. $\bar{S}(s) = 0$ for all s. Furthermore, because the orders of the zeroes at infinity of $T(s)$ and of $[T(s)\, S(s)]$ are the same, we have that deg $\tilde{s}_{i,j}(s) \leq -t_i$, for all $i = 1,..,r$ and $j = 1,..,l$, where we have denoted $\tilde{s}_{i,j}(s)$ for the entry of $\tilde{S}(s)$ in the i-th row and the j-th column. To prove this, we assume that deg $\tilde{s}_{i_0,j_0}(s) > -t_{i_0}$ for some $i_0 \in \{1,..,r\}$ and $j_0 \in \{1,..,l\}$, with i_0 as small as possible. Next we restrict our attention to the first i_0 rows of $[T(s)\ S(s)]$, and we consider the $i_0 \times i_0$ submatrix made up of the entries in these rows, and in the first i_0-1 columns of $T(s)$ and the j_0-th column of $S(s)$. The determinant of this square matrix is equal to $\tilde{s}_{i_0,j_0}(s)\ s^{\kappa}$ with $\kappa = -m_{i_0-1}$. So, by assumption we have found a i_0-th order minor of $[T(s)\ S(s)]$ with a degree strictly larger than $-m_{i_0}$, while from lemma 3.2 we know that every i_0-th order minor of $T(s)$ has a degree less than or equal to $-m_{i_0}$. By theorem 3.1 and lemma 3.2 it is clear that $T(s)$ and $[T(s)\ S(s)]$ can *not* have zeroes at infinity that have the same orders. Hence, we have a contradiction with the assumption that the ranks and the orders of the zeroes at infinity of $T(s)$ and of $[T(s)\ S(s)]$ are the same, and consequently, we have proved that deg $\tilde{s}_{i,j}(s) \leqslant -t_i$, for all $i = 1,..,r$ and $j = 1,..,l$. To complete the proof of the theorem, we define the $r \times l$ matrix $\tilde{X}(s)$ as $\tilde{x}_{i,j}(s) = \tilde{s}_{i,j}(s)\ s^{t_i}$ for all $i = 1,..,r$ and $j = 1,..,l$. It then follows that the matrix $\tilde{X}(s)$ is proper and it can be easily verified that $\Gamma(s)\tilde{X}(s) = \tilde{S}(s)$. By extending $\tilde{X}(s)$ with a suitable number of rows each containing l proper rational entries, we obtain a proper rational matrix $X(s)$ that satisfies $T(s)\,X(s) = S(s)$. □

4. Generic structure at infinity

In this section we introduce the generic rank and the generic orders of the zeroes at infinity of the transfer matrix of a structured system of type (1.1), by giving new meanings to the integers r, t_i and m_i, for $i = 1,2,...,r$. We start with the introduction of the generic rank.

Given a structured system of type (1.1), parametrized by $\lambda \in \mathbb{R}^k$, we denote

$$K_\lambda(s) = C_\lambda(sI-A_\lambda)^{-1}B_\lambda, \qquad M_\lambda(s) = \begin{bmatrix} A_\lambda - sI & B_\lambda \\ C_\lambda & 0 \end{bmatrix} \tag{4.1}$$

and we define

$$r = \max_{\lambda\in\mathbb{R}^k}\{\text{rank } K_\lambda(s)\}, \qquad R = \{\lambda\in\mathbb{R}^k \mid \text{rank } K_\lambda(s)<r\}. \tag{4.2}$$

Note that if $r = 0$, then $R = \varnothing$, where $\varnothing$ denotes the empty set.

Following Wonham [20], we call a subset L in $\mathbb{R}^k$ a *algebraic variety* in $\mathbb{R}^k$ if L can be described as the locus of common zeroes of a finite number of polynomials $\psi_1,\psi_2,....,\psi_t$ in the indeterminate $\tau=(\tau_1,\tau_2,....,\tau_k)$, i.e. $L = \{(\tau_1,\tau_2,....,\tau_k) \in \mathbb{R}^k \mid \psi_i(\tau_1,\tau_2,....,\tau_k)=0 \text{ for all } i=1,2,....,t\}$. We say that an algebraic variety L in $\mathbb{R}^k$ is *proper* if $L \neq \mathbb{R}^k$. Now we can state the following (cf. van der Woude [21]).

Theorem 4.1. *R is a proper algebraic variety in $\mathbb{R}^k$.*

Proof. If $r = 0$, then $R = \varnothing$, and R clearly is a proper algebraic variety. If $r > 0$, then using the identity

$$M_\lambda(s) = \begin{bmatrix} I & 0 \\ C_\lambda(A_\lambda - sI)^{-1} & I \end{bmatrix} \begin{bmatrix} A_\lambda - sI & 0 \\ 0 & K_\lambda(s) \end{bmatrix} \begin{bmatrix} I & (A_\lambda - sI)^{-1}B_\lambda \\ 0 & I \end{bmatrix}, \tag{4.3}$$

it follows that rank $K_\lambda(s)$ = rank $M_\lambda(s) - n$. From the definition of R and our notion of rank it is now clear that

$$R = \{\lambda\in\mathbb{R}^k \mid \text{every } n+r\text{-th order minor of } M_\lambda(s) \text{ is identically equal to zero}\}.$$

Next observe that any minor of the matrix $M_\lambda(s)$ is a polynomial in the indeterminate s with coefficients that are polynomials in $\lambda=(\lambda_1,\lambda_2,.......,\lambda_k)$. Furthermore, recall that a polynomial in the indeterminate s is identically equal to zero if and only if all its coefficients are zero. Therefore, it follows that R is the locus of common zeroes of a finite number of polynomials in λ. By the definition of r it is clear that $R \neq \mathbb{R}^k$. So, also if $r > 0$, the set R is a proper algebraic variety in $\mathbb{R}^k$. □

The above theorem implies that rank $K_\lambda(s) = r$ for almost all $\lambda\in\mathbb{R}^k$, where almost all is to be interpreted as everywhere except for a proper algebraic variety. Hence, we can think of r as the *generic rank* of $K(s)$, where $K(s)$ formally denotes the transfer matrix of the structured system, i.e.

$K(s) = C(sI - A)^{-1}B.$

By lemma 3.2 we know that in the unstructured case the orders of the zeroes at infinity of a transfer matrix are closely related to the degrees of its minors. Hence, for the introduction of the generic orders of the zeroes at infinity of $K(s)$, it seems natural that we first consider a *square* structured system, i.e. $m = p$, and that we introduce the generic degree of the determinant of the transfer matrix of such a square structured system. To do this, we consider a square structured system of type (1.1), parametrized by $\lambda \in \mathbb{R}^k$, and we assume that the generic rank of its transfer matrix is r.

If $m = p = r$, we define

$$q = \max_{\lambda \in \mathbb{R}^k}\{\deg(\det K_\lambda(s))\}, \qquad Q = \{\lambda \in \mathbb{R}^k \mid \deg(\det K_\lambda(s)) < q\}, \tag{4.4.a}$$

and if $m = p > r$, which means that $\det K_\lambda(s) = 0$ for all $\lambda \in \mathbb{R}^k$ and all s, we define

$$q = -\infty, \qquad\qquad Q = \varnothing. \tag{4.4.b}$$

In the above deg stands for degree and det for determinant. Now we can state the following.

THEOREM 4.2. *Q is contained in a proper algebraic variety in $\mathbb{R}^k$.*

PROOF. If $m = p > r$, then $Q = \varnothing$, and Q clearly is contained in a proper algebraic variety. If $m = p = r$, then it easily follows from (4.3) and (4.4.a) that

$$q + n = \max_{\lambda \in \mathbb{R}^k}\{\deg(\det M_\lambda(s))\}\,(\geqslant 0), \tag{4.5.a}$$

and that

$$Q = \{\lambda \in \mathbb{R}^k \mid \deg(\det M_\lambda(s)) < q + n\}. \tag{4.5.b}$$

As in the proof of theorem 4.1, it is clear that $\det M_\lambda(s)$ is a polynomial in the indeterminate s with coefficients that are polynomials in $\lambda \in \mathbb{R}^k$. By the above description of Q, it is therefore clear that Q is contained in the algebraic variety in $\mathbb{R}^k$ defined as the set of $\lambda \in \mathbb{R}^k$ for which the coefficient of s^{q+n} in the $q+n$-th order polynomial $\det M_\lambda(s)$ is equal to zero. By the definition of q it moreover follows that this algebraic variety is proper. So, also if $m = p = r$, the set Q is contained in a proper algebraic variety. □

Theorem 4.2 implies that for a square structured system of type (1.1), parametrized by $\lambda \in \mathbb{R}^k$, we have that if $m = p > r$, then $\deg(\det K_\lambda(s)) = q = -\infty$ for all $\lambda \in \mathbb{R}^k$, and that if $m = p = r$, then $\deg(\det K_\lambda(s)) = q$, with $-\infty < q < 0$, for almost all $\lambda \in \mathbb{R}^k$. Hence, for a square structured system of type (1.1), we can think of q as the *generic degree* of the determinant of $K(s)$.

In the remainder of the present section we return to a general structured system of type (1.1), and we only assume that the system is square. Then, using the above, we can introduce the generic orders of the zeroes at infinity of the transfer matrix $K(s)$. To that end, we note that for every $\lambda \in \mathbb{R}^k$, any minor of $K_\lambda(s)$ corresponds to the determinant of the transfer matrix of a square subsystem of system (1.1) at the parameter value λ. Therefore, it is clear that we can consider the generic degree of such a minor to be the generic degree of the determinant of the transfer matrix of the corresponding square structured subsystem. Since there are only a finite number of minors of the same order, we can take the maximum of the generic degrees of all these minors. We denote m_i for *minus* the maximum of the generic degrees of all i-th order minors of $K(s)$, where $1 \leqslant i \leqslant r$, with r the generic rank of $K(s)$. We can now easily prove that

$$m_i = -\max_{\lambda \in \mathbb{R}^k}\{\max\{\deg(K_\lambda^i(s)) \mid K_\lambda^i(s) \text{ is an } i\text{-th order minor of } K_\lambda(s)\}\} \tag{4.6}$$

for $i = 1,2,\ldots,r$, and that $0 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_r$. In addition, we can prove in the same way as in theorems 4.1 and 4.2 that the set of parameter values $\lambda \in \mathbb{R}^k$ for which all i-th order minors of $K_\lambda(s)$ have a degree less than $-m_i$ is contained in a proper algebraic variety in $\mathbb{R}^k$.

Now, in the spirit of theorem 3.1 and lemma 3.2, we define the integers t_i, given by

$$t_i = m_i - m_{i-1} \tag{4.7}$$

with $m_0 = 0$, to be the *generic orders of the zeroes at infinity* of $K(s)$.

5. GRAPHS

In the previous section we introduced the generic rank and the generic orders of the zeroes at infinity of the transfer matrix $K(s)$. In the next section we describe how these notions can be related to the structure of systems of type (1.1). For this purpose, we represent a structured system of type (1.1) by a *directed graph*. This graph, denoted $G(V,E)$, consists of a vertex set V with $n+m+p$ vertices and an edge set E of k directed edges (ordered pairs). The set V is defined as $V = U \cup X \cup Y$ where $U = \{u_1, u_2, ..., u_m\}$, $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_p\}$ and $\cup$ denotes the union. The set E is defined as $E = \{(u_j, x_i) \,|\, b_{i,j} \neq 0\} \cup \{(x_j, x_i) \,|\, a_{i,j} \neq 0\} \cup \{(x_j, y_i) \,|\, c_{i,j} \neq 0\}$. Here, for instance, the ordered pair (u_j, x_i) represents a directed edge from the vertex u_j to the vertex x_i and $b_{i,j} \neq 0$ means that the entry $b_{i,j}$ in the matrix B is a structural unknown entry. U, X and Y are called the set of input vertices, state vertices and output vertices, respectively. In figure 1 we have depicted the graph $G(V,E)$ corresponding to the structured system in the example of section 2.

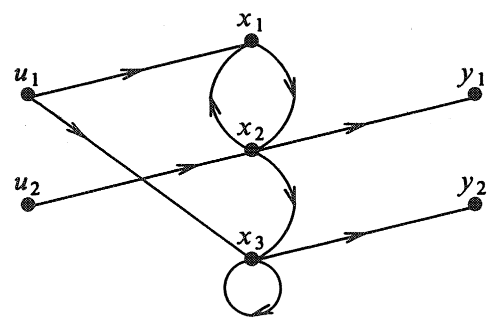

Figure 1.

In section 2 we considered a structured system of type (1.1), and we assumed that the system contained k structural unknowns, numbered from 1 upto k. We parametrized all nominal systems that correspond to the same structured system by a parameter $\lambda \in \mathbb{R}^k$ by writing λ_i at the i-th structural unknown. Using this numbering, we can also number the edges in E from 1 to k, and we can introduce the nominal (or weighted) directed graph $G_\lambda(V,E)$ at the parameter value $\lambda \in \mathbb{R}^k$ by weighting the i-th edge of the graph $G(V,E)$ by λ_i. In figure 2 we have depicted the graph $G_\lambda(V,E)$ that is obtained from the graph in figure 1 by weighting the edges in accordance to the parametrization described in the example of section 2.

Given the graph $G(V,E)$ (or $G_\lambda(V,E)$), we say that there is a *self loop* at the vertex $v \in V$, if $(v,v) \in E$. We say that there exists a *path* from the vertex v to the vertex v', if there are vertices $w_1, w_2, ..., w_\tau$ in V such that $v = w_1$, $v' = w_\tau$ and $(w_i, w_{i+1}) \in E$ for $i = 1,2,...,\tau-1$. If, in addition, $v \in U$ and $v' \in Y$, we say that there is a path from U to Y. If we have a path from v to v' with $v = v'$, we say that the path is *closed*. If a path consists of *distinct* vertices, we say that the path is *simple*, and if a path is both simple and closed, we call it a *cycle*. Clearly, a self loop is a cycle. We say that an l-tuple of paths (cycles) in $G(V,E)$ are *disjoint* if each pair of paths (cycles) of the l-tuple has no vertices in common.

The weighted graph $G_\lambda(V,E)$ can considered to be a special case of a so-called *Coates graph* associated to a real square matrix (cf. Chen [2]). For a real nominal $\tau \times \tau$ matrix M that has l nonzero entries

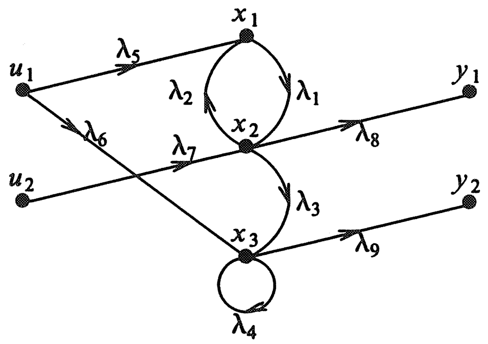

Figure 2.

the associated Coates graph, denoted G_M, is a graph with a vertex set V_M of τ vertices and an edge set E_M of l directed and weighted edges. If the vertex set is given by $V_M = \{v_1, \ldots\ldots, v_\tau\}$, then the edge set E_M consists of edges weighted $m_{i,j}$ and directed from v_j to v_i, precisely if $m_{i,j} \neq 0$, i.e. $E_M = \{(v_j, v_i) \mid m_{i,j} \neq 0\}$.

We define paths, cycles, disjoint paths and disjoint cycles for Coates graphs in the same way as done for $G(V,E)$, and we define a *cycle family* for a Coates graph to be a number of *disjoint* cycles such that each vertex of the graph precisely belongs to one cycle, in which case we say that the cycle family *spans* the graph. We define the weight of a cycle family to be the product of the weights of the edges that constitute the cycle family. If Cy denotes a cycle family, we denote its weight by $W(Cy)$ and we denote $n(Cy)$ for the total number of disjoint cycles the cycle family consists of. Now we can state the following classical result (cf. Chen [2], theorem 3.1), where we denote Cf for the set of all cycle families in the Coates graph G_M associated to the $\tau \times \tau$ matrix M, and where, as before, det stands for determinant.

THEOREM 5.1. $\det M = (-1)^\tau \sum_{Cy \in Cf} (-1)^{n(Cy)} W(Cy).$

As an example, we have depicted in figure 3 the Coates graph $G_{M_\lambda(0)}$ corresponding to the square matrix $M_\lambda(0)$ defined in (4.1), with A_λ, B_λ and C_λ as described in the example of section 2.

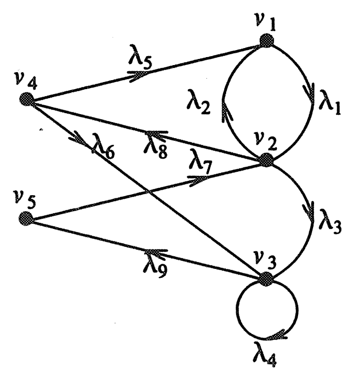

Figure 3.

In the remainder of the present section we consider a structured system of type (1.1), parametrized by $\lambda \in \mathbb{R}^k$, for which the number of inputs and the number of outputs are equal, i.e. $m = p$. Hence, the system is square and also the nominal matrix $M_\lambda(0)$ is square. We can now obtain the Coates graph $G_{M_\lambda(0)}$, corresponding to the matrix $M_\lambda(0)$, directly from the graph $G_\lambda(V,E)$. We can do this by identifying in the vertex set V the i-th input vertex with the i-th output vertex, for $i = 1,2,...,m$. We then obtain a graph with a vertex set consisting of $n+m$ vertices, and with an weighted edge set similar to $E_{M_\lambda(0)}$. Conversely, if we have a Coates graph $G_{M_\lambda(0)}$ with $V_{M_\lambda(0)} = \{v_1,.....,v_n,v_{n+1},.....,v_{n+m}\}$, we can obtain the graph $G_\lambda(V,E)$ by replacing each vertex v_{n+i} by two vertices u_i and y_i, for $i = 1,2....,m$, and by replacing each edge of the form $(v_{n+i},v) \in E_{M_\lambda(0)}$ by the edge (u_i,v), and each edge of the form $(v',v_{n+i}) \in E_{M_\lambda(0)}$ by the edge (v',y_i) where $i = 1,2,....m$. We then obtain a graph with a vertex set consisting of $n+m+m$ vertices and with an edge set similar to E.

Furthermore, with the vertex set of the Coates graph $G_{M_\lambda(0)}$ given as $V_{M_\lambda(0)} = \{v_1,v_2,....,v_{n+m}\}$, we can make the following observation. If there exists a *cycle* in $G_{M_\lambda(0)}$ that contains exactly μ vertices of the set $\{v_{n+1},....,v_{n+m}\}$, then there exists in $G_\lambda(V,E)$ (and also in $G(V,E)$) a μ-tuple of disjoint paths from U to Y. To see this, we may assume without loss of generality, that the μ vertices of the cycle in $\{v_{n+1},....,v_{n+m}\}$ are in fact the vertices v_{n+1}, v_{n+2},....,$v_{n+\mu}$, and that in the cycle v_{i+n} preceeds v_{i+n+1}, for $i = 1,2,...,\mu-1$, and that $v_{n+\mu}$ preceeds v_{n+1}. Here we say that the vertex v preceeds the vertex v', if there is a part of the cycle that constitutes a simple path from v to v' that besides v and v' does not contain any other vertex of the set $\{v_{n+1},v_{n+2},....,v_{n+m}\}$. Then it easily follows that in the graph $G_\lambda(V,E)$ the paths from the vertex u_i to the vertex y_{i+1}, for $i = 1,2,...,\mu-1$, and together with the path from the vertex u_μ to the vertex y_1 constitute a μ-tuple of disjoint paths in $G_\lambda(V,E)$ from U to Y. Using the same reasoning, it follows that if there is a *cycle-family* in $G_{M_\lambda(0)}$ that contains (necessarily) all the vertices of the set $\{v_{n+1},....,v_{n+m}\}$, then there is an m-tuple of disjoint paths in $G_\lambda(V,E)$ (and in $G(V,E)$) from U to Y.

6. Main results

In this section we state the main results of this paper. The results describe relations between the graph $G(V,E)$ associated to a structured system of type (1.1), and the generic rank and the generic orders of the zeroes at infinity of the corresponding transfer matrix. We recall that X, U and Y denote the sets of state vertices, input vertices and output vertices, respectively, of the graph $G(V,E)$. As a first result we state the following theorem in which we use r as defined in (4.2) (cf. van der Woude [21]).

Theorem 6.1. *The maximal number of disjoint paths in $G(V,E)$ from U to Y is equal to r.*

Proof. We start the proof by considering the case that $r > 0$. By the definition of r and the notion of rank introduced in section 3, it follows that there is a parameter $\bar{\lambda} \in \mathbb{R}^k$ for which there is an r-th order minor of $K_\lambda(s)$ unequal to zero. Without loss of generality we may assume that this r-th order minor is $\det C'_{\bar{\lambda}}(sI-A_{\bar{\lambda}})^{-1}B'_{\bar{\lambda}}$, where B'_λ denotes the first r columns of B_λ, C'_λ denotes the first r rows of C_λ and where we have substituted $\lambda=\bar{\lambda}$. Since the minor is nonzero, there exists a real number $\bar{s}$ such that $\det C'_{\bar{\lambda}}(\bar{s}I-A_{\bar{\lambda}})^{-1}B'_{\bar{\lambda}} \neq 0$, and $\det(A_{\bar{\lambda}}-\bar{s}I) \neq 0$. Using (4.3) it now follows that $\det M'_{\bar{\lambda}}(\bar{s}) \neq 0$, where

$$M'_\lambda(s) = \begin{bmatrix} A_\lambda - sI & B'_\lambda \\ C'_\lambda & 0 \end{bmatrix}.$$

By theorem 5.1 this implies that in the Coates graph associated to the nominal matrix $M'_{\bar{\lambda}}(\bar{s})$ there is at least one spanning cycle family. Now we let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be structured matrices for which $A_{\bar{\lambda}}-\bar{s}I$, $B'_{\bar{\lambda}}$ and $C'_{\bar{\lambda}}$, respectively, can occur as the nominal values. Clearly, we can take $\hat{A} = A + E$, $\hat{B} = B'$ and $\hat{C} = C'$, where B' denotes the first r columns of B, C' denotes the first r rows fo C and E denotes a structured matrix with only structural unknowns on its diagonal. By the remarks at the end of the previous section it now follows that in the graph of the structured system described by $\hat{A}$, $\hat{B}$ and $\hat{C}$ there exists an r-tuple of disjoint paths from the set of input vertices to the set of output vertices. Since in the context of disjoint paths the self loops, introduced by E, are of no interest, it follows that in the graph of the structured system described by A, B' and C' there are r disjoint paths from the set of input vertices to the set of output vertices. Because B' is a part of B and C' is a part of C, it now follows that $n_+ \geqslant r > 0$, where we have denoted n_+ for the maximal number of disjoint paths in $G(V,E)$ from U to Y. The latter also implies that if we have the case that $n_+ = 0$, which means that there is no path in $G(V,E)$ from U to Y, then $r = 0$.

Next we consider the case that we have a n_+-tuple disjoint paths in $G(V,E)$ and in $G_\lambda(V,E)$ from U to Y with $n_+ > 0$, where n_+ is as defined above, and we concentrate on the subgraph build up from the vertices and edges in the n_+-tuple of paths only. It is easy to see that this subgraph corresponds to n_+ totally decoupled structured single-input single-output systems that each have a transfer function with a generic rank equal to 1. The n_+ subsystems can be obtained from the original system by specifying that some of the structural unknowns are in fact zero. This comes down to saying that the parameter λ, which parametrizes $G_\lambda(V,E)$ and also system (1.1), is restricted to some proper subset L in $\mathbb{R}^k$. Therefore, since $L \subseteq \mathbb{R}^k$, it is clear that $0 < n_+ = \max_{\lambda \in L}[\text{rank } K_\lambda(s)] \leq r$. This also implies that if $r = 0$, meaning that $K_\lambda(s) = 0$ for all $\lambda \in \mathbb{R}^k$ and all s, then $n_+ = 0$.

The proof of the present theorem can now be completed by combining all the obtained relations between r and n_+. □

From section 4 it is immediate that theorem 6.1 implies that the generic rank of $K(s)$ is equal to the maximal number of disjoint paths in $G(V,E)$ from U to Y. Hence, we have obtained a graph theoretic characterization of the generic rank of the transfer matrix $K(s)$. To obtain a graph theoretic characterization of the generic orders of the zeroes at infinity of the transfer matrix $K(s)$, we need the following theorem, formulated for a square structured system with a transfer matrix that has a generic rank r equal to $m = p$. In the theorem we use q as defined in (4.4.a).

THEOREM 6.2. *If $m = p = r$, then the minimal number of state vertices in any r-tuple of disjoint paths in $G(V,E)$ from U to Y is equal to $-q$.*

PROOF. By the definition of q in (4.4.a), there exists a $\bar{\lambda} \in \mathbb{R}^k$ for which $\deg(\det K_{\bar{\lambda}}(s)) = q$ with $-\infty < q < 0$. Then, from (4.3) it follows that $\deg(\det M_{\bar{\lambda}}(s)) = n + q$ with $0 \leq n + q < n$. Hence, the value of $\det(M_{\bar{\lambda}}(s))$ is equal to an $n+q$-th order polynomial in the indeterminate s. By theorem 5.1 this means that in the Coates graph $G_{M_{\bar{\lambda}}(s)}$ there is at least one cycle family with a weight that is equal to a $n+q$-th order polynomial in the indeterminate s. Now note that a factor $s + \alpha$ in the product making up the weight of a cycle family precisely corresponds to a self loop at one of the vertices $v_1, \ldots, v_n$ in the vertex set $V_{M_{\bar{\lambda}}(s)} = \{v_1, \ldots, v_{n+m}\}$ of $G_{M_{\bar{\lambda}}(s)}$ This implies that the above cycle family consists of at least $n+q+1$ disjoint cycles of which exactly $n+q$ are self loops at $n+q$ vertices in the subset $\{v_1, \ldots, v_n\}$. The other $n-(n+q) = -q$ vertices in the subset $\{v_1, \ldots, v_n\}$ appear in the remaining cycles of the cycle family. These remaining cycles can not be self loops, and have weights that are independent of the indeterminate s. Also these cycles contain all the vertices of the set $\{v_{n+1}, \ldots, v_{n+m}\}$. By the remarks at the end of the previous section, it now follows that these cycles correspond with r disjoint paths in the graph $G(V,E)$ from U to Y. It is clear that these r disjoint paths contain at most $-q$ state vertices. Hence, $n_- \leq -q$, where we have denoted n_- for the minimal number of state vertices in any r-tuple of disjoint paths in $G(V,E)$ from U to Y.

Conversely, suppose that for a given $\bar{\lambda} \in \mathbb{R}^k$, there is a set of r *simple* and disjoint paths in $G_{\bar{\lambda}}(V,E)$ from U to Y, and that the r paths contain n_- state vertices with n_- as defined above. Then consider the Coates graph associated the matrix $M_{\bar{\lambda}}(s)$ obtained by identifying the vertices u_i with y_i for $i = 1,2,\ldots,r$. The r-tuple disjoint paths in $G_{\bar{\lambda}}(V,E)$ induce a number of disjoint cycles in $G_{M_{\bar{\lambda}}(s)}$. It is clear that for almost all values of s there is a self loop with a nonzero weight at each of the vertices $v_1, \ldots, v_n$ in the vertex set $\{v_1, \ldots, v_{n+m}\}$ of $G_{M_{\bar{\lambda}}(s)}$. So clearly, at the $n - n_-$ vertices in $\{v_1, \ldots, v_n\}$ that do not appear in the disjoint cycles induced by the r disjoint paths in $G_{\bar{\lambda}}(V,E)$ from U to Y, there is a self loop with a nonzero weight for almost all values of s. The product of the weights of these self loop is a polynomial of degree $n - n_-$ in the value of s. Since the weights of the disjoint cycles induced in $G_{M_{\bar{\lambda}}(s)}$ by the r disjoint paths in $G_{\bar{\lambda}}(V,E)$ from U to$Y$ are independent of the value of s (the paths are simple), it follows that the cycle family constituted by the self loops and the disjoint cycles has a weight that is a polynomial of degree $n-n_-$ in the value s. Because each cycle family contributes to det $M_{\bar{\lambda}}(s)$, the latter implies that $n-n_- \leqslant n+q$, which in turn implies that $-q \leqslant n_-$. □

The above theorem was formulated for a square structured system. We now return to a general system of type (1.1) that is not necessarilly square. If the transfer matrix of the system $K(s)$ has a generic rank equal to r, then for $i = 1,2,...,r$, every i-th order minor of $K(s)$ has a generic rank less than or equal to $-m_i$, and there exists at least one i-th order minor of $K(s)$ that has a generic degree equal to $-m_i$. This immediately follows from the properties of the numbers m_i defined in (4.6). Since each minor of $K(s)$ corresponds to the determinant of the transfer matrix of a square subsystem, the next theorem immediately follows from theorem 6.2 and the way in which the numbers m_i were introduced in section 4. In the theorem we assume that the generic orders of the zeroes at infinity are defined by (4.7).

THEOREM 6.3. *Let the generic rank of $K(s)$ be equal to r. Then for $i = 1,2,...,r$, the minimal number of state vertices in any i-tuple of disjoint paths in $G(V,E)$ from U to Y is equal to $m_i = \sum_{j=1}^{i} t_j$.*

To conclude this section, we return to the structured system in the example of section 2. From the graph in figure 1 and theorem 6.1, it follows that ~~the~~ the generic rank r of the transfer matrix $K(s)$ equals 2. In addition, it follows from theorem 6.3 that the generic orders of the zeroes at infinity satisfy $t_1 = 1$ and $t_2 = 1$. The generic rank and the generic orders of the zeroes at infinity could also have been determined by considering

$$K_\lambda(s) = \frac{1}{s^3-s^2\lambda_4-s\lambda_1\lambda_2+\lambda_1\lambda_2\lambda_4}\begin{bmatrix} (s-\lambda_4)\lambda_1\lambda_5\lambda_8 & s(s-\lambda_4)\lambda_7\lambda_8 \\ (s^2-\lambda_1\lambda_2)\lambda_6\lambda_9+\lambda_1\lambda_3\lambda_5\lambda_9 & s\lambda_3\lambda_7\lambda_9 \end{bmatrix},$$

but it is clear that computing and manipulating with $K_\lambda(s)$ may be more cumbersome than working with the simple graph in figure 1. However, for systems larger than the one in the example of section 2, the graphs may become more complicated and it may not be possible to determine the generic rank and the generic orders of the zeroes at infinity by hand. In such cases, we can use some efficient algorithms from combinatorial optimization. We discuss these algorithms in the next section.

7. COMPUTATIONAL ASPECTS

In the previous section we derived a graph theoretic characterization of the generic rank and the generic orders of the zeroes at infinity of the transfer matrix of a structured system. In the present section, we describe how these generic rank and generic orders can be computed by means of algorithms from combinatorial optimization.

Therefore, in addition to the graph $G(V,E)$, we introduce a second type of graph corresponding to a structured system of type (1.1). This new type of graph, denoted $G(\tilde{V},\tilde{E})$, consists of a vertex set $\tilde{V}$ and an edge set $\tilde{E}$. The set $\tilde{V}$ is given by $\tilde{V} = \{a\} \cup \{u_1,u_2,....,u_m\} \cup \{x_1^I,x_2^I,....,x_n^I\} \cup \{x_1^O,x_2^O,....,x_n^O\} \cup \{y_1,y_2,....,y_p\} \cup \{b\}$, and the edge set $\tilde{E}$ is given by $\{(a,u_i) \mid i = 1,..,m\} \cup \{(u_j,x_i^I) \mid b_{i,j} \neq 0\} \cup \{(x_j^O,x_i^I) \mid a_{i,j} \neq 0\} \cup \{(x_j^O,y_i) \mid c_{i,j} \neq 0\} \cup \{(y_j,b) \mid j = 1,...,p\} \cup \{(x_i^I,x_i^O) \mid i = 1,..,n\}$. Again, for instance, the ordered pair (u_j,x_i^I) represents a directed edge from the vertex u_j to the vertex x_i^I and $b_{i,j} \neq 0$ means that the entry $b_{i,j}$ in the matrix B is a structural unknown entry. We call the vertices a and b in $G(\tilde{V},\tilde{E})$, the source and the sink, respectively. It is easy to see that any i-tuple of disjoint paths in $G(V,E)$ from U to Y is in one-to-one correspondence to an i-tuple of paths in $G(\tilde{V},\tilde{E})$ from a to b, in which each pair of paths, apart from a and b, have no vertices in common (compare figure 1 with figure 4 below).

In the remainder of this section we think of the graph $G(\tilde{V},\tilde{E})$ as a network in which there is a flow from the source a to the sink b. We only allow non-negative flows in the direction of the edges of the network, and we assume that all edges have a (maximal flow) capacity equal to 1. In figure 4 we have depicted the graph $G(\tilde{V},\tilde{E})$ associated to the structured system of the example in section 2. The number above each edge denotes its capacity.

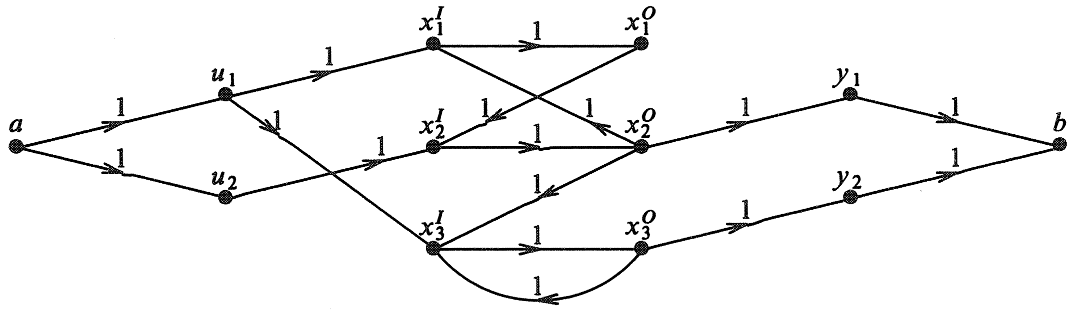

Figure 4.

Using algorithms based on the celebrated max-flow min-cut theorem, we can compute the maximal flow in the network $G(\tilde{V},\tilde{E})$ from a to b (cf. Lawler [10], section 4.3). Moreover, using standard results we can prove the following (cf. Lawler [10]).

THEOREM 7.1. *The maximal flow in $G(\tilde{V},\tilde{E})$ from a to b is equal to the maximal number of disjoint paths in $G(V,E)$ from U to Y.*

Hence, by algorithms based on the max-flow min-cut theorem, we can compute the maximal number of disjoint paths in $G(V,E)$ from U to Y, and, consequently, we can compute the generic rank of the transfer matrix of the underlying structured system. As before, we denote this generic rank by r.

To compute the generic orders of the zeroes at infinity of the transfer matrix of the structured system, we proved in the previous section that we have to compute, for $i = 1,..,r$, the minimal number of state vertices appearing in any i-tuple of disjoint paths in $G(V,E)$ from U to Y. Also these numbers can be computed by means of a well-known algorithm from combinatorial optimization. To do this, we modify the graph $G(\tilde{V},\tilde{E})$ by also attaching costs to flows along the edges. To flows along each of the n edges from x_i^I to x_i^O we attach a cost factor 1, and to flows along all the other edges we attach cost factor 0. The actual costs of a flow along an edge are then given by the product of the cost factor and the strength of the flow, and the costs associated to a flow in the network are given by the sum of the costs of the flows along the edges. More precisely, the above means that, if for $i = 1,2,..,n$, the flow along the edge $(x_i^I,x_i^O) \in \tilde{E}$ has a strength α_i, then the costs associated to the total flow in $G(\tilde{V},\tilde{E})$ from a to b is equal to $\sum_{i=1}^{n}(\alpha_i \times 1) = \sum_{i=1}^{n}\alpha_i$. In figure 5 we have depicted the graph of figure 4 where in addition to the capacity also a cost factor is attached to each edge. The two numbers above each edge denote the capacity and the cost factor, respectively.

Let the maximal flow in $G(\tilde{V},\tilde{E})$ from a to b have strength r. Then it is easy to see that, if the minimal number of state vertices in any r-tuple of disjoint paths in $G(V,E)$ from U to Y is equal to l, the minimal costs associated to a maximal flow in $G(\tilde{V},\tilde{E})$ from a to b is less than or equal to l. And also the converse is true. In fact, using standard results we can prove the following. (cf. Lawler [10]).

THEOREM 7.2. *For $i = 1,2,....,r$, the minimal costs associated to a flow of strength i in $G(\tilde{V},\tilde{E})$ from a to b is equal to the minimal number of state vertices appearing in any i-tuple of disjoint paths in $G(V,E)$ from U to Y.*

Hence, based on theorem 7.2, we can apply a well-known algorithm that computes the minimal costs of successive flows in $G(\tilde{V},\tilde{E})$ from a to b, starting with a flow of strength zero upto the maximal flow r

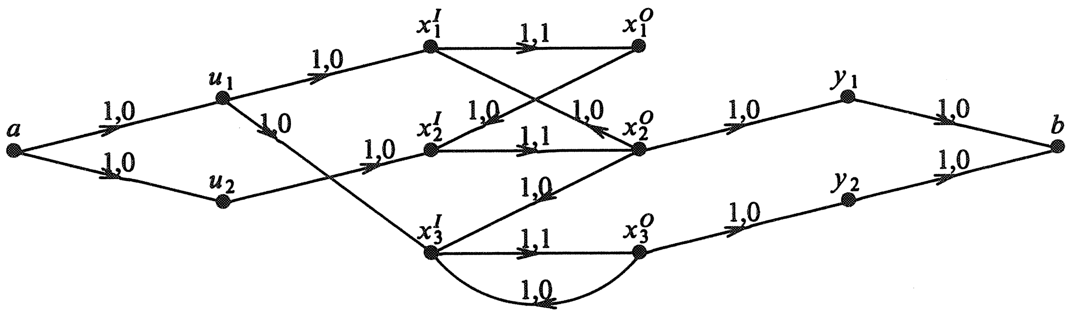

Figure 5.

(cf. Lawler [10], section 4.7). The obtained sequence of minimal costs can then be used to compute the generic orders of the zeroes at infinity of $K(s)$, using theorem 6.3 and (4.7).

8. APPLICATION

In the present section we propose structural versions of the well-known disturbance decoupling problem and the so-called modified disturbance decoupling problem, and we apply our main results to obtain graph theoretic conditions for the solvability of each of the two problems. To formulate the problems, we consider the following extension of system (1.1)

$$\dot{x}(t)=Ax(t)+Bu(t)+Gd(t), \tag{8.1.a}$$

$$y(t)=Cx(t). \tag{8.1.b}$$

Here $x(t),u(t),y(t),A,B$ and C are as in the description of system (1.1), $d(t)\in\mathbb{R}^l$ denotes the disturbance input and G is an $n\times l$ matrix. Like A,B and C, we assume that G is a structured matrix. We denote the total number of structural unknowns in A,B,C and G by k'. Parametrizing the structural unknowns and collecting all parameters in the vector $\lambda'\in\mathbb{R}^{k'}$, we denote $A_{\lambda'}$, $B_{\lambda'}$, $C_{\lambda'}$ and $G_{\lambda'}$ for the nominal values of A,B,C and G for a given $\lambda'\in\mathbb{R}^{k'}$. Note that the compound matrix $[B\ G]$ can considered to be the input matrix for system (8.1) in the same way as the matrix B is the input matrix for system (1.1). Like to system (1.1), we can associate graphs $G(V',E')$ and $G_{\lambda'}(V',E')$ to system (8.1). The graph $G(V',E')$ consists of a vertex set $V'=V\cup D$ and an edge set $E'=E\cup\{(d_j,x_i)\,|\,g_{i,j}\neq 0\}$. Here $D=\{d_1,d_2,\ldots,d_l\}$, called the set of disturbance vertices, and V and E are the vertex set and the edge set, respectively, of the graph $G(V,E)$ associated to system (1.1). The graph $G_{\lambda'}(V',E')$ is obtained from $G(V',E')$ by weighting each edge in E' with the appropriate component of $\lambda'\in\mathbb{R}^{k'}$.

Following Emre and Hautus [4], we say that for a given $\lambda'\in\mathbb{R}^{k'}$ the *modified disturbance decoupling problem* for system (8.1) is solvable if there is a real $m\times n$ matrix F and a real $m\times l$ matrix H, representing a feedback law $u(t)=Fx(t)+Hd(t)$, such that $C_{\lambda'}(sI-(A_{\lambda'}+B_{\lambda'}F))^{-1}(G_{\lambda'}+B_{\lambda'}H)=0$ (see also Wonham [20], exercise 4.10). Using the results of Emre and Hautus [4], it can be shown that for a given $\lambda'\in\mathbb{R}^{k'}$ the modified disturbance decoupling problem for the system (8.1) is solvable if and only if there exists a *proper* rational $m\times l$ matrix $X(s)$ such that $K_{\lambda'}(s)X(s)=L_{\lambda'}(s)$. Here we have denoted $K_{\lambda'}(s)=C_{\lambda'}(sI-A_{\lambda'})^{-1}B_{\lambda'}$ and $L_{\lambda'}(s)=C_{\lambda'}(sI-A_{\lambda'})^{-1}G_{\lambda'}$.

From theorem 3.3 it is now immediate that for a given $\lambda'\in\mathbb{R}^{k'}$ the modified disturbance decoupling problem for system (8.1) is solvable if and only if the rank and the orders of the zeroes at infinity of $K_{\lambda'}(s)$ and $[K_{\lambda'}(s)\ L_{\lambda'}(s)]$ are equal.

In the spirit of the present paper, we say that the modified disturbance decoupling problem for the structured system (8.1) is *generically solvable* if the set of parameter values $\lambda'\in\mathbb{R}^{k'}$ for which the rank and/or the orders of the zeroes at infinity of $K_{\lambda'}(s)$ and of $[K_{\lambda'}(s)\ L_{\lambda'}(s)]$ are *not* equal, is contained in a proper variety in $\mathbb{R}^{k'}$. The following theorem is now an immediate consequence of theorems 6.1 and 6.3.

THEOREM 8.1. *The modified disturbance decoupling problem for the structured system (8.1) is generically solvable if and only if*
a) the maximal number of disjoint paths in $G(V,E)$ from U to Y is equal to the maximal number of disjoint paths in $G(V',E')$ from $U \cup D$ to Y, say r, and
b) for $i = 1,2,...,r$, the minimal number of state vertices (vertices in the set X) in any i-tuple of disjoint paths in $G(V,E)$ from U to Y is equal to the minimal number of state vertices in any i-tuple of disjoint paths in $G(V',E')$ from $U \cup D$ to Y.

We continue with the disturbance decoupling problem. Following Wonham [20], we say that for a given $\lambda' \in \mathbf{R}^{k'}$ the *disturbance decoupling problem* for system (8.1) is solvable if there is a real $m \times n$ matrix F, representing a feedback law $u(t) = Fx(t)$, such that $C_{\lambda'}(sI-(A_{\lambda'}+B_{\lambda'}F))^{-1}G_{\lambda'} = 0$.

By the results of Hautus [8], it follows that the disturbance decoupling problem for the system (8.1) for a given $\lambda' \in \mathbf{R}^{k'}$ is solvable if and only if there exists a *strictly proper* rational $m \times l$ matrix $X(s)$ such that $K_{\lambda'}(s)X(s) = L_{\lambda'}(s)$. Clearly, the latter is equivalent to the existence of a *proper* rational matrix $X'(s)$ such that $K_{\lambda'}(s)\Delta s^{-1}X'(s) = L_{\lambda'}(s)$, where Δ is an arbitrary constant nonsingular diagonal matrix. Therefore, to derive conditions for the solvability of a structural version of the disturbance decoupling problem, it turns out to be useful to extend the structured system (8.1) as follows

$$\dot{u}(t) = Nw(t), \tag{8.2}$$

where N denotes a square structured matrix with only structural unknowns on the diagonal. The compound system made up of (8.1) and (8.2) is again a structured system and is described by

$$\begin{bmatrix}\dot{x}(t)\\ \dot{u}(t)\end{bmatrix} = \begin{bmatrix}A & B\\ 0 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}0\\ N\end{bmatrix}w(t) + \begin{bmatrix}G\\ 0\end{bmatrix}d(t) \tag{8.3.a}$$

$$y(t) = Cx(t). \tag{8.3.b}$$

Note that the control input $u(t)$ of system (8.1) is part of the state of system (8.3), and that $w(t)$ is the control input of (8.3). We denote the total number of structural unknowns in A,B,C,G and N by k'', i.e. $k'' = k' + m$. Parametrizing the structural unknowns and collecting all the parameters in $\lambda'' \in \mathbf{R}^{k''}$, we denote $A_{\lambda''}$, $B_{\lambda''}$, $C_{\lambda''}$, $G_{\lambda''}$ and $N_{\lambda''}$ for the nominal values of A,B,C,G and N for a given $\lambda'' \in \mathbf{R}^{k''}$. Furthermore, like before, we denote $K_{\lambda''}(s) = C_{\lambda''}(sI-A_{\lambda''})^{-1}B_{\lambda''}$ and $L_{\lambda''}(s) = C_{\lambda''}(sI-A_{\lambda''})^{-1}G_{\lambda''}$.

It is now easy to see that there is a *strictly proper* rational matrix $X(s)$ such that $K_{\lambda''}(s)X(s) = L_{\lambda''}(s)$ if and only if there is a *proper* rational matrix $X'(s)$ such that $K_{\lambda''}(s)N_{\lambda''}s^{-1}X'(s) = L_{\lambda''}(s)$, for all $\lambda'' \in \mathbf{R}^{k''}$ for which $N_{\lambda''}$ is nonsingular. Furthermore, it is easy to see that $K_{\lambda''}(s)N_{\lambda''}s^{-1}$ and $L_{\lambda''}(s)$ are the transfer martrices of the system (8.3) from the control input to the output, and the disturbance input to the output, respectively, at $\lambda'' \in \mathbf{R}^{k''}$ Moreover, note that system (8.3) is of the same type as system (8.1) for which we have formulated a structural version of the modified disturbance decoupling problem.

Based on the above, we say that the disturbance decoupling problem for the structured system (8.1) is *generically solvable* if the modified disturbance decoupling problem for the structured system (8.3) is *generically solvable*.

We can now apply theorem 8.1 to system (8.3) to obtain a graph theoretic characterization for the generic solvability of the disturbance decoupling problem for the structured system (8.1) in terms of the graph of system (8.3). To do this, we have to modify the graphs $G(V',E')$ and $G_\lambda(V',E')$ to make them correspond to the structured system (8.3). For instance, we have to add the set W of 'new' input vertices. However, it is easy to see that the characterization obtained in this way is equivalent to the following characterization which is entirely in terms of the graphs $G(V,E)$ and $G(V',E')$.

THEOREM 8.2. *The disturbance decoupling problem for the structured system (8.1) is generically solvable if and only if*
a) the maximal number of disjoint paths in $G(V,E)$ *from* U *to* Y *is equal to the maximal number of disjoint paths in* $G(V',E')$ *from* $U \cup D$ *to* Y*, say* r*, and*
b) for $i = 1,2,...,r$*, the minimal number of vertices in* $X \cup U$ *in any i-tuple of disjoint paths in* $G(V,E)$ *from* U *to* Y *is equal to the minimal number of vertices in* $X \cup U$ *in any i-tuple of disjoint paths in* $G(V',E')$ *from* $U \cup D$ *to* Y.

Note that for the generic solvability of the modified disturbance decoupling problem the number of only *state* vertices in a tuple of disjoint paths is a relevant number. This in contrast with the generic solvability of the disturbance decoupling problem in which the number of both *state* and *input* vertices in a tuple of disjoint paths is significant.

9. REMARKS AND CONCLUSIONS

In this paper we studied a general type of structured systems. We introduced structured systems in section 2 as systems of which only the zero-nonzero structure is given, where we assumed the nonzeroes, called structural unknowns, to be entries of the system matrices of which the values are unknown and independent of each other. Also in section 2 we described how structured systems can be parametrized by a parameter $\lambda \in \mathbf{R}^k$, where k denoted the number of structural unknowns.

In section 4 we used the parametrization to introduce the *generic rank* and *generic orders of the zeroes at infinity* of the transfer matrix of a structured system. For the introduction of these notions, we assumed that the parameter space was $\mathbf{R}^k$. However, it is easy to see that we could have restricted ourselves to parameter spaces that are *open non empty* subsets in $\mathbf{R}^k$. Such parameter spaces are sometimes more realistic, because in practical situations, there may be components of the parameter vector λ that only can have values in a (open) subset of $\mathbf{R}$. Then, using the techniques of section 4, we can show that if the overall parameter space is an open non empty subset in $\mathbf{R}^k$, the main results of this paper are still valid.

The main results of this paper, presented in section 6, relate the generic rank and the generic orders of the zeroes at infinity of the transfer matrix of a structured system (1.1) to properties of the corresponding graph $G(V,E)$. This graph was introduced in section 5. We showed that the generic rank of the transfer matrix of structured system can be determined by calculating the maximal number of disjoint paths in $G(V,E)$ from the set of input vertices U to the set of output vertices Y. The generic orders of the zeroes at infinity of the transfer matrix can be determined by calculating the minimal number of state vertices in any i-tuple of disjoint paths in $G(V,E)$ from U to Y, for $i = 1,2,...,r$, where r is the generic rank of the transfer matrix of the structured system. For simple systems these numbers can be determined by hand, for complicated systems we indicated in section 7 that these numbers can be determined by means of *max-flow min-cut* and *minimal cost flow* algorithms (cf. Lawler [10]).

As an application of our results we proposed structural versions of the well-known disturbance decoupling problem and the so-called modified disturbance decoupling problem for a structured system of type (8.1) (cf. Wonham [20], Emre and Hautus [4]), and we derived necessary and sufficient conditions for the structural solvability of the problems in terms of the graphs $G(V,E)$ and $G(V',E')$. Results concerning the solvability of a structural version of dual problems like, for instance, the *disturbance decoupled estimation problem* (cf. Schumacher [16]), can be obtained in a similar way.

In Van der Woude [21], we derived conditions for the solvability of a structural version of the *almost disturbance decoupling problem* (cf. Willems [18]). We also indicated there that with the obtained results, conditions for the solvability of structural versions for the *almost disturbance decoupled estimation problem* and *the almost disturbance decoupling problem by measurement feedback* (cf. Willems [19]) can be derived in a straightforward way.

Conditions concerning the solvability of structural versions of the *disturbance decoupling problem by measurement feedback* (cf. Akashi and Imai [1], Schumacher [16]), and the *disturbance decoupling problem with pole assignment* (cf. Wonham [20]) are topics of future investigation. Clearly, motivated by the unstructured case, in the latter problem the notion of structural controllability will play an important role (cf. Glover and Silverman [6], Lin [12], Murota [14] and Shields and Pearson [17]). Also a topic of future investigation is how the results of the present paper can be extended to descriptor or singular systems (cf. Lewis [11]). Another important matter for future investigation is, once the generic solvability of a control problem has been established, is it possible to actually solve the control problem by a structured control law and, if so, how can it be determined (cf. Linneman [13])?

References

1. H. Akashi and H. Imai (1979). Disturbance localization and output deadbeat control through an observer in discrete time linear multivariable systems, *IEEE Trans. Automat. Contr.*, AC-24, 621-627.
2. W.K. Chen (1971). *Applied graph theory*, North Holland, Amsterdam.
3. J. Descusse and J.M. Dion (1982). On the structure at infinity of linear square decoupled systems, *IEEE Trans. Automat. Contr.*, AC-27, 971-974.
4. E. Emre and M.L.J. Hautus (1980). A polynomial characterization of (A,B)-invariant and reachability subspaces, *SIAM J. Control and Optimization*, 18, 420-436.
5. F.R. Gantmacher (1977). *Matrix Theory*, Chelsea, New York.
6. K. Glover and L.M. Silverman (1976). Characterization of structural controllability, *IEEE Trans. Automat. Contr.*, AC-21, 534-357.
7. M.L.J. Hautus (1976). The formal Laplace transform for smooth linear systems, in *Lecture Notes in Economics and Mathematical Systems Theory*, 29-47, Springer Verlag, New York.
8. M.L.J. Hautus (1980). (A,B)-invariant and stabilizability subspaces, a frequency domain description, *Automatica*, 16, 703-707.
9. M.L.J. Hautus and M. Heymann (1978). Linear Feedback - an algebraic approach, *SIAM J. Contr. Optimiz.*, 16, 83-105.
10. E.L. Lawler (1976). *Combinatorial Optimization : Networks and Matroids*, Holt, Rinehart and Winston, New York.
11. F.L. Lewis (1986). A survey of linear singular systems, *Circuits, systems and signal processing*, 5, 3-36.
12. C.T. Lin (1974). Structural controllability, *IEEE Trans. Automat. Contr.*, AC-19, 201-208.
13. A. Linneman (1983). *Entwurf von dynamischen Reglern vorgegebenen Musters,* Report No. 109, University of Bremen.
14. K. Murota (1987). *Systems Analysis by Graphs and Matroids*, Springer Verlag, New York.
15. M. Newman (1972). *Integral Matrices*, Academic Press, New York.
16. J.M. Schumacher (1980). Compensator synthesis using (C,A,B)-pairs, *IEEE Trans. Automat. Contr.*, AC-25, 1133-1138.
17. R.W. Shields and J.B. Pearson (1976). Structural controllability of multi-input linear systems, *IEEE Trans. Automat. Contr.*, AC-21, 203-212.
18. J.C. Willems (1981). Almost invariant subspaces : An approach to high gain feedback design - part I : Almost controlled invariant subspaces, *IEEE Trans. Automat. Contr.*, AC-26, 235-252.
19. J.C. Willems (1982). Almost invariant subspaces : An approach to high gain feedback design - part II : Almost conditionally invariant subspaces, *IEEE Trans. Automat. Contr.*, AC-27, 1071-1085.
20. W.M. Wonham (1985). *Linear Multivariable Control : a geometric approach (3-rd ed.)*, Springer Verlag, New York.
21. J.W. van der Woude (1988). A graph theoretic characterization for the rank of the transfer matrix of a structured system, *Report OS-R8819*, Centre for Mathematics and Computer Science, to appear in *Mathematics of Control, Signals and Systems.*