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Sequentiality in Orthogonal Term Rewriting Systems*

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ABSTRACT

For orthogonal term rewriting systems, G. Huet and J.-J. Lévy have introduced the property of 'strong sequentiality'. A strongly sequential orthogonal term rewriting system admits an efficiently computable normalizing one-step reduction strategy. As shown by Huet and Lévy, strong sequentiality is a decidable property. In this paper we present an alternative analysis of strongly sequential term rewriting systems, leading to two simplified proofs of the decidability of this property. We also compare some related notions of sequentiality that recently have been proposed.

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Key Words and Phrases: orthogonal term rewriting systems, normalizing reduction strategy, sequentiality, needed redex.

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Introduction

The analysis of term rewriting systems is of growing interest for a large number of applications having to do with computing with equations. Two main streams can be distinguished in the study of term rewriting systems: (1) theory and applications of Knuth-Bendix completion procedures — here the point of departure is a given set of equations for which one tries to generate a complete (i.e. confluent and terminating) term rewriting system, and (2) theory and applications of orthogonal term rewriting systems; here the term rewriting system is fixed but subject to the restrictions of being 'left-linear' and 'non-ambiguous', for short 'orthogonal'. (Previously, we used 'regular' instead of 'orthogonal'.) The restriction of orthogonality enables one to develop a quite sizeable amount of theory, for a large part due to the efforts of the 'French school' (Berry, Boudol, Huet, Lévy e.a.; see [3], [4], [8]).

The present paper is exclusively concerned with orthogonal term rewriting systems. In an admirable paper ([8]), Huet and Lévy investigated the issue of parallel versus sequential reduction in an orthogonal term rewriting system. More specifically, they formulated a criterion 'strong sequentiality', guaranteeing the existence of an effective sequential normalizing reduction strategy, that is a strategy Φ such that its iteration on a given term t leads to a reduction sequence

$$t \to \Phi(t) \to \Phi^2(t) \to \dots$$

which ends in the (unique) normal form of t if it exists and is infinite otherwise. The sequentiality is in the fact that the strategy indicates in each step just one redex to be rewritten, rather than a set of redexes to be rewritten in parallel. Actually, Huet and Lévy prove that every orthogonal term rewriting system possesses a sequential normalizing 'call-by-need' strategy: a deep theorem in [8] says that every term t in an orthogonal term rewriting system contains a 'needed' redex, that is one which has to be rewritten in any reduction to normal form. A call-by-need strategy is then obtained by rewriting in each step such a needed redex, and it is proved in [8] that such a strategy is normalizing. Unfortunately, it is undecidable in general whether a redex is needed or not. However, Huet and Lévy go on to show that in 'strongly sequential' term rewriting systems, a needed redex can be found effectively. This does not mean that in a strongly sequential term rewriting system *all* needed redexes can be determined effectively. For instance Combinatory Logic

$$\mathrm{CL} = \begin{cases} Ap(Ap(Ap(S,x),y),z) & \to \; Ap(Ap(x,z),Ap(y,z)), \\ Ap(Ap(K,x),y) & \to \; x, \\ Ap(I,x) & \to \; x, \end{cases}$$

is a strongly sequential term rewriting system where this is impossible; cf. the analogous statement for λ-calculus in [1]. In fact, a needed redex is very easy to determine in the case of CL: the leftmost redex is always needed. By contrast, consider CL$\oplus$B, that is CL extended with B ('Berry's term rewriting system', also called 'Gustave's term rewriting system' in [7]):

$$\mathrm{B} = \begin{cases} F(A,B,x) & \to \; C, \\ F(B,x,A) & \to \; C, \\ F(x,A,B) & \to \; C. \end{cases}$$

In the term rewriting system CL$\oplus$B it is not clear at all how to find a needed redex: in a term $F(t_1,t_2,t_3)$ the redexes in t_1 may be non-needed because t_2,t_3 reduce to the constants A,B respectively, and likewise for redexes in t_2 and t_3. Actually, we do not know whether there is an algorithm to determine a needed redex in a term of CL$\oplus$B (cf. the surprising fact in [10] where it is shown that every orthogonal term rewriting system, including CL$\oplus$B, has a computable normalizing one-step reduction strategy), but it seems safe to conjecture that if such an algorithm exists, it will not be very 'feasible'.

However, in strongly sequential term rewriting systems a needed redex can be found really effectively, as shown in [8]. Moreover, it is decidable whether a term rewriting system is strongly sequential. This brings us to the point dealt with in this paper: in [8] a proof of the decidability of strong sequentiality is given with great ingenuity; but it is also very complicated, and in the present paper our endeavour is to analyze the notion of a strongly sequential term rewriting system in order to arrive at a simplified proof of the decidability. We present two proofs of which the first is the most direct; but the corresponding decision procedure itself is only of mathematical relevance as its computational complexity forbids a practical application. We feel however that this proof is conceptually simple and gives a good insight in the structure of a strongly sequential term rewriting system. Some of the underlying notions in [8] are eliminated here; notably: the 'matching dag', 'directions', 'increasing indices' and 'Δ-sets' (or: 'properties Q_1, Q_2'). Also our proof is direct in the sense that it does not take the form of a correctness proof of some algorithm. The second proof is of comparable computational complexity as the one in [8]; conceptually it is harder than the first, though still simpler than the one in [8]. This proof is essentially already in [8] and uses their notions of increasing indices and Δ-sets (the latter with a slight simplification by us). In both proofs our concepts of an 'preredex' and of a 'tower of preredexes' play a crucial role. We construct a term rewriting system which is 'inherently difficult' with respect to deciding strong sequentiality, and we make the simple but useful observation that strong sequentiality is a 'modular' property, i.e. depends on the 'disjoint pieces' of a term rewriting system. In the last section we give an overview of other notions of sequentiality proposed in the literature.

Especially in the first part of our paper we follow [8] quite closely; also some proofs there are repeated for the sake of completeness. Although our paper is self-contained, familiarity with term rewriting systems might be helpful (e.g. [5], [9], [11]).

Contents

1. Orthogonal Term Rewriting Systems: Preliminaries

We start with a number of definitions. Let $\mathcal{V}$ be a countably infinite set of variables. A *term rewriting system* is a pair $(\mathcal{F},\mathcal{R})$. $\mathcal{F}$ is a set of *function symbols*; associated to every $f\in\mathcal{F}$ is its arity $n\geq 0$. Function symbols of arity 0 are called *constants*. The set of terms built from $\mathcal{F}$ and $\mathcal{V}$, notation $\mathcal{T}(\mathcal{F},\mathcal{V})$, is the smallest set such that

- $\mathcal{V}\subset\mathcal{T}(\mathcal{F},\mathcal{V})$,
- if $F\in\mathcal{F}$ has arity n and $t_1,\ldots,t_n\in\mathcal{T}(\mathcal{F},\mathcal{V})$ then $F(t_1,\ldots,t_n)\in\mathcal{T}(\mathcal{F},\mathcal{V})$.

A *ground* term is a term without variables. $\mathcal{R}$ is a finite set of pairs (l,r) with $l,r\in\mathcal{T}(\mathcal{F},\mathcal{V})$ subject to two constraints:

(1) the left-hand side l is not a variable,

(2) the variables which occur in the right-hand side r also occur in l.

Pairs (l,r) are called *rewriting rules* or *reduction rules* and will henceforth be written as $l\rightarrow r$. We usually write $\mathcal{R}$ instead of $(\mathcal{F},\mathcal{R})$, assuming that $\mathcal{F}$ contains no function symbols which do not occur in the rewriting rules $\mathcal{R}$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$. Substitutions are extended to $\mathcal{T}(\mathcal{F},\mathcal{V})$ in the obvious way; we denote by t^σ the term obtained from t by applying the substitution σ. We call t^σ an *instance* of t. An instance of a left-hand side of a rewriting rule is called a *redex* (reducible expression).

A *context* $C[\,]$ is a 'term' which contains exactly one occurrence of a special symbol $\square$. Contexts are inductively defined by:

- $\square$ is a context,
- if $C[\,]$ is a context then $F(t_1,\ldots,t_{i-1},C[\,],t_{i+1},\ldots,t_n)$ is a context for every n-ary function symbol F and terms $t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n\in\mathcal{T}(\mathcal{F},\mathcal{V})$.

If $C[\,]$ is a context and $t\in\mathcal{T}(\mathcal{F},\mathcal{V})$ then $C[t]$ is the result of replacing the symbol $\square$ by t; t is said to be a *subterm* of $C[t]$.

The *rewriting relation* $\rightarrow_{\mathcal{R}}\subset\mathcal{T}(\mathcal{F},\mathcal{V})\times\mathcal{T}(\mathcal{F},\mathcal{V})$ is defined by $s\rightarrow_{\mathcal{R}} t$ iff there exists a rewriting rule $l\rightarrow r$, a substitution σ and a context $C[\,]$ such that $s\equiv C[l^\sigma]$ and $t\equiv C[r^\sigma]$ (the symbol $\equiv$ stands for syntactic equality). The transitive-reflexive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$; if $s\twoheadrightarrow_{\mathcal{R}} t$ we say that s *reduces* to t and we call t a *reduct* of s. We often omit the subscript $\mathcal{R}$.

EXAMPLE 1.1. Let

$$\mathcal{R}=\begin{cases} A(x,0) & \rightarrow\; x,\\ A(x,S(y)) & \rightarrow\; S(A(x,y)).\end{cases}$$

Consider the term $A(A(0,0),A(S(0),0))$. To this term we can apply the following reduction sequence (at each step the rewritten redex is underlined):

$$A(\underline{A(0,0)},A(S(0),0))\rightarrow A(0,\underline{A(S(0),0)})\rightarrow \underline{A(0,S(0))}\rightarrow S(\underline{A(0,0)})\rightarrow S(S(0)).$$

The term $S(S(0))$ is a *normal form*, i.e. a term which contains no redexes.

We denote the set of normal forms of $\mathcal{R}$ by $\mathrm{NF}_{\mathcal{R}}$ (NF for short). A precise formalism is obtained through the notion of *occurrences*. For any term $t\in\mathcal{T}(\mathcal{F},\mathcal{V})$, the set $O(t)$ of its occurrences is inductively defined as follows

- $\lambda\in O(t)$ (the empty occurrence),
- if $t\equiv F(t_1,\ldots,t_n)$ and $u\in O(t_i)$ then $i.u\in O(F(t_1,\ldots,t_n))$.

If we view terms as trees, an occurrence of t denotes a unique node in the tree of t. If $u \in O(t)$, the *subterm of t at u*, notation t/u, is defined by

- $t/\lambda = t$,
- $F(t_1,\ldots,t_n)/i.u = t_i/u$.

The *symbol of t at occurrence u*, notation $t(u)$, is defined for $u \in O(t)$ by

- $x(\lambda) = x$,
- $F(t_1,\ldots,t_n)(\lambda) = F$,
- $F(t_1,\ldots,t_n)(i.u) = t_i(u)$.

Finally, if $u \in O(t)$, for every term s the *replacement in t of the subterm at u by s*, notation $t[u \leftarrow s]$, is defined by

- $t[\lambda \leftarrow s] = s$,
- $F(t_1,\ldots,t_n)[i.u \leftarrow s] = F(t_1,\ldots,t_i[u \leftarrow s],\ldots,t_n)$.

EXAMPLE 1.2. Consider again the term rewriting system of Example 1.1. The occurrences of $t = S(A(S(0),0))$ are exhibited in the following figure.

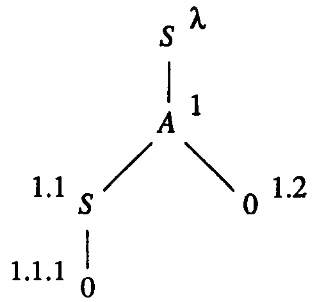


FIGURE 1.1.

We have $t/1 \equiv A(S(0),0)$, $t(1.1.1) \equiv 0$ and $t[1.1 \leftarrow t/1.2] = S(A(0,0))$.

The set of occurrences $O(t)$ of a term t is partially ordered by the *prefix ordering* $\leq$, i.e. $u \leq v$ iff there exists a w such that $uw = v$ (if such a w exists, it is unique). In this case we define $v/u = w$. We say that two occurrences u and v are *disjoint*, notation $u \perp v$, if neither $u \leq v$ nor $v \leq u$. If $u \leq v$ and $u \neq v$, we write $u < v$. If $u_1,\ldots,u_n \in O(t)$ are pairwise disjoint, we write $t[u_i \leftarrow s_i \mid 1 \leq i \leq n]$ as an alternative for $t[u_1 \leftarrow s_1]\ldots[u_n \leftarrow s_n]$ (the order of the u_i's is irrelevant). Sometimes we write $t[s \leftarrow s']$ instead of $t[u \leftarrow s' \mid t/u \equiv s]$.

The *depth* $|u|$ of an occurrence u is defined by

$$|\lambda| = 0,$$
$$|i.u| = 1 + |u|.$$

In this paper we restrict ourselves to the subclass of *orthogonal* term rewriting systems. A term rewriting system is orthogonal if it satisfies the following two constraints:

(1) *left-linearity*: the left-hand side l of a rewriting rule $l \rightarrow r$ does not contain more than one occurrence of the same variable ($\forall u,v \in O(l)$ if $l/u \equiv x \equiv l/v$ then $u = v$).

(2) *non-ambiguity*: the left-hand sides of the rewriting rules do not overlap. This means that whenever $l_1 \rightarrow r_1$, $l_2 \rightarrow r_2$ are rewriting rules and $u \in O(l_1)$ such that $l_1/u \notin \mathcal{V}$, there are no substitutions σ, τ such that $(l_1/u)^\sigma \equiv l_2^\tau$, except in the case where $l_1 \rightarrow r_1$, $l_2 \rightarrow r_2$ are the same rewriting

rule and $u = \lambda$.

EXAMPLE 1.3. The system

$$\mathcal{R} = \begin{cases} IF(T,x,y) & \to & x, \\ IF(F,x,y) & \to & y, \\ IF(x,y,y) & \to & y, \end{cases}$$

is neither left-linear (the left-hand side of the rule $IF(x,y,y) \to y$ contains two occurrences of the variable y) nor non-ambiguous (take $l_1 \equiv IF(T,x,y)$, $l_2 \equiv IF(x,y,y)$ and $u = \lambda$ in the above definition). The system of Example 1.1 is orthogonal.

Orthogonal term rewriting systems have some very nice properties. Among these is the important Church-Rosser property. A term rewriting system is *confluent* or has the *Church-Rosser property* if for all terms s, t_1, t_2 with $s \twoheadrightarrow t_1$ and $s \twoheadrightarrow t_2$ we can find a term t_3 such that $t_1 \twoheadrightarrow t_3$ and $t_2 \twoheadrightarrow t_3$ (see Figure 1.2). Such a term t_3 is called a *common reduct* of t_1 and t_2.

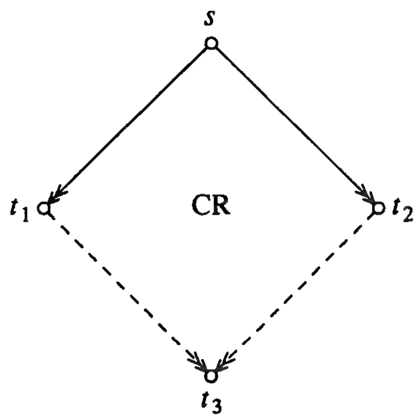


FIGURE 1.2.

THEOREM 1.4 (Rosen [17]). *Every orthogonal term rewriting system has the Church-Rosser property.* □

An immediate consequence of Theorem 1.4 is the fact that if a term s has a normal form t (i.e. $s \twoheadrightarrow t$ with $t \in \mathrm{NF}$), then t is the unique normal form of s (i.e. if $s \twoheadrightarrow t'$ and $t' \in \mathrm{NF}$ then $t \equiv t'$). In the next section we will see some more important properties of orthogonal term rewriting systems.

2. Strongly Sequential Term Rewriting Systems

There are orthogonal term rewriting systems in which some terms have a normal form, but also admit an infinite reduction sequence.

EXAMPLE 2.1 Let

$$\mathcal{R} = \begin{cases} F(x,B) & \to & B, \\ A & \to & B, \\ C & \to & C. \end{cases}$$

The term $F(C,A)$ has a normal form

$$F(C,\underline{A}) \to \underline{F(C,B)} \to B,$$

but always choosing the leftmost redex fails

$$F(\underline{C},A) \to F(\underline{C},A) \to F(\underline{C},A) \to \ldots .$$

Therefore, it is important to have a "good" *reduction strategy*. Informally, a reduction strategy tells us, when presented a term, which redex(es) to rewrite. To be more precise, a *one-step reduction strategy* is a function $\Phi: \mathcal{T}(\mathcal{F},\mathcal{V}) \to \mathcal{T}(\mathcal{F},\mathcal{V})$ such that

- $\Phi(t) \equiv t$ if $t \in \mathrm{NF}$,
- $t \to \Phi(t)$ otherwise.

A *many-step reduction strategy* is a function $\Phi: \mathcal{T}(\mathcal{F},\mathcal{V}) \to \mathcal{T}(\mathcal{F},\mathcal{V})$ such that

- $\Phi(t) \equiv t$ if $t \in \mathrm{NF}$,
- $t \to^{+} \Phi(t)$ otherwise ($\to^{+}$ is the transitive closure of $\to$).

A reduction strategy Φ is *normalizing* if for each term t having a normal form, the sequence

$$t, \Phi(t), \Phi(\Phi(t)), \ldots, \Phi^{n}(t), \ldots$$

contains a normal form. We are only interested in effective normalizing strategies. (A reduction strategy Φ is *effective* if $\Phi(t)$ can be computed from t.)

An important normalizing many-step reduction strategy for orthogonal term rewriting systems is the *parallel-outermost* strategy: rewrite simultaneously all maximal (outermost) redexes. (In a term t a redex at occurrence u is *maximal* if for every v with $\lambda \le v < u$, t/v is not a redex.) For a proof that the parallel-outermost strategy is normalizing for orthogonal term rewriting systems, see [15] or the appendix of [2]. Alternatively, this fact can be obtained as a corollary of Theorem 2.3 below. The following example shows that the parallel-outermost strategy not always gives the shortest reduction sequence to normal form.

EXAMPLE 2.2. Let

$$\mathcal{R} = \begin{cases} IF(T,x,y) & \to & x, \\ IF(F,x,y) & \to & y, \\ A & \to & B. \end{cases}$$

Consider the term $IF(IF(T,F,T),A,A)$. The parallel-outermost strategy rewrites a total of 4 redexes

$$IF(\underline{IF(T,F,T)},\underline{A},\underline{A}) \twoheadrightarrow \underline{IF(F,B,B)} \to B,$$

but the following sequence uses only 3 redexes

$$IF(\underline{IF(T,F,T)},A,A) \to \underline{IF(F,A,A)} \to \underline{A} \to B.$$

In the example above it is not necessary to rewrite the redex at occurrence 2 in the term $IF(IF(T,F,T),A,A)$ in order to get a normal form. Before we make this more precise, we introduce the notion of "descendants" in reductions.

Consider the reduction rule $F(x,y) \to G(F(x,x))$. When instantiated to, say, $F(t_1,t_2) \to G(F(t_1,t_1))$ it is clear that t_1 in this step is doubled and that t_2 has been erased. Obviously we have an intuition of the subterms in t_1 as propagating to the right. We say that a subterm s of t_1 has (two) *descendants* in $G(F(t_1,t_1))$. In general, suppose we have a reduction $t_1 \twoheadrightarrow t_2$ and a subterm s of t_1. The descendants of s in t_2 are computed as follows:

> Underline the head (leftmost) symbol of s and perform the reduction $t_1 \twoheadrightarrow t_2$. Then the descendants of s in t_2 are the subterms of t_2 which have an underlined head symbol.

A formal definition can be found in [8]. Orthogonal term rewriting systems have the property that descendants of redexes remain redexes.

A redex s in a term t is called *needed* if in every reduction of t to normal form a descendant of s is rewritten. A needed redex must eventually be rewritten in order to get a normal form. In Example 2.2 the underlined redex in $IF(IF(T,F,T),\underline{A},A)$ is not needed. Huet and Lévy proved the following very important theorem.

THEOREM 2.3. *Let $\mathcal{R}$ be an orthogonal term rewriting system. For all $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$ we have*

(1) *if t contains redexes then t contains a needed redex;*

(2) *if t has a normal form, repeated rewriting of needed redexes leads to that normal form.*

□

This theorem gives us a normalizing one-step reduction strategy: just contract some needed redex. However, the definition of 'needed' refers to all reductions to normal form, so in order to determine what the needed redexes are, we have to inspect the normalizing reductions first, which is not a very good recipe for a reduction strategy. In other words, the determination of needed redexes involves *look-ahead*, and it is this necessity for look-ahead that we wish to eliminate.

Every term t not in normal form can be written as $t \equiv C[r_1,\ldots,r_n]$ where $C[\,,\ldots,\,]$ denotes a context in normal form with $n \geq 1$ holes (occurrences of □) and $r_1,\ldots,r_n$ are the outermost redexes of t. It is not difficult to see, using Theorem 2.3 and the orthogonality of the term rewriting system under consideration, that one of the r_i is needed. The actual i such that r_i is needed depends on the 'substitution' of the redexes $r_1,\ldots,r_n$ for the □'s in $C[\,,\ldots,\,]$. A more pleasant state of affairs is expressed in the following definition.

DEFINITION 2.4. An orthogonal term rewriting system is *sequential** if for every context $C[\,,\ldots,\,]$ in normal form there exists an i such that for every substitution of redexes $r_1,\ldots,r_n$ for the □'s in $C[\,,\ldots,\,]$, r_i is needed in $C[r_1,\ldots,r_n]$.

This concept is only introduced for expository purposes. It is not a satisfactory property as it is undecidable. The property defined in the next definition turns out to be more tractable.

DEFINITION 2.5. Let $\mathcal{R}$ be an orthogonal term rewriting system.

(1) The reduction relation $\to_?$ (*arbitrary reduction*) is defined as follows:

$$C[s] \rightarrow_? C[t]$$

for every context $C[\,]$, redex s and *arbitrary* term t. As usual, $\twoheadrightarrow_?$ denotes the transitive-reflexive closure of $\rightarrow_?$.

(2) A redex s in a term t is *strongly needed* if in every reduction $t \twoheadrightarrow_? t'$ to normal form, a descendant of s is rewritten.

(3) $\mathcal{R}$ is *strongly sequential** if for every context $C[\,,\ldots,]$ in normal form there exists an i such that for every substitution of redexes $r_1,\ldots,r_n$ for the $\square$'s in $C[\,,\ldots,]$, r_i is strongly needed in $C[r_1,\ldots,r_n]$.

Notice that the property of being strongly sequential* is determined by the left-hand sides of a term rewriting system only. Because $\twoheadrightarrow \subseteq \twoheadrightarrow_?$, every strongly needed redex is needed. Hence, every strongly sequential* term rewriting system is also sequential*. The reverse is not true, as the following example of Huet and Lévy shows.

EXAMPLE 2.6. Let

$$\mathcal{R} = \begin{cases} F(G(A,x),B) & \rightarrow \; x, \\ F(G(x,A),C) & \rightarrow \; x, \\ F(D,x) & \rightarrow \; x, \\ G(E,E) & \rightarrow \; E. \end{cases}$$

We leave it to the reader to show that every redex of a given term is needed. Therefore, $\mathcal{R}$ is sequential*. Consider the context $F(G(\square,\square),\square)$. The following arbitrary reductions show that $\mathcal{R}$ is not strongly sequential* (r is an arbitrary redex):

$$F(G(r,\underline{r}),\underline{r}) \twoheadrightarrow_? \underline{F(G(r,A),C)} \rightarrow_? A,$$
$$F(G(\underline{r},r),\underline{r}) \twoheadrightarrow_? \underline{F(G(A,r),B)} \rightarrow_? A,$$
$$F(G(\underline{r},\underline{r}),r) \twoheadrightarrow_? F(\underline{G(E,E)},r) \rightarrow_? \underline{F(D,r)} \rightarrow_? A.$$

Huet and Lévy defined the properties 'sequentiality' and 'strong sequentiality' in a different way. Our 'sequentiality*' does not exactly coincide with 'sequentiality', but 'strong sequentiality*' is equivalent to 'strong sequentiality'. We will define these concepts now, but first we introduce some more formalism.

We add a new constant Ω to our alphabet. Ω will represent an unknown part of a term. We abbreviate $\mathcal{T}(\mathcal{F} \cup \{\Omega\},\mathcal{V})$ to $\mathcal{T}_\Omega$. The *prefix ordering* $\leq$ on $\mathcal{T}_\Omega$ is defined by

- $x \leq x$ for every $x \in \mathcal{V}$.
- $\Omega \leq t$ for every $t \in \mathcal{T}_\Omega$,
- if $s_i \leq t_i$ for $i=1,\ldots,n$ then $F(s_1,\ldots,s_n) \leq F(t_1,\ldots,t_n)$,

Clearly, $s \leq t$ iff $s \equiv C[\Omega,\ldots,\Omega]$ and $t \equiv C[t_1,\ldots,t_n]$ for some context $C[\,,\ldots,]$ not containing Ω and Ω-terms $t_1,\ldots,t_n$. If $s \leq t$ and $s \not\equiv t$, we write $s < t$. If $t \in \mathcal{T}_\Omega$ we write $O_\Omega(t)$ for the Ω-occurrences of t and $\overline{O}(t)$ for the other occurrences. The *greatest lower bound* of two Ω-terms s and t, notation $s \cap t$, is defined by

- $x \cap x = x$,
- $F(s_1,\ldots,s_n) \cap F(t_1,\ldots,t_n) = F(s_1 \cap t_1,\ldots,s_n \cap t_n)$,
- $s \cap t = \Omega$ in all other cases.

We partition the set of Ω-terms without redexes according to the presence or absence of Ω's:

- a *normal form* is a term $t \in \mathcal{T}_\Omega$ such that t does neither contain redexes nor Ω's ($O_\Omega(t) = \varnothing$);
- an *Ω-normal form* is a term $t \in \mathcal{T}_\Omega$ without redexes containing at least one Ω ($O_\Omega(t) \neq \varnothing$).

The set of all normal forms is denoted by NF and NF_Ω denotes the set of all Ω-normal forms.

DEFINITION 2.7.

(1) A predicate P on $\mathcal{T}_\Omega$ is *monotonic* if $P(t)$ implies $P(t')$ whenever $t \leq t'$.

(2) The predicate *nf* is defined on $\mathcal{T}_\Omega$ as follows:

$$nf(t) \text{ holds } \Leftrightarrow t \twoheadrightarrow t' \text{ for some } t' \in \mathrm{NF}.$$

(3) The predicate $nf_?$ is defined on $\mathcal{T}_\Omega$ as follows:

$$nf_?(t) \text{ holds } \Leftrightarrow t \twoheadrightarrow_? t' \text{ for some } t' \in \mathrm{NF}.$$

It is easily proved that *nf* and $nf_?$ are monotonic predicates.

DEFINITION 2.8.

(1) Let P be a monotonic predicate on $\mathcal{T}_\Omega$ and let $t \in \mathcal{T}_\Omega$. An occurrence $u \in O_\Omega(t)$ is called an *index with respect to P* if for every $t' \geq t$, $P(t')$ implies $t'/u \not\equiv \Omega$. (In particular, if t has an index with respect to P then $P(t)$ is false.) The set of indices of t with respect to P is denoted by $I_P(t)$.

(2) An orthogonal term rewriting system is ***sequential*** if every $t \in \mathrm{NF}_\Omega$ has an index with respect to *nf*.

(3) An orthogonal term rewriting system is ***strongly sequential*** if every $t \in \mathrm{NF}_\Omega$ has an index with respect to $nf_?$.

To link the beginning of this section, which used the terminology of contexts, with the present set-up via Ω-terms, we note that a context in normal form corresponds to a term in NF_Ω (by viewing $\square$ as Ω).

Figure 2.1 exhibits the relationship between the properties introduced so far. The inclusions in this picture are easily proved. Notice that not every sequential* term rewriting system is sequential. For example, one can show that

$$\mathcal{R} = \begin{cases} F(A,B,x) \to C, \\ F(B,x,A) \to C, \\ F(x,A,B) \to C, \end{cases}$$

is sequential*, but $\mathcal{R}$ is not sequential: the term $F(\Omega,\Omega,\Omega)$ does not have an index with respect to *nf*.

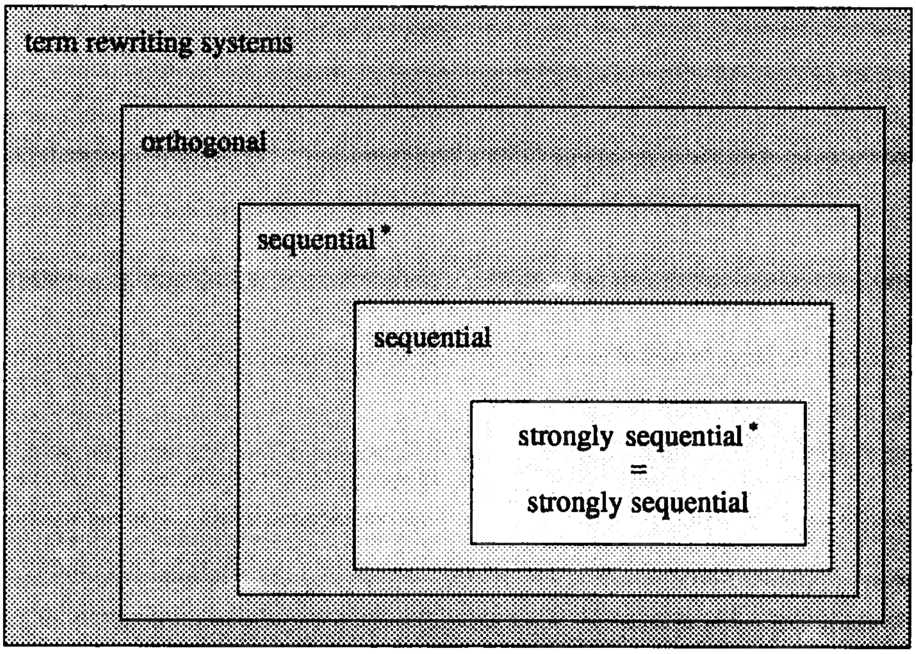

term rewriting systems
orthogonal
sequential*
sequential
strongly sequential*
=
strongly sequential

FIGURE 2.1.

3. Indices with respect to Strong Sequentiality

In this section we will describe a procedure of Huet and Lévy to compute the indices of a given Ω-term with respect to $nf_?$. But first we prove two useful properties of indices, not necessarily with respect to $nf_?$.

PROPOSITION 3.1. *Let P be a monotonic predicate on $\mathcal{T}_\Omega$ and let $t \in \mathcal{T}_\Omega$.*

(1) *If $u \in I_P(t)$, $t \leq t'$ and $t'/u \equiv \Omega$ then $u \in I_P(t')$.*

(2) *If $uv \in I_P(t)$ then $u \in I_P(t[u \leftarrow \Omega])$.*

PROOF.

(1) Suppose $u \notin I_P(t')$. Then there exists a term $t'' \geq t'$ such that $t''/u \equiv \Omega$ and $P(t'')$ is true. Clearly $t'' \geq t$ and therefore $u \notin I_P(t)$.

(2) If $u \notin I_P(t[u \leftarrow \Omega])$ then there exists a term $t' \geq t[u \leftarrow \Omega]$ such that $t'/u \equiv \Omega$ and $P(t')$ holds. Let $t'' \equiv t'[u \leftarrow t/u]$. From $t'' \geq t'$ and the monotonicity of P we obtain $P(t'')$. Together with $t''/uv \equiv \Omega$, this implies $uv \notin I_P(t'')$.

□

The two properties are depicted in Figure 3.1, where an arrow points to an index with respect to P.

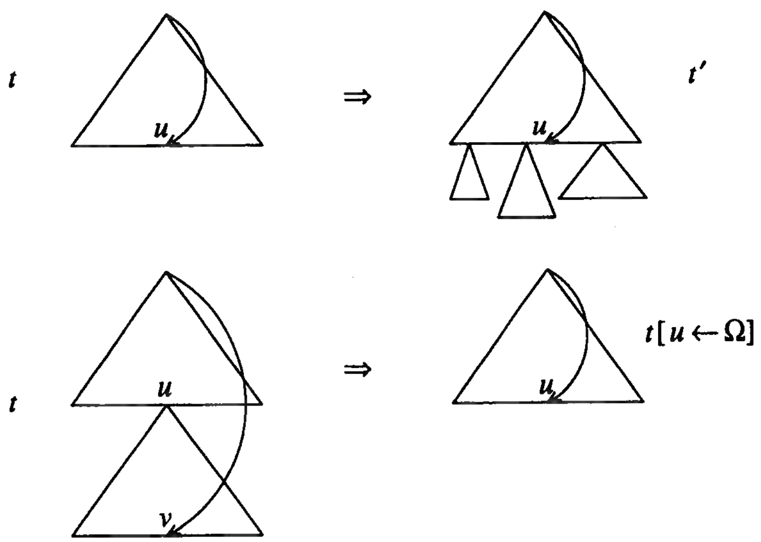


FIGURE 3.1.

In the remainder of this paper index means index with respect to $nf_?$, unless stated otherwise. Furthermore, we abbreviate $I_{nf_?}$ to I.

DEFINITION 3.2.

(1) An Ω-term t is *redex compatible* if t can be refined to a redex (i.e. $t \leq t'$ for some redex t').

(2) The reduction relation $\rightarrow_\Omega$ (*Ω-reduction*) is defined as follows:

$$C[t] \rightarrow_\Omega C[\Omega]$$

for every context $C[\;]$ and redex compatible term $t \not\equiv \Omega$.

EXAMPLE 3.3. Let

$$\mathcal{R} = \begin{cases} F(F(A,x),y) & \to \ x, \\ G(B,B) & \to \ A, \end{cases}$$

and $t \equiv F(F(\Omega,A),G(B,\Omega))$. Figure 3.2 shows all Ω-reductions starting from t.

$$\begin{tikzcd}
 & t \arrow[dl] \arrow[d] \arrow[dr] & \\
F(\Omega,G(B,\Omega)) \arrow[r] \arrow[dr] & \Omega & F(F(\Omega,A),\Omega) \arrow[l] \arrow[dl] \\
 & F(\Omega,\Omega) \arrow[u] &
\end{tikzcd}$$

FIGURE 3.2.

The next proposition relates Ω-reduction to arbitrary reduction.

PROPOSITION 3.4.
(1) *If* $s \twoheadrightarrow_\Omega t$ *then* $s' \twoheadrightarrow_? t$ *for some* $s' \geq s$.
(2) *If* $s \twoheadrightarrow_? t$ *then* $s \twoheadrightarrow_\Omega t'$ *for some* $t' \leq t$.

PROOF.
(1) We use induction on the length (number of $\to_\Omega$-steps) of $s \twoheadrightarrow_\Omega t$. The case of zero length is trivial. Suppose $s \to_\Omega t_1 \twoheadrightarrow_\Omega t$. We have $s \equiv C[s_1] \to_\Omega C[\Omega] \equiv t_1$ for some redex compatible subterm $s_1 \not\equiv \Omega$ of s. From the induction hypothesis we obtain the existence of a term $t_2 \geq t_1$ such that $t_2 \twoheadrightarrow_? t$. Because $t_2 \geq t_1 \equiv C[\Omega]$ we can write $t_2 \equiv C'[t_3]$ for some context $C'[\,] \geq C[\,]$ and term $t_3 \geq \Omega$. Let r be any redex with $s_1 \leq r$. Define $s' \equiv C'[r]$. Now we have the following arbitrary reduction:

$$s' \equiv C'[r] \to_? C'[t_3] \equiv t_2 \twoheadrightarrow_? t.$$

(2) Similar to (1).
□

PROPOSITION 3.5.
(1) *Ω-reduction is confluent:* $\forall s,t_1,t_2 \in \mathcal{T}_\Omega$ *if* $s \twoheadrightarrow_\Omega t_1$ *and* $s \twoheadrightarrow_\Omega t_2$ *then* $\exists t_3 \in \mathcal{T}_\Omega$ *such that* $t_1 \twoheadrightarrow_\Omega t_3$ *and* $t_2 \twoheadrightarrow_\Omega t_3$.
(2) *Ω-reduction is terminating: there are no infinite reductions*

$$t_0 \to_\Omega t_1 \to_\Omega t_2 \to_\Omega \cdots.$$

PROOF.
(1) Let $\to_\Omega^{=}$ be the reflexive closure of $\to_\Omega$. Suppose $s \to_\Omega t_1$ and $s \to_\Omega t_2$. By considering the relative positions of the redex compatible subterms rewritten in each step, one easily proves that there exists a term $t_3 \in \mathcal{T}_\Omega$ such that $t_1 \to_\Omega^{=} t_3$ and $t_2 \to_\Omega^{=} t_3$. From this the confluence of Ω-reduction follows by induction.
(2) If $s \to_\Omega t$ then the length of t is less than the length of s, so every sequence of Ω-reductions must eventually terminate.
□

DEFINITION 3.6. The *fixed part* $\omega(t)$ of an Ω-term t is the normal form of t with respect to Ω-reduction. Notice that $\omega(t)$ is well-defined by Proposition 3.5.

The following properties are heavily used in the sequel. Their proofs are left to the reader as simple exercises.

PROPOSITION 3.7. *Let* $s, t \in \mathcal{T}_\Omega$.

- $\omega(t) \leq t$;
- $\omega(t) \equiv \omega(t[u \leftarrow \omega(t/u)])$ *for all* $u \in O(t)$;
- *if* $s \leq t$ *then* $\omega(s) \leq \omega(t)$;
- $\omega(\omega(t)) \equiv \omega(t)$;
- *if* $s \rightarrow_? t$ *then* $\omega(s) \leq \omega(t)$;
- *if* t *is redex compatible then* $\omega(t) \equiv \Omega$.

□

Let $t \in \mathcal{T}_\Omega$ and $u \in O_\Omega(t)$. Let $\bullet$ be a fresh constant symbol. The following procedure determines whether u is an index of t:

(1)	replace in t the Ω at occurrence u by $\bullet$, result $t' \equiv t[u \leftarrow \bullet]$;
(2)	compute the normal form of t' with respect to $\rightarrow_\Omega$, result $\omega(t')$;
(3)	u is an index of $t \Leftrightarrow \bullet$ occurs in $\omega(t')$.

Graphically:

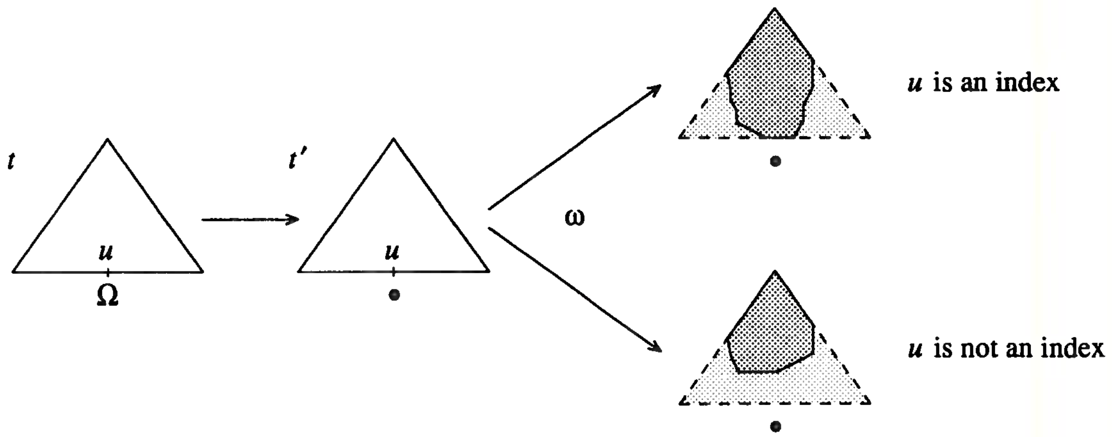


FIGURE 3.3.

Intuitively, the persistence of the 'test symbol' $\bullet$ in $\omega(t')$ means that whatever the redexes in the other (Ω-) places are and whatever their reducts might be, the $\bullet$ does not vanish. So if instead of $\bullet$ an actual redex r was present, the only way to ($\rightarrow_?$-) normalize the term at hand is to reduce r itself, eventually. The formal justification of the above procedure is given by the following lemma.

LEMMA 3.8. *Let* $t \in \mathcal{T}_\Omega$ *and* $u \in O_\Omega(t)$. *The following three statements are equivalent:*

(a) $u \in I(t)$;

(b) $\omega(t[u \leftarrow \bullet]) \not\equiv \omega(t)$;

(c) $u \in O(\omega(t[u \leftarrow \bullet]))$.

PROOF.

(a) ⇒ (b) If $\omega(t[u \leftarrow \bullet]) \equiv \omega(t)$ then $t[u \leftarrow \bullet] \twoheadrightarrow_\Omega \omega(t)$. Proposition 3.4(1) yields a term t' such that $t' \twoheadrightarrow_? \omega(t)$ and $t' \geq t[u \leftarrow \bullet]$. Let $t'' \equiv t'[\Omega \leftarrow x][u \leftarrow \Omega]$ and $\omega(t)' \equiv \omega(t)[\Omega \leftarrow x]$. It is not difficult to see that we can transform the reduction $t' \twoheadrightarrow_? \omega(t)$ into $t'' \twoheadrightarrow_? \omega(t)'$. Because $\omega(t)$ is an Ω-normal form, $\omega(t)'$ is a normal form and hence $nf_?(t'')$ is true. Clearly $t'' \geq t$ and $t''/u \equiv \Omega$. Therefore $u \notin I(t)$.

(b) ⇒ (c) If $u \notin O(\omega(t[u \leftarrow \bullet]))$ then $\omega(t[u \leftarrow \bullet]) \leq t$ and thus $\omega(t[u \leftarrow \bullet]) \leq \omega(t)$. Because $t \leq t[u \leftarrow \bullet]$ we also have $\omega(t) \leq \omega(t[u \leftarrow \bullet])$. Combining these two facts, we obtain $\omega(t[u \leftarrow \bullet]) \equiv \omega(t)$.

(c) ⇒ (a) If $u \notin I(t)$ then there exists a term $t' \geq t$ such that $t'/u \equiv \Omega$ and $nf_?(t)$ is true. Thus we have an arbitrary reduction $t' \twoheadrightarrow_? n$ from t' to some $n \in \mathrm{NF}$. Because n does not contain any occurrences of Ω, we can transform this reduction into $t'[u \leftarrow \bullet] \twoheadrightarrow_? n$. Using Proposition 3.7 or the second part of Proposition 3.4, we obtain $\omega(t'[u \leftarrow \bullet]) \leq n$. Now suppose $u \in O(\omega(t[u \leftarrow \bullet]))$. Using the fact that $\bullet$ is not redex compatible, it is not difficult to show that $\omega(t[u \leftarrow \bullet])/u \equiv \bullet$. But this is contradictory to $\omega(t'[u \leftarrow \bullet]) \leq n$ and therefore $u \notin O(\omega(t[u \leftarrow \bullet]))$.

□

The decision procedure for strong sequentiality is much more difficult. The main problem is that we do *not* have the following transitivity property for indices, which at first sight one might expect to hold: if $u \in I(s)$ and $v \in I(t)$ then $uv \in I(s[u \leftarrow t])$.

EXAMPLE 3.9. Consider the term rewriting system

$$\mathcal{R} = \begin{cases} F(G(A,x),B) & \rightarrow \; x, \\ F(G(x,A),C) & \rightarrow \; x, \\ G(D,D) & \rightarrow \; D. \end{cases}$$

Occurrence 1 is an index of $F(\Omega,\Omega)$, as is easily seen by applying the '$\bullet$-test': $\omega(F(\bullet,\Omega)) \equiv F(\bullet,\Omega)$. Similarly, occurrence 1 is an index of $G(\Omega,\Omega)$. However, occurrence 1.1 is not an index of $F(G(\Omega,\Omega),\Omega)$: $\omega(F(G(\bullet,\Omega),\Omega)) \equiv \Omega$.

The next two propositions express properties of indices which are used in the proof of the decidability of strong sequentiality.

PROPOSITION 3.10. *If $uv \in I(t)$ then $v \in I(t/u)$.*

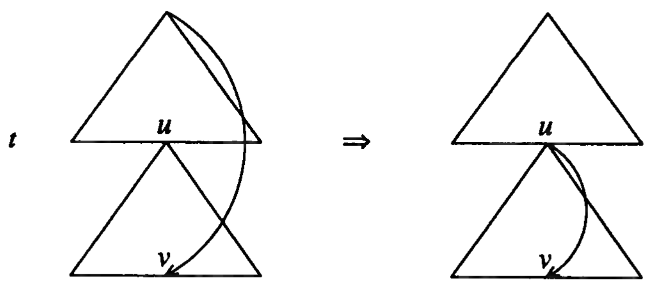


FIGURE 3.4.

PROOF. If $v \notin I(t/u)$ then $\omega((t/u)[v \leftarrow \bullet]) \equiv \omega(t/u)$ by Lemma 3.8. Therefore $\omega(t[uv \leftarrow \bullet]) \equiv \omega(t[u \leftarrow \omega((t/u)[v \leftarrow \bullet])]) \equiv \omega(t[u \leftarrow \omega(t/u)]) \equiv \omega(t)$ and from Lemma 3.8 we obtain $uv \notin I(t)$. □

PROPOSITION 3.11. *If* $u \in I(t)$, $u \perp v$ *and* $\omega(t/v) \equiv \Omega$ *then* $u \in I(t[v \leftarrow \Omega])$.

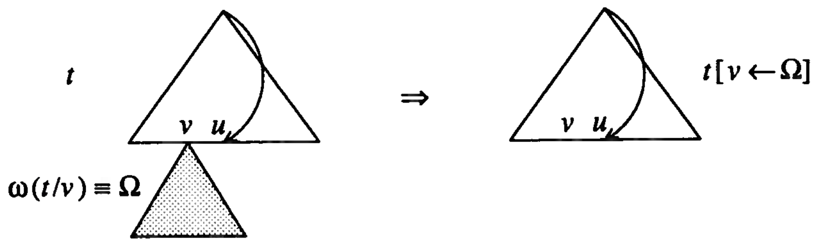


FIGURE 3.5.

PROOF. If $u \notin I(t[v \leftarrow \Omega])$ then $\omega(t[v \leftarrow \Omega][u \leftarrow \bullet]) \equiv \omega(t[v \leftarrow \Omega])$ by Lemma 3.8. Proposition 3.7 yields $\omega(t) \equiv \omega(t[v \leftarrow \omega(t/v)]) \equiv \omega(t[v \leftarrow \Omega])$ and likewise $\omega(t[u \leftarrow \bullet]) \equiv \omega(t[u \leftarrow \bullet][v \leftarrow \Omega]$. Hence $\omega(t[u \leftarrow \bullet]) \equiv \omega(t)$. Another application of Lemma 3.8 gives $u \notin I(t)$. □

The next example shows that the condition $\omega(t/v) \equiv \Omega$ in Proposition 3.11 is necessary.

EXAMPLE 3.12. Let $\mathcal{R}$ be the same as in Example 3.9. We have $1.1 \in I(F(G(\Omega,\Omega),B))$, $1.1 \perp 2$ and $\omega(B) \equiv B$, but occurrence 1.1 is not an index of $F(G(\Omega,\Omega),\Omega)$.

4. Decidability of Strong Sequentiality

A term $t \in \mathrm{NF}_\Omega$ is called *free* if t does not have indices. By definition, a term rewriting system $\mathcal{R}$ is strongly sequential if and only if $\mathcal{R}$ does not have free terms. In this section we will show that we can restrict the search for a free term to a finite subset of NF_Ω, thus proving the decidability of strong sequentiality. We first prove that we only have to consider terms $t \in \mathrm{NF}_\Omega$ with $\omega(t) \equiv \Omega$.

DEFINITION 4.1. An Ω-term t is called *rigid* if $\omega(t) \equiv t$; t is called *soft* if $\omega(t) \equiv \Omega$. The subset of soft terms of NF_Ω is denoted by NF_s.

Soft terms 'melt away' completely by Ω-reduction. Because $\omega(t) \leq t$, every Ω-term t can be written as $t \equiv \omega(t)[u_i \leftarrow t_i \mid 1 \leq i \leq n]$ where $\{u_1,\ldots,u_n\} = O_\Omega(\omega(t))$ and $t_i \equiv t/u_i$ $(i=1,\ldots,n)$. Notice that $\omega(t)$ is rigid and $t_1,\ldots,t_n$ are soft.

PROPOSITION 4.2. *Let* $t \equiv \omega(t)[u_i \leftarrow t_i \mid 1 \leq i \leq n]$ *with* $O_\Omega(\omega(t)) = \{u_1,\ldots,u_n\}$. *If* $v \in I(t_i)$ *then* $u_i v \in I(t)$.

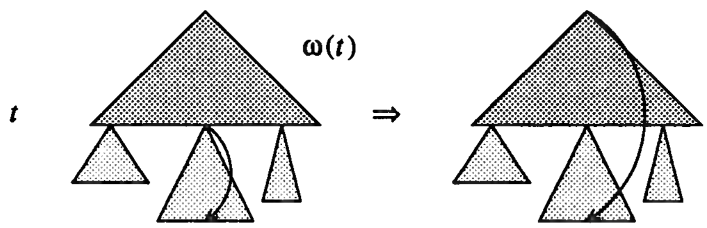


FIGURE 4.1.

PROOF. By Lemma 3.8 it is sufficient to show that $\omega(t[u_i v \leftarrow \bullet])$ and $\omega(t)$ are different. We have

$$\omega(t[u_i v \leftarrow \bullet]) \equiv \omega(t[u_i \leftarrow \omega(t_i[v \leftarrow \bullet])]) \equiv \omega(t)[u_i \leftarrow \omega(t_i[v \leftarrow \bullet])]$$

where the first identity follows from Proposition 3.7 and the second identity is due to the fact that $u_i \in O_\Omega(\omega(t))$ and $\omega(t)$, $\omega(t_i[v \leftarrow \bullet])$ are rigid. Because $v \in I(t_i)$ and t_i is a soft term, $\omega(t_i[v \leftarrow \bullet]) \not\equiv \Omega$. Therefore $\omega(t[u_i v \leftarrow \bullet]) \not\equiv \omega(t)$. □

COROLLARY 4.3. *A term rewriting system is strongly sequential if and only if every term* $t \in \mathrm{NF}_s$ *has an index.* □

Let t be a soft term. Every Ω-reduction $t \twoheadrightarrow_\Omega \Omega$ induces a partition of t into redex compatible subterms.

EXAMPLE 4.4. Let

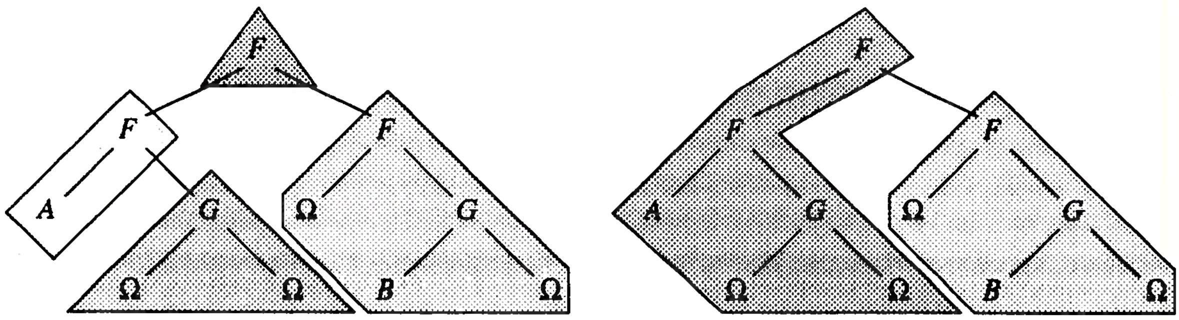
$$\mathcal{R} = \begin{cases} F(x,G(y,A)) & \to & x, \\ G(A,B) & \to & A, \end{cases}$$

and $t \equiv F(F(A,G(\Omega,\Omega)),F(\Omega,G(B,\Omega)))$. Figure 4.2(a) shows the decomposition of t into redex compatible terms with respect to the Ω-reduction

$$F(F(A,\underline{G(\Omega,\Omega)}),F(\Omega,G(B,\Omega))) \to_\Omega F(F(A,\Omega),\underline{F(\Omega,G(B,\Omega))})$$
$$\to_\Omega F(\underline{F(A,\Omega)},\Omega) \to_\Omega \underline{F(\Omega,\Omega)} \to_\Omega \Omega,$$

and Figure 4.2(b) shows the decomposition corresponding to the Ω-reduction

$$F(F(A,G(\Omega,\Omega)),\underline{F(\Omega,G(B,\Omega))}) \to_\Omega \underline{F(F(A,G(\Omega,\Omega)),\Omega)} \to_\Omega \Omega.$$



(a) (b)

FIGURE 4.2.

A possible formalization of the decomposition of a term into redex compatible terms is given in the next definition.

DEFINITION 4.5. Let $t \in \mathcal{T}_\Omega$ be a soft term. Suppose

$$t \equiv t_0 \rightarrow_\Omega t_1 \rightarrow_\Omega \cdots \rightarrow_\Omega t_n \equiv \Omega$$

is an Ω-reduction from t to Ω such that in each step $t_i \rightarrow_\Omega t_{i+1}$ the redex compatible term at occurrence u_i is replaced by Ω. Then the set $\{(u_i, t_i/u_i) \mid 0 \le i \le n-1\}$ is a *decomposition* of t.

EXAMPLE 4.6. The Ω-reductions of the previous example correspond to the following two decompositions of $F(F(A,G(\Omega,\Omega)),F(\Omega,G(B,\Omega)))$:

$$\{(\lambda, F(\Omega,\Omega)), (1, F(A,\Omega)), (1.2, G(\Omega,\Omega)), (2, F(\Omega, G(B,\Omega)))\},$$
$$\{(\lambda, F(F(A,G(\Omega,\Omega)),\Omega)), (2, F(\Omega, G(B,\Omega)))\}.$$

In an attempt to decide whether a term rewriting system is strongly sequential, we will try to construct a free term. We are particularly interested in a *minimal* free term, minimal with respect to its length. According to Corollary 4.3 we may suppose that a minimal free term, if it exists, is soft. So a minimal free term is built from redex compatible terms. However, this observation is not yet sufficient for a sensible attempt to construct a minimal free term, for there are in general infinitely many redex compatible terms. Fortunately, we may even suppose that a minimal free term is built from a special kind of redex compatible terms, the so-called *preredexes*, of which only finitely many exist.

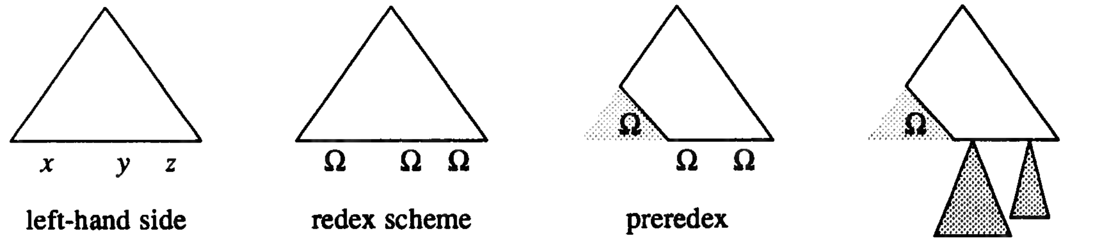
DEFINITION 4.7.

(1) A *redex scheme* is a left-hand side of a reduction rules in which all variables are replaced by Ω.

(2) A *preredex* is a term which can be refined to a redex scheme.

(3) Two Ω-terms t_1, t_2 are *compatible*, notation $t_1 \uparrow t_2$, if there exists an Ω-term t_3 such that $t_1 \le t_3$ and $t_2 \le t_3$.

Clearly, t is redex compatible if and only if there is a redex scheme r such that $t \uparrow r$. Notice that every preredex is redex compatible and every redex scheme is a preredex. Because we assumed that a term rewriting system has a finite number of reduction rules, there are only finitely many preredexes.



left-hand side   redex scheme   preredex   redex compatible term

FIGURE 4.3.

EXAMPLE 4.8. Let

$$\mathcal{R} = \begin{cases} F(A,F(A,x)) & \to\ x, \\ F(B,x) & \to\ x. \end{cases}$$

The preredexes are listed below:

$$\Omega,\ F(\Omega,\Omega),\ F(A,\Omega),\ F(\Omega,F(\Omega,\Omega)),\ F(A,F(\Omega,\Omega)),\ F(\Omega,F(A,\Omega)),$$
$$F(A,F(A,\Omega)),\ F(B,\Omega).$$

The last two are redex schemes.

DEFINITION 4.9. Let $t \in \mathcal{T}_\Omega$ be redex compatible. Like Procrustes, we cut off all parts of t that stick out:

$$\text{✂}(t) = t \cap r_1 \cap \ldots \cap r_n,$$
$$O_{\text{✂}}(t) = \overline{O}(t) \cap O_\Omega(\text{✂}(t)),$$

where $\{r_1,\ldots,r_n\}$ is the set of all redex schemes compatible with t. Notice that $O_{\text{✂}}(t)$ is the set of Ω-occurrences that are created in cutting down t to $\text{✂}(t)$.

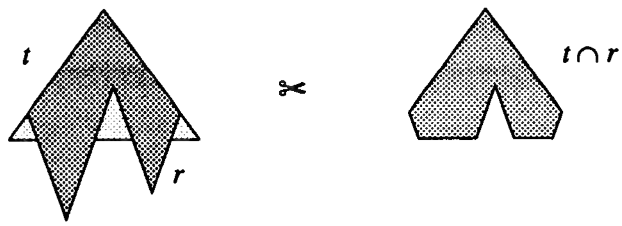


FIGURE 4.4.

PROPOSITION 4.10. *Let $t \in \mathcal{T}_\Omega$ be redex compatible. If $u \in O_{\text{✂}}(t)$ then $u \notin I(\text{✂}(t))$.*
PROOF. Suppose $u \in O_{\text{✂}}(t)$. Let R be the non-empty set of redex schemes compatible with t. It is easy to show that there exists a $r \in R$ such that $u \in O_\Omega(r)$. Because $r \geq \text{✂}(t)$ and $I(r) = \varnothing$ we obtain $u \notin I(\text{✂}(t))$ from Proposition 3.1. □

The next proposition states that the 'Procrustes procedure' does not create new indices.

PROPOSITION 4.11. *If $t \in \mathcal{T}_\Omega$ is redex compatible then $I(\text{✂}(t)) \subseteq I(t)$.*
PROOF. If $u \in I(\text{✂}(t))$ then $u \in O_\Omega(\text{✂}(t))$. According to the previous proposition we cannot have $u \in O_{\text{✂}}(t)$, hence $u \in O_\Omega(t)$. Proposition 3.1 yields $u \in I(t)$. □

We may however loose some indices.

EXAMPLE 4.12. Let

$$\mathcal{R} = \begin{cases} F(A,F(x,A,A),A) & \to x, \\ F(B,x,B) & \to x. \end{cases}$$

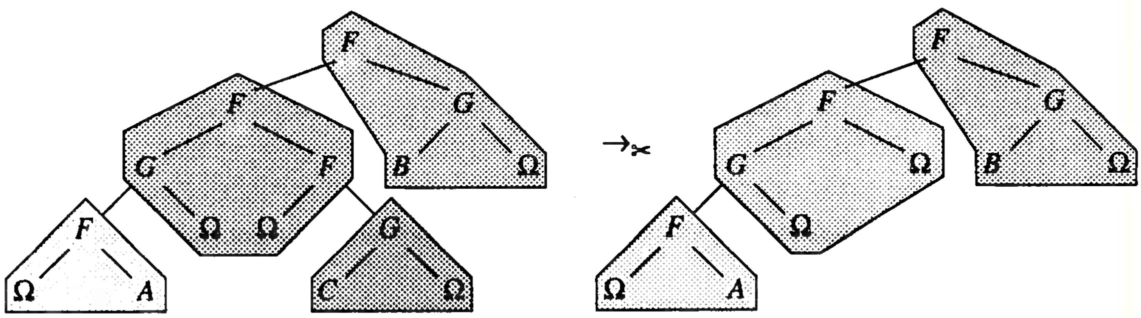
The term $t \equiv F(A,F(A,\Omega,\Omega),A)$ is redex compatible. We have $I(t) = \{2.2, 2.3\}$, $\text{✂}(t) \equiv F(A,F(\Omega,\Omega,\Omega),A)$ and $I(\text{✂}(t)) = \{2.3\}$.

We now extend the 'Procrustes procedure' to soft terms.

EXAMPLE 4.13. Let

$$\mathcal{R} = \begin{cases} F(G(A,x),y) & \to x, \\ F(G(B,x),G(B,x)) & \to x, \\ G(C,C) & \to C, \end{cases}$$

and $t \equiv F(F(G(F(\Omega,A),\Omega),F(\Omega,G(C,\Omega))),G(B,\Omega))$. Figure 4.5(a) shows a decomposition of t. If we replace the redex compatible term $t' \equiv F(G(\Omega,\Omega),F(\Omega,\Omega))$ at occurrence 1 by $\text{✂}(t') = F(G(\Omega,\Omega),\Omega)$ we obtain Figure 4.5(b). Notice that we have lost one redex compatible term, viz. $G(C,\Omega)$ at occurrence 1.2.2.



(a) (b)

FIGURE 4.5.

DEFINITION 4.14. Let D be a decomposition of a soft term t. We write $t \to_{\text{✂}} t'$ if $t' \equiv t[uv \leftarrow \Omega \mid v \in O_{\text{✂}}(s)]$ for some $(u,s) \in D$ such that $\text{✂}(s) \not\equiv s$.

PROPOSITION 4.15. *If $t \to_{\text{✂}} t'$ then $t' < t$ and $I(t') \subseteq I(t)$.*
PROOF. The first part is obvious. Suppose $w \in I(t')$. If $w \in O_\Omega(t)$ then $w \in I(t)$ by Proposition 3.1. So let us assume $w \notin O_\Omega(t)$. We know that $t' \equiv t[uv \leftarrow \Omega \mid v \in O_{\text{✂}}(s)]$ for some (u,s) in some decomposition of t, and hence $w = uv$ for some $v \in O_{\text{✂}}(s)$. From Proposition 3.10 we obtain $v \in I(t'/u)$. Together with $\text{✂}(s) \le t'/u$ and $v \in O_{\text{✂}}(s)$ this gives us $v \in I(\text{✂}(s))$, by repeated application of Proposition 3.11. This is contradictory to Proposition 4.10. □

PROPOSITION 4.16. *Let t be a soft term. If $t \twoheadrightarrow_{\text{✂}} t'$ and t' is a $\to_{\text{✂}}$-normal form, then $t' \le t$, $I(t') \subseteq I(t)$ and all decompositions of t contain only preredexes.*
PROOF. This is an immediate consequence of Proposition 4.15 and Definition 4.14. □

The subset of $NF_s - \{\Omega\}$ containing all normal forms with respect to $\rightarrow_{\times}$ is denoted by $NF_{\times}$. The reason for excluding Ω is only a matter of convenience. Notice that $I(\Omega) = \{\lambda\}$ because the left-hand side of a rewriting rule is not a variable.

COROLLARY 4.17. *A term rewriting system is strongly sequential if and only if every term $t \in NF_{\times}$ has an index.* □

We will now show that we only have to consider terms of $NF_{\times}$ with a bounded height, in order to decide whether a term rewriting system is strongly sequential.

DEFINITION 4.18. The *height* of an Ω-term t, notation $\rho(t)$, is defined by

$$\rho(t) = \begin{cases} 1+\max\{\rho(t_1),\ldots,\rho(t_n)\} & \text{if } t \equiv F(t_1,\ldots,t_n) \text{ and } n \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

The maximal height of the left-hand sides of the reduction rules of a term rewriting system $\mathcal{R}$ is denoted by $\rho_{\mathcal{R}}$:

$$\rho_{\mathcal{R}} = \max\{\rho(l) \mid l \rightarrow r \in \mathcal{R}\}.$$

As usual, we often omit the subscript $\mathcal{R}$.

The following lemma states a partial transitivity result for index propagation. It plays a crucial role in our first proof of the decidability of strong sequentiality, because it enables us to restrict the search for a free term to a finite set of Ω-terms which are entirely built from preredexes.

LEMMA 4.19. *Let $t \in \mathcal{T}_{\Omega}$ and $u,v,w \in O(t)$ such that $u < v < w$. If $v \in I(t[v \leftarrow \Omega])$, $w/u \in I(t/u)$ and $|v/u| \geq \rho - 1$, then $w \in I(t)$.*

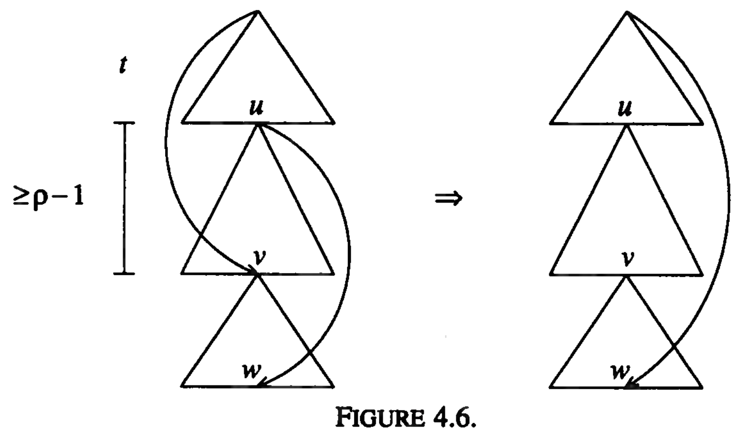


FIGURE 4.6.

PROOF. Suppose $w \notin I(t)$. According to Lemma 3.8 $w \notin O(\omega(t[w \leftarrow \bullet]))$ and hence there exists an Ω-reduction

$$t[w \leftarrow \bullet] \twoheadrightarrow_{\Omega} t_1 \rightarrow_{\Omega} t_2 \twoheadrightarrow_{\Omega} \omega(t[w \leftarrow \bullet])$$

such that $t_1/w \equiv \bullet$ and $w \notin O(t_2)$. Let t_1/u' be the redex compatible subterm rewritten in the step $t_1 \rightarrow_{\Omega} t_2$. We have $u' < w$. We distinguish two cases: (1) $u \leq u' < w$ and (2) $u' < u$.

(1) Because $u \in O(t_2)$ we can transform the Ω-reduction $t[w \leftarrow \bullet] \twoheadrightarrow_{\Omega} t_1 \rightarrow_{\Omega} t_2$ into

$$t[w \leftarrow \bullet]/u \equiv t/u[w/u \leftarrow \bullet] \twoheadrightarrow_{\Omega} t_1/u \rightarrow_{\Omega} t_2/u.$$

Clearly $w/u \notin O(t_2/u)$ and therefore $w/u \notin O(\omega(t_2/u)) = O(\omega(t/u[w/u \leftarrow \bullet]))$. This contradicts the assumption $w/u \in I(t/u)$.

(2) Let r be a redex scheme with $t_1/u' \uparrow r$. Consider the term $t_1' \equiv t_1[v \leftarrow \bullet]$. We have $|v/u'| > |v/u| \geq \rho - 1$, so if t_1'/u' is not compatible with r, then $v/u' \in \overline{O}(r)$. Because t_1/v is not a constant, $r(v/u')$ must be a function symbol of arity greater than zero. But then $\rho(r) \geq \rho + 1$, which is impossible. So t_1'/u' is redex compatible. Noting that occurrence v is preserved in $t[w \leftarrow \bullet] \twoheadrightarrow_\Omega t_1$, we now transform the Ω-reduction $t[w \leftarrow \bullet] \twoheadrightarrow_\Omega t_1 \rightarrow_\Omega t_2$ into

$$t[v \leftarrow \bullet] \twoheadrightarrow_\Omega t_1[v \leftarrow \bullet] \equiv t_1' \rightarrow_\Omega t_1'[u' \leftarrow \Omega] \equiv t_2.$$

A similar argument as in the previous case shows the impossible $v \notin I(t[v \leftarrow \Omega])$.

□

The bound $\rho - 1$ in Lemma 4.19 cannot be relaxed, as the following example shows.

EXAMPLE 4.20. Let

$$\mathcal{R} = \begin{cases} F(G(H(x), A)) & \rightarrow \ x, \\ G(H(B), C) & \rightarrow \ B, \end{cases}$$

and $t \equiv F(G(H(\Omega), \Omega))$. Take $u = 1$, $v = 1.1$ and $w = 1.1.1$. We have

$$v \in I(t[v \leftarrow \Omega]) = I(F(G(\Omega, \Omega))) = \{1.1\},$$
$$w/u \in I(t/u) = I(G(H(\Omega), \Omega)) = \{1.1, 1.2\}$$

and $|v/u| = 1 = \rho - 2$, but $w \notin I(t) = \{1.2\}$.

We will now try to construct a minimal free term t in a tree-like procedure, as suggested in Figure 4.7. Because we want t to be in Ω-normal form, we start with the finitely many proper preredexes, where a preredex is *proper* if it is neither a redex scheme not equal to Ω. In the next construction step we attach at every index position again a proper preredex, such that the resulting term is in Ω-normal form. (According to Proposition 4.23 below, there is no need to attach proper preredexes at non-index positions.) A branch in the thus originating tree of construction terminates 'successfully' if a free term is reached. In that case the term rewriting system under consideration is not strongly sequential. However, there may arise infinite branches in the construction tree.

DEFINITION 4.21. Let D be a decomposition of a term $t \in \mathrm{NF}_{\asymp}$.

(1) A non-empty subset D' of D is a *tower of preredexes* if the following two conditions are satisfied:
- if (u_1, s_1) and (u_2, s_2) are different elements of D' then either $u_1 < u_2$ or $u_2 < u_1$;
- if $(u_1, s_1), (u_2, s_2) \in D'$ and $(u, s) \in D$ such that $u_1 < u < u_2$ then $(u, s) \in D'$.

For convenience, we will assume that $u_1 < u_2 < \ldots < u_n$ whenever $\{(u_i, s_i) \mid 1 \leq i \leq n\}$ is a tower of preredexes.

(2) A tower of preredexes $\{(u_i, s_i) \mid 1 \leq i \leq n\}$ is a *main tower* if $u_1 = \lambda$ and there does not exists an element $(u, s) \in D$ with $u_n < u$.

(3) Let $D' = \{(u_i, s_i) \mid 1 \leq i \leq n\}$ be a tower of preredexes. The term $\pi(D')$ is defined by

$$\pi(D') = \begin{cases} s_1 & \text{if } n = 1, \\ \pi(\{(u_i, s_i) \mid 1 \leq i \leq n-1\})[u_n/u_1 \leftarrow s_n] & \text{if } n > 1. \end{cases}$$

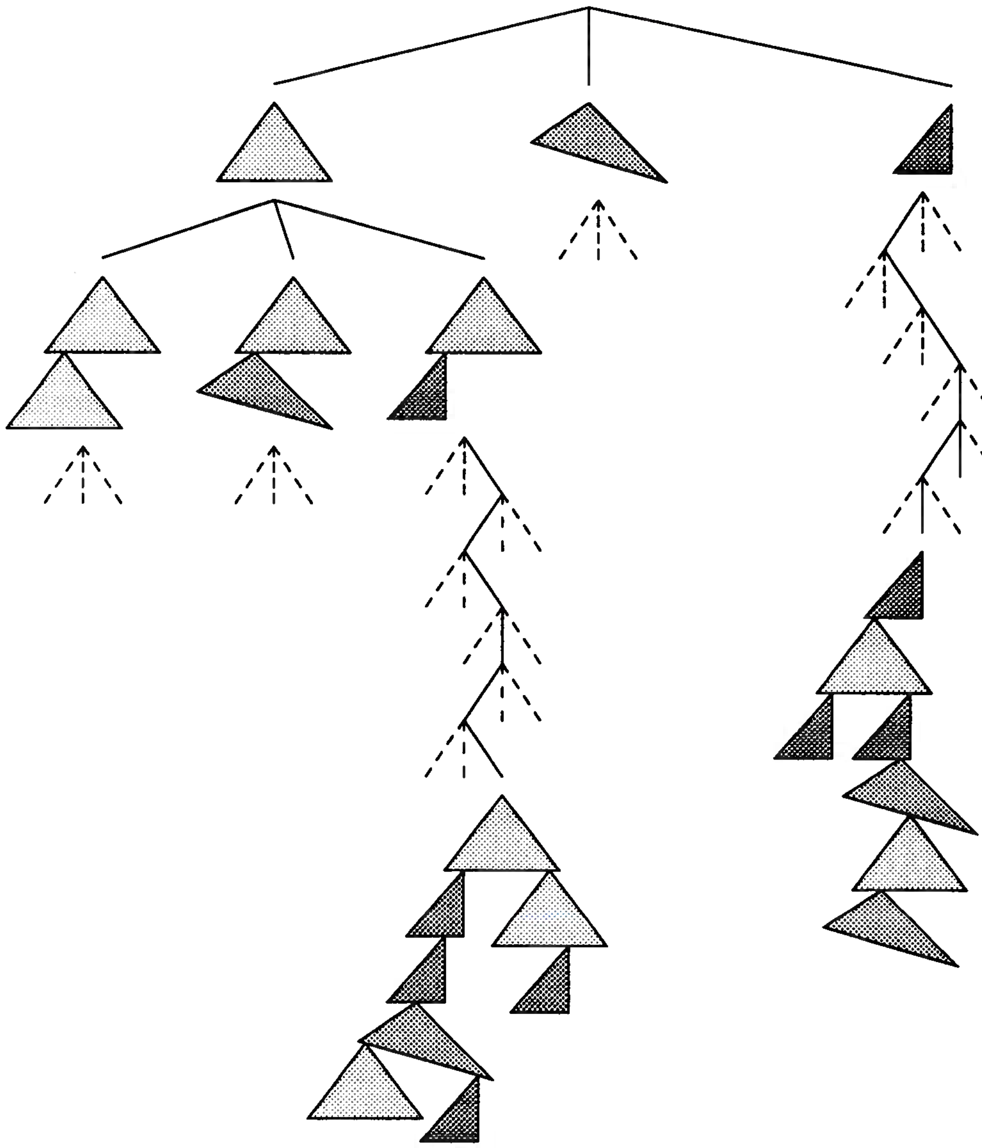


FIGURE 4.7.

(4) A tower of preredexes $\{(u_i, s_i) \mid 1 \le i \le n\}$ is *sufficiently high* if $n \ge 2$, $|u_n/u_1| \ge \rho - 1$ and $|u_{n-1}/u_1| < \rho - 1$.

EXAMPLE 4.22. Let

$$\mathcal{R} = \begin{cases} F(G(x,F(y,A)),B) & \to \; x, \\ G(x,A) & \to \; x. \end{cases}$$

Consider the term $F(F(G(\Omega,G(\Omega,\Omega)),G(\Omega,\Omega)),G(\Omega,\Omega))$ with decomposition

$$\{(\lambda, F(\Omega,\Omega)), (1, F(G(\Omega,\Omega),\Omega)), (1.1.2, G(\Omega,\Omega)), (1.2, G(\Omega,\Omega)), (2, G(\Omega,\Omega))\},$$

see Figure 4.8. Table 4.9 lists all towers of preredexes consisting of at least two elements. In this table, s means that the tower under consideration is sufficiently high and m means that it is a main tower.

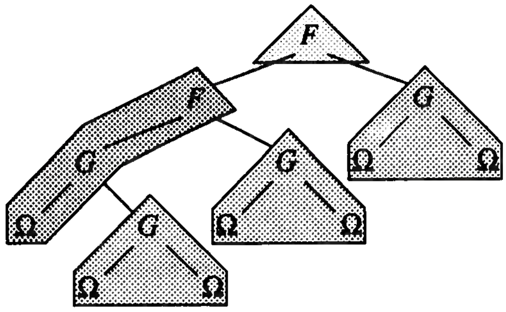


FIGURE 4.8.

tower D	$\pi(D)$	
$\{(\lambda, F(\Omega,\Omega)), (1, F(G(\Omega,\Omega),\Omega))\}$	$F(F(G(\Omega,\Omega),\Omega),\Omega)$	
$\{(1, F(G(\Omega,\Omega),\Omega)), (1.1.2, G(\Omega,\Omega))\}$	$F(G(\Omega,G(\Omega,\Omega)),\Omega)$	s
$\{(1, F(G(\Omega,\Omega),\Omega)), (1.2, G(\Omega,\Omega))\}$	$F(G(\Omega,\Omega),G(\Omega,\Omega))$	
$\{(\lambda, F(\Omega,\Omega)), (2, G(\Omega,\Omega))\}$	$F(\Omega,G(\Omega,\Omega))$	m
$\{(\lambda, F(\Omega,\Omega)), (1, F(G(\Omega,\Omega),\Omega)), (1.1.2, G(\Omega,\Omega))\}$	$F(F(G(\Omega,G(\Omega,\Omega)),\Omega),\Omega)$	$m\,s$
$\{(\lambda, F(\Omega,\Omega)), (1, F(G(\Omega,\Omega),\Omega)), (1.2, G(\Omega,\Omega))\}$	$F(F(G(\Omega,\Omega),G(\Omega,\Omega)),\Omega)$	$m\,s$

TABLE 4.9.

If we observe at some branch in the construction tree the arising of a term which has a main tower containing two disjoint occurrences of a sufficiently high tower of preredexes, that branch is stopped unsuccessfully. So every branch of the construction tree terminates, either successfully in a free term, or unsuccessfully. Because the construction is finitely branching, we obtain a finite construction tree. A term rewriting system is strongly sequential if and only if all branches in the construction tree terminate unsuccessfully. The justification for unsuccessfully closing branches at which a repetition of sufficiently high towers occurs is given in Lemma 4.24.

PROPOSITION 4.23. *If t is a minimal free term then $I(t[u \leftarrow \Omega]) = \{u\}$ for all $u \in O(t)$ such that the length of t/u is greater than* 1.
PROOF. Because the length of $t[u \leftarrow \Omega]$ is less than the length of t, we have $I(t[u \leftarrow \Omega]) \neq \varnothing$. Let $v \in I(t[u \leftarrow \Omega])$. By Proposition 3.1, v cannot be disjoint from u, hence $I(t[u \leftarrow \Omega]) = \{u\}$. □

LEMMA 4.24. *Suppose t is a minimal free term and let D be a decomposition of t. If a main tower $D' \subseteq D$ contains two distinct sufficiently high towers of preredexes $D_1, D_2 \subseteq D'$, then $\pi(D_1) \not\equiv \pi(D_2)$.*
PROOF. Suppose a main tower $D' = \{(u_i, s_i) \mid 1 \le i \le n\}$ in a decomposition of t contains two sufficiently high towers of preredexes

$$D_1 = \{(u_i, s_i) \mid j \le i \le k\}$$
$$D_2 = \{(u_i, s_i) \mid l \le i \le m\}$$

such that $1 \le j < k < l < m \le n$ and $\pi(D_1) \equiv \pi(D_2)$. Let $t' \equiv t[u_j v \leftarrow t/u_l v]$ where occurrence v is defined by $v \in O_\Omega(\pi(D_1))$ and $u_j v \in \{u_{k+1}, \ldots, u_l\}$, see Figure 4.10. It is not difficult to show that $t' \in \mathrm{NF}_\Omega$. In order to arrive at a contradiction, we will prove that t' is a free term. Suppose $I(t') \neq \varnothing$. Let $w \in I(t')$. If $w \perp u_j v$ then $w \in I(t'[u_j v \leftarrow \Omega]) = I(t[u_j v \leftarrow \Omega])$ by Proposition 3.11, and therefore $w \in I(t)$ using Proposition 3.1. This is impossible because t is a free term. So if $w \in I(t')$ then $w \ge u_j v$.

From Proposition 3.10 we obtain $w/u_j \in I(t'/u_j)$. Repeated application of Proposition 3.11 and a single application of Proposition 3.1 yields $w/u_j \in I(t/u_l)$. Proposition 4.23 gives us $u_m \in I(t[u_m \leftarrow \Omega])$. Because D_2 is sufficiently high, we have $|u_m/u_l| \geq \rho - 1$. Finally we can use Lemma 4.19 to obtain the impossible $w \in I(t)$. Therefore, t' is a free term and the lemma is proved. □

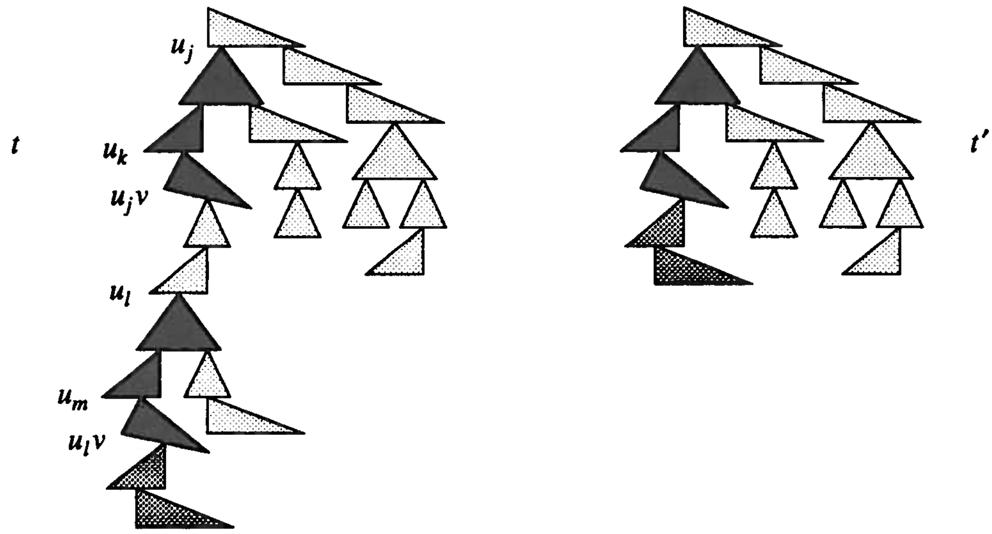


FIGURE 4.10.

COROLLARY 4.25. *Strong sequentiality is a decidable property of orthogonal term rewriting systems.*
PROOF. Because there are only finitely many sufficiently high towers of preredexes, a repetition of a sufficiently high tower along a main tower must occur within a computable bound. Inspection of all terms $t \in NF_{\times}$ with height up to this bound reveals strong sequentiality or the absence of it. □

5. Δ-sets and Increasing Indices

Huet and Lévy proved the decidability of strong sequentiality by showing the equivalence of strong sequentiality and the existence of so-called Δ-*sets*:

> for every proper preredex t, $\Delta(t)$ is a non-empty subset of $I(t)$ subject to the following constraint: for all $u \in \Delta(t)$, if s is a proper preredex such that $t[u \leftarrow s]$ is again a proper preredex, then $\{v \mid uv \in \Delta(t[u \leftarrow s])\}$ is a non-empty subset of $\Delta(s)$.

Assuming the existence of Δ-sets, Huet and Lévy constructed a 'matching dag', a special kind of graph on which they defined an efficient algorithm to find a strongly needed redex in a given term. (In [8] it is proved that strong sequentiality is equivalent to the existence of a function Q satisfying two constraints Q_1 and Q_2. The equivalent notion of Δ-sets stems from [7].) Actually, the notion of Δ-sets in [7], [8] is more complicated than the one we use, since in [7], [8] it involves so-called 'directions', not introduced in the present paper.

The second part of the equivalence proof (existence of Δ-sets ⇒ strong sequentiality) is in essence a correctness proof of their algorithm. In this section we will give a direct proof of this implication. For the other implication (strong sequentiality ⇒ existence of Δ-sets) we use the increasing indices of [8].

DEFINITION 5.1. Let $t \in \mathcal{T}_\Omega$. An occurrence $u \in I(t)$ is an *increasing index* if for every term $s \in \mathrm{NF}_s$ there exists an index $v \in I(t[u \leftarrow s])$ such that $u \leq v$. The set of all increasing indices of t is denoted by $J(t)$.

The following proposition shows that every term $t \in \mathrm{NF}_\Omega$ has at least one increasing index, provided $\mathcal{R}$ is strongly sequential.

PROPOSITION 5.2. *If $\mathcal{R}$ is strongly sequential then for any term $t \in \mathrm{NF}_\Omega$ we have $J(t) \neq \varnothing$.*
PROOF. Suppose $\mathcal{R}$ is strongly sequential and let $t \in \mathrm{NF}_\Omega$. We have $I(t) \neq \varnothing$, say $I(t) = \{u_1, \ldots, u_n\}$. If $J(t) = \varnothing$ then for every $i \in \{1, \ldots, n\}$ there exists a term $s_i \in \mathrm{NF}_s$ such that $\{v \in I(t[u_i \leftarrow s_i]) \mid v \geq u_i\} = \varnothing$. Consider $t' \equiv t[u_i \leftarrow s_i \mid 1 \leq i \leq n]$. It is not difficult to show that $t' \in \mathrm{NF}_\Omega$. Hence $I(t') \neq \varnothing$. Let $v \in I(t')$. If $v \geq u_i$ for some $i \in \{1, \ldots, n\}$ then $v \in I(t[u_i \leftarrow s_i])$ by $n-1$ applications of Proposition 3.11. This is impossible, so $v \perp u_i$ for all $i \in \{1, \ldots, n\}$. Now we have $v \in I(t)$, again by applications of Proposition 3.11. But $v \notin \{u_1, \ldots, u_n\}$. We conclude that $J(t) \neq \varnothing$. □

The 'suffix property' (Proposition 3.10) also holds for increasing indices.

PROPOSITION 5.3. *If $uv \in J(t)$ then $v \in J(t/u)$.*
PROOF. If $v \notin J(t/u)$ then there exists a term $s \in \mathrm{NF}_s$ such that $\{w \in I(t/u[v \leftarrow s]) \mid w \geq v\} = \varnothing$. Let $t' \equiv t[uv \leftarrow s]$. We have $\{w \in I(t') \mid w \geq uv\} = \varnothing$ by Proposition 3.10 and therefore $uv \notin J(t)$. □

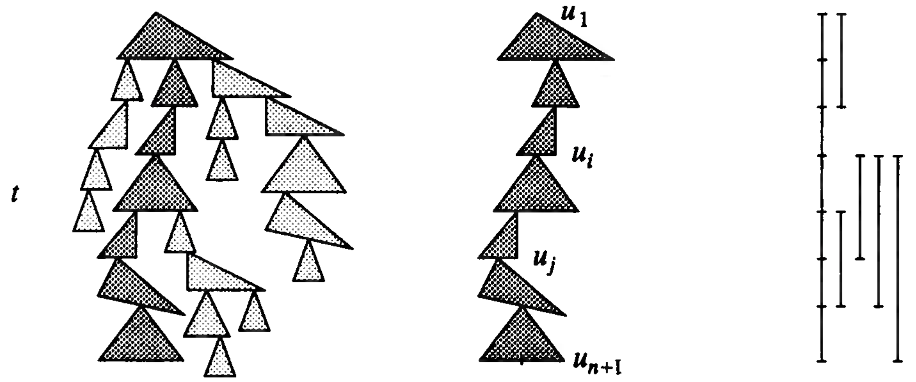
PROPOSITION 5.4. *Suppose $\mathcal{R}$ is strongly sequential. Let $t \in \mathrm{NF}_\Omega$ and $s \in \mathrm{NF}_s$. If $u \in J(t)$ then there exists a $v \in J(t[u \leftarrow s])$ with $u \leq v$.*
PROOF. Suppose $\{v \in J(t[u \leftarrow s]) \mid v \geq u\} = \varnothing$. From Proposition 5.2 we know that the set $\{v \in I(t[u \leftarrow s]) \mid v \geq u\}$ is non-empty, say $\{v \in I(t[u \leftarrow s]) \mid v \geq u\} = \{u_1, \ldots, u_n\}$. For every $i \in \{1, \ldots, n\}$ there exists a term $s_i \in \mathrm{NF}_s$ such that $\{v \in I(t[u \leftarrow s][u_i \leftarrow s_i]) \mid v \geq u_i\} = \varnothing$. Consider the term $s' \equiv s[u_i/u \leftarrow s_i \mid 1 \leq i \leq n]$. We obtain a contradiction like in the proof of Proposition 5.2. □

DEFINITION 5.5. A proper preredex t is called *atomic* if t does not contain other proper preredexes, i.e. t/u is not a proper preredex for all $u \in O(t) - \{\lambda\}$.

PROPOSITION 5.6. *Let t be an atomic preredex. If $u \in I(t)$ then* $\omega(t[u \leftarrow \bullet]) \equiv t[u \leftarrow \bullet]$.
PROOF. Easy. □

We call a decomposition D of a term $t \in \mathrm{NF}_{\times}$ *elementary* if D consists only of atomic preredexes, i.e. s is an atomic preredex whenever $(u,s) \in D$. Clearly every decomposition of a term $t \in \mathrm{NF}_{\times}$ can be refined to an elementary decomposition.

We are now ready for the main theorem of this section. First we will give an intuitive description of the proof idea. As noted before, the problem with indices is that they are not 'transitive'. However, 'partial transitivity' properties do hold; in our first proof of the decidability of strong sequentiality this was embodied by Lemma 4.19, in the following proof this is embodied by the Δ-sets. To show that the existence of Δ-sets guarantees the existence of an index in a term $t \in \mathrm{NF}_{\times}$, we consider an elementary decomposition of t and we select a main tower as in Figure 5.1(a) which has the property that Δ-indices are transmitted along the tower, in the following sense. The main tower in Figure 5.1(b) may contain next to the atomic preredexes, larger preredexes formed by some consecutive atomic pieces of the tower, e.g. as indicated in Figure 5.1(c) where every line segment denotes a preredex between some u_i, u_j. Now for every such preredex between u_i, u_j we have that u_j/u_i is a Δ-index of that preredex. The result is that the main tower leads indeed to a position u_{n+1} which is an index of that tower, and hence of the whole term t. This can be seen as follows: if the test symbol • is inserted at u_{n+1}, then the tower is perfectly rigid, no chunk can be melted away. First by our use of atomic preredexes, so no chunk away from the main path $u_1 - u_2 - \ldots - u_{n+1}$ of the main tower can be melted away, and second by the arrangement that all preredexes in the tower 'looking at' the test symbol • at position u_{n+1} have an index at that point. We will now give the formal proof.



(a) (b) (c)

FIGURE 5.1.

THEOREM 5.7. *$\mathcal{R}$ is strongly sequential if and only if there exists Δ-sets for $\mathcal{R}$.*
PROOF.

⇒ If $\mathcal{R}$ is strongly sequential then the increasing indices satisfy the conditions for being Δ-sets, by Propositions 5.2, 5.3 and 5.4.

⇐ We have to prove that every term $t \in \mathrm{NF}_{\Omega}$ has an index. By previous results (Corollary 4.17) it is sufficient to prove that every term $t \in \mathrm{NF}_{\times}$ has an index. Let $t \in \mathrm{NF}_{\times}$ and suppose D is an elementary decomposition of t. We will construct a sequence of towers of preredexes $D_1 \subseteq D_2 \subseteq \ldots \subseteq D_n \subseteq D$ and an occurrence u_{n+1} such that $D_n = \{(u_i, s_i) \mid 1 \le i \le n\}$ is a main tower and the following property (*) holds:

if $D_k^j = \{(u_i,s_i) \mid j \le i \le k\}$ is a tower of preredexes such that $\pi(D_k^j)$ is a preredex, then $u_{k+1}/u_j \in \Delta(\pi(D_k^j))$.

D_1 is the singleton set $\{(u_1,s_1)\}$ where $u_1=\lambda$ and $(\lambda,s_1)\in D$. Because s_1 is a proper preredex, $\Delta(s_1)$ is non-empty, and hence we can take $u_2\in\Delta(s_1)$. Suppose we have defined $D_1,\ldots,D_{j-1}$ and occurrence u_j. If D_{j-1} is a main tower then we end the construction and set $n=j-1$. Otherwise we extend D_{j-1} with the unique element $(u_j,s_j)\in D$ to obtain D_j. Let $k\in\{1,\ldots,j\}$ be minimal under the restriction that $\pi(D_j)/u_k$ is a preredex. In order to define u_{j+1} we consider two cases: (1) $k=j$ and (2) $k<j$.

(1) If $k=j$ then we choose some $u_{j+1}\in\Delta(s_j)$. In this case the hypothesis (*) is clearly satisfied.

(2) If $k<j$ then $\pi(D_{j-1})/u_k \equiv \pi(D_{j-1}^k)$ also is a preredex. From the induction hypothesis we obtain $u_j/u_k\in\Delta(\pi(D_{j-1}^k))$ and the existence of Δ-sets implies the existence of an occurrence $u'>u_j/u_k$ such that $u'\in\Delta(\pi(D_j^k))$ and $u'/(u_{j+1}/u_k)\in\Delta(\pi(D_j^j))=\Delta(s_j)$. Now we define $u_{j+1}=u_ku'$. We still have to show that the hypothesis (*) is satisfied. Suppose $\pi(D_m^l)$ is a preredex. If $m<j$ the result follows by induction. So assume $m=j$. We have $k\le l$ by the definition of k. If $k=l$ then we already know that $u_{m+1}/u_l=u'\in\Delta(\pi(D_m^l))$. If $k<l$ then $u_l/u_k\in\Delta(\pi(D_{l-1}^k))$ by the induction hypothesis. Because $\pi(D_j^k)\equiv \pi(D_{l-1}^k)[u_l/u_k\leftarrow\pi(D_j^l)]$ and $u_{j+1}/u_k\in\Delta(\pi(D_j^k))$, we obtain $(u_{j+1}/u_k)/(u_l/u_k)= u_{j+1}/u_l\in\Delta(D_j^l)$ from the definition of Δ-sets.

We will now show that $u_{n+1}\in I(\pi(D_n))$. Suppose $\pi(D_n)[u_{n+1}\leftarrow\bullet]$ contains a redex compatible subterm $s\not\equiv\Omega$ at occurrence v. Because $\pi(D_n)[u_{n+1}\leftarrow\bullet]$ is a normal form with respect to $\rightarrow_{\bowtie}$, s must be a preredex. If v is disjoint from u_{n+1} then s is a proper subterm of an atomic preredex, which is impossible. For similar reasons v cannot be distinct from $u_1,\ldots,u_n$. So $v=u_i$ for some $i\le n$. Clearly $s[u_{n+1}/u_i\leftarrow\Omega]\equiv\pi(D_n^i)$ is also a preredex. From (*) we obtain $u_{n+1}/u_i\in\Delta(\pi(D_n^i))\subseteq I(\pi(D_n^i))$ and hence $\omega(s)\equiv\omega(\pi(D_n^i)[u_{n+1}/u_i\leftarrow\bullet])\not\equiv\omega(\pi(D_n^i))\equiv\Omega$. This contradicts the assumption that s is redex compatible. Therefore $\pi(D_n)[u_{n+1}\leftarrow\bullet]$ does not contain redex compatible subterms $\not\equiv\Omega$ and thus $\omega(\pi(D_n)[u_{n+1}\leftarrow\bullet])\equiv\pi(D_n)[u_{n+1}\leftarrow\bullet]$. We conclude that $u_{n+1}\in I(\pi(D_n))$. Finally, Proposition 3.1 yields $u_{n+1}\in I(t)$.

□

Because it is straightforward to give an (inefficient) algorithm for finding Δ-sets, Theorem 5.7 gives a decision procedure for strong sequentiality.

6. Further Remarks on Deciding Strong Sequentiality

We conjectured for some time that, with the help of Lemma 4.19, it should be possible to prove that the height of a minimal free term is bounded by 2ρ or perhaps 3ρ (where ρ is the maximum height of the redex schemes as defined in Section 4), which would imply a very simple decision procedure for strong sequentiality: just check all terms with height up to 2ρ (3ρ). Unfortunately, this is not the case.

DEFINITION 6.1.

(1) The term rewriting systems $\mathcal{R}_n$ $(n \geq 2)$ are defined by induction:

$$\mathcal{R}_2 = \begin{cases} F_0(A,B,x) & \to x, \\ F_1(F_0(x,A,B),A) & \to x, \\ F_2(F_1(F_0(B,x,A),B),A) & \to x, \end{cases}$$

$$\mathcal{R}_{n+1} = \mathcal{R}_n \cup \{F_{n+1}(F_n(F_{n-1}(A,x),B),A) \to x\}.$$

(2) The term rewriting systems $\mathcal{S}_n$ $(n \geq 2)$ are defined as follows:

$$\mathcal{S}_n = \mathcal{R}_n \cup \{F_{n+1}(F_n(F_{n-1}(A,x),y),z) \to x\}.$$

PROPOSITION 6.2. *The term rewriting system $\mathcal{R}_n$ is strongly sequential for every $n \geq 2$.*

PROOF. We will inductively define collections Δ_i for $i \geq 2$, satisfying the conditions for being Δ-sets with respect to $\mathcal{R}_i$. The collection Δ_2 is defined by (the underlined Ω's denote the Δ-indices):

$$F_1(\Omega,\underline{\Omega}),$$
$$F_2(\Omega,\underline{\Omega}),$$
$$F_2(F_1(\Omega,\underline{\Omega}),\underline{\Omega}),$$
$$F_2(F_1(\Omega,\underline{\Omega}),A)$$

and

$$\Delta_2(t) = I(t)$$

for all other proper preredexes t of $\mathcal{R}_2$. It is straightforward to show that Δ_2 satisfies the conditions for being Δ-sets with respect to $\mathcal{R}_2$. Suppose we have defined $\Delta_2,\ldots,\Delta_i$. Let t be a proper preredex of $\mathcal{R}_{i+1}$. If t is a proper preredex of $\mathcal{R}_i$ then we define

$$\Delta_{i+1}(t) = \begin{cases} \{1,2\} & \text{if } t \equiv F_i(\Omega,\Omega), \\ \Delta_i(t) & \text{otherwise}, \end{cases}$$

and if t is not a proper preredex of $\mathcal{R}_i$ then $\Delta_{i+1}(t)$ is given below:

$$F_{i+1}(\Omega,\underline{\Omega}),$$
$$F_{i+1}(F_i(\Omega,\underline{\Omega}),\underline{\Omega}),$$
$$F_{i+1}(F_i(\Omega,\underline{\Omega}),A),$$
$$F_{i+1}(F_i(F_{i-1}(\Omega,\Omega),\underline{\Omega}),\underline{\Omega}),$$
$$F_{i+1}(F_i(F_{i-1}(\Omega,\Omega),\underline{\Omega}),A)$$

and

$$\Delta_{i+1}(t) = I(t)$$

if t is not listed above. Although very tedious, it is not difficult to verify that Δ_{i+1} indeed satisfies the conditions for being Δ-sets with respect to $\mathcal{R}_{i+1}$. Theorem 5.7 yields the strong sequentiality of $\mathcal{R}_n$, for every $n \geq 2$. □

PROPOSITION 6.3. *Let $n \geq 2$. The term rewriting system $\mathcal{S}_n$ is not strongly sequential; its minimal free term is $F_{n+1}(F_n(\ldots(F_2(F_1(F_0(\Omega,\Omega,\Omega),\Omega),\Omega)\ldots),\Omega)$.*

PROOF. Because $I(F_{n+1}(F_n(\ldots(F_2(F_1(F_0(\Omega,\Omega,\Omega),\Omega),\Omega)\ldots),\Omega)) = \varnothing$, $\mathcal{S}_n$ is not strongly sequential. Let t be a minimal free term. The following observation is easily proved:

if $t(u) \equiv F_j$ and $t(u \cdot i) \equiv F_k$ then $i = 1$ and $j = k+1$.

From this one obtains $t \equiv F_{n+1}(F_n(\ldots(F_2(F_1(F_0(\Omega,\Omega,\Omega),\Omega),\Omega)\ldots),\Omega)$ by a sequence of routine arguments. □

COROLLARY 6.4. *For every $n \geq 1$ there exists a term rewriting system $\mathcal{R}$ which is not strongly sequential such that every free term t of $\mathcal{R}$ has height $\rho(t) \geq n\, \rho_{\mathcal{R}}$.* □

The above gives evidence that deciding strong sequentiality is not a trivial matter. Indeed, there is no known efficient method for finding Δ-sets. (We conjecture that deciding strong sequentiality is NP-complete.) Huet and Lévy pointed out that for the practically relevant case of constructor systems, deciding strong sequentiality is easy. Laville [12] showed the close connection between strong sequentiality of constructor systems and the existence of lazy pattern matching algorithms for functional programming languages.

DEFINITION 6.5. A *constructor system* is a term rewriting system $(\mathcal{F},\mathcal{R})$ in which the set of function symbols $\mathcal{F}$ can be partitioned into a set $\mathcal{D}$ of *defined* function symbols and a set $\mathcal{C}$ of *constructors* such that every left-hand side of $\mathcal{R}$ has the form $F(t_1,\ldots,t_n)$ with $F \in \mathcal{D}$ and $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{C},\mathcal{V})$.

The nice thing about constructor systems is the transitivity of index propagation for terms starting with a defined function symbol.

PROPOSITION 6.6. *Let $\mathcal{R}$ be a constructor system. Let $s,t \in \mathcal{T}_\Omega$ such that $s(\lambda), t(\lambda) \in \mathcal{D}$. If $u \in I(s)$ and $v \in I(t)$ then $uv \in I(s[u \leftarrow t])$.*

PROOF. If $uv \notin I(s[u \leftarrow t])$ then $uv \notin O(\omega(s[u \leftarrow t][uv \leftarrow \bullet]))$ and hence there exists an Ω-reduction

$$s[u \leftarrow t][uv \leftarrow \bullet] \twoheadrightarrow_\Omega t_1 \rightarrow_\Omega t_2$$

such that $t_1/uv \equiv \bullet$ and $uv \notin O(t_2)$. Let t_1/u' be the redex compatible subterm rewritten in the step $t_1 \rightarrow_\Omega t_2$. Clearly $u' < uv$. We distinguish two cases: (1) $u \leq u' < uv$ and (2) $u' < u$.

(1) The proof is the same as the first case of the proof of Lemma 4.19.

(2) Let r be a redex scheme compatible with t_1/u'. Because $t_1(u) \in \mathcal{D}$ we have either $u/u' \notin O(r)$ or $r(u/u') \equiv \Omega$. In both cases the term $t_1[u \leftarrow \bullet]/u'$ also is compatible with r. We obtain a contradiction as in the second case of the proof of Lemma 4.19.

□

COROLLARY 6.7. *A constructor system is strongly sequential if and only if every proper preredex has an index.*

PROOF.

$\Rightarrow$ Trivial.

$\Leftarrow$ We have to show that every term $t \in NF_{\mathscr{K}}$ has an index. Because every $t \in NF_{\mathscr{K}}$ can be partitioned into proper preredexes, this follows from Proposition 6.6.

□

Alternatively, this fact can be obtained from Theorem 5.7 and the definition of Δ-sets, noting that if s, t are proper preredexes and $u \in \Delta(t)$ then $t[u \leftarrow s]$ can never be a proper preredex.

In order to decide whether a constructor system $\mathscr{R}$ is strongly sequential, we only have to compute the indices of its proper preredexes. According to the next proposition, this is very easy.

PROPOSITION 6.8. *Let t be a proper preredex in a constructor system. An Ω-occurrence u of t is an index if and only if $t[u \leftarrow \bullet]$ is not redex compatible.*
PROOF. Easy. □

We conclude this section with the observation that strong sequentiality is a modular property, i.e. depends on the disjoint pieces of a term rewriting system.

DEFINITION 6.9.

(1) The *direct sum* $\mathscr{R}_1 \oplus \mathscr{R}_2$ of two term rewriting systems $\mathscr{R}_1, \mathscr{R}_2$ is the system obtained by taking the disjoint union of $\mathscr{R}_1$ and $\mathscr{R}_2$. That is, if the sets of function symbols of $\mathscr{R}_1$ and $\mathscr{R}_2$ are disjoint, then $\mathscr{R}_1 \oplus \mathscr{R}_2$ is the union of $\mathscr{R}_1$ and $\mathscr{R}_2$; otherwise we take renamed copies $\mathscr{R}_1', \mathscr{R}_2'$ of $\mathscr{R}_1, \mathscr{R}_2$ such that $\mathscr{R}_1'$ and $\mathscr{R}_2'$ have disjoint sets of function symbols, and define $\mathscr{R}_1 \oplus \mathscr{R}_2 = \mathscr{R}_1' \cup \mathscr{R}_2'$.

(2) A property $\mathscr{P}$ of term rewriting systems is called *modular* if the following holds for all $\mathscr{R}_1, \mathscr{R}_2$:

$$\mathscr{R}_1 \oplus \mathscr{R}_2 \text{ has the property } \mathscr{P} \iff \text{both } \mathscr{R}_1 \text{ and } \mathscr{R}_2 \text{ have the property } \mathscr{P}.$$

A well-known example of a modular property is the Church-Rosser property (Toyama [20]). The modular aspects of other properties of term rewriting systems have been studied in [13], [14], [18], [21], [22].

PROPOSITION 6.10. *Strong sequentiality is a modular property of orthogonal term rewriting systems.*
PROOF. Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be orthogonal term rewriting systems with disjoint sets of function symbols. We have to show that $\mathscr{R}_1 \cup \mathscr{R}_2$ is strongly sequential if and only if both $\mathscr{R}_1$ and $\mathscr{R}_2$ are strongly sequential.

$\Rightarrow$ If $\mathscr{R}_1 \cup \mathscr{R}_2$ is strongly sequential then, according to Theorem 5.7, we can find Δ-sets for preredexes of $\mathscr{R}_1 \cup \mathscr{R}_2$, say $\Delta_{\mathscr{R}_1 \cup \mathscr{R}_2}$. The restriction of $\Delta_{\mathscr{R}_1 \cup \mathscr{R}_2}$ to preredexes of $\mathscr{R}_i$ clearly satisfies the conditions for being Δ-sets with respect to $\mathscr{R}_i$ $(i=1,2)$. Theorem 5.7 yields the strong sequentiality of $\mathscr{R}_1$ and $\mathscr{R}_2$.

$\Leftarrow$ If $\mathscr{R}_i$ is strongly sequential, there exists Δ-sets $\Delta_{\mathscr{R}_i}$ for proper preredexes of $\mathscr{R}_i$ $(i=1,2)$. Define $\Delta_{\mathscr{R}_1 \cup \mathscr{R}_2}$ by

$$\Delta_{\mathscr{R}_1 \cup \mathscr{R}_2}(t) = \begin{cases} \Delta_{\mathscr{R}_1}(t) & \text{if } t \text{ is a proper preredex of } \mathscr{R}_1, \\ \\ \Delta_{\mathscr{R}_2}(t) & \text{if } t \text{ is a proper preredex of } \mathscr{R}_2. \end{cases}$$

It is very easy to show that $\Delta_{\mathcal{R}_1 \cup \mathcal{R}_2}$ satisfies the conditions for being Δ-sets with respect to $\mathcal{R}_1 \cup \mathcal{R}_2$. Therefore $\mathcal{R}_1 \cup \mathcal{R}_2$ is strongly sequential.

□

It should be noted that in order to apply the previous proposition for deciding the strong sequentiality of a term rewriting system $\mathcal{R}$, it is sufficient that $\mathcal{R}$ can be partitioned into $\mathcal{R}_1 \cup \mathcal{R}_2$ such that the left-hand sides of $\mathcal{R}_1$ and $\mathcal{R}_2$ do not have any function symbols in common.

REMARK. Sequentiality*, as defined in Definition 2.4, is not a modular property. For instance, the trivial term rewriting system $\mathrm{I} = \{I(x) \to x\}$ is strongly sequential (and hence sequential*, cf. Figure 2.1). We have already observed that Berry's term rewriting system

$$\mathrm{B} = \begin{cases} F(A,B,x) & \to & C, \\ F(B,x,A) & \to & C, \\ F(x,A,B) & \to & C, \end{cases}$$

is sequential*, but $\mathrm{I} \oplus \mathrm{B}$ is not sequential*:

$$\begin{aligned} &F(\underline{I(A)},\underline{I(B)},r) \twoheadrightarrow \underline{F(A,B,r)} \to C, \\ &F(\underline{I(B)},r,\underline{I(A)}) \twoheadrightarrow \underline{F(B,r,A)} \to C, \\ &F(r,\underline{I(A)},\underline{I(B)}) \twoheadrightarrow \underline{F(r,A,B)} \to C. \end{aligned}$$

7. Different Notions of Sequentiality

In this last section we discuss two different notions of sequentiality. The first one is *left sequentiality* introduced by Thatte [19] (not to be confused with the notion of left sequentiality by Hoffmann and O'Donnell [6]). Left sequentiality is intuitively more satisfactory than strong sequentiality, but Thatte showed that the notions coincide for the subclass of constructor systems. We will give a simple proof of this fact. Thatte also showed that left sequentiality is necessary for safe computation based on the analysis of left-hand sides alone, again for the subclass of constructor systems. The second notion of sequentiality we discuss is *sufficient sequentiality* introduced by Oyamaguchi [16]. Sufficient sequentiality is not only based on the analysis of the left-hand sides of the term rewriting systems (as is the case for strong and left sequentiality) but also on the non-variable parts of the right-hand sides. Oyamaguchi showed that the class of sufficiently sequential term rewriting systems properly includes the class of strongly sequential systems. Furthermore, he established the decidability of sufficient sequentiality.

The following example from Thatte motivates the introduction of left sequentiality.

EXAMPLE 7.1. Let

$$\mathcal{R} = \begin{cases} F(A,B,x) & \to & x, \\ F(B,x,A) & \to & x, \\ F(x,A,B) & \to & x, \\ G(A) & \to & A. \end{cases}$$

Consider the term $t \equiv F(G(\Omega),G(\Omega),\Omega)$. The third occurrence of Ω in t is not an index with respect to strong sequentiality (r_1, r_2 and r_3 are arbitrary redexes):

$$\begin{aligned} F(G(r_1),G(r_2),r_3) &\to_? F(G(A),G(r_2),r_3) \to_? F(A,G(r_2),r_3) \\ &\to_? F(A,G(A),r_3) \to_? F(A,B,r_3) \to_? A. \end{aligned}$$

In the second step we replaced the redex $G(A)$ by A and in the fourth step we replaced the same redex by B. However, using Theorem 1.4 one easily shows that there does not exists a term rewriting system $\mathcal{R}'$ with the same left-hand sides as $\mathcal{R}$ such that $G(r_1) \twoheadrightarrow_{\mathcal{R}'} A$ and $G(r_2) \twoheadrightarrow_{\mathcal{R}'} B$. Therefore, the above arbitrary reduction sequence is impossible for any system based on the left-hand sides of $\mathcal{R}$.

DEFINITION 7.2.

(1) Two term rewriting systems $\mathcal{R}_1,\mathcal{R}_2$ are *left equivalent*, notation $\mathcal{R}_1 \sim_l \mathcal{R}_2$, if they have the same left-hand sides, i.e. $\mathcal{R}_1 = \{l_i \to r_i^1 \mid 1 \le i \le n\}$ and $\mathcal{R}_2 = \{l_i \to r_i^2 \mid 1 \le i \le n\}$ for some terms l_i, r_i^1, r_i^2 $(i = 1,\ldots,n)$.

(2) The monotonic predicate Inf is defined on $\mathcal{T}_\Omega$ by

$$\mathit{Inf}(t) \text{ holds } \Leftrightarrow\ t \twoheadrightarrow_{\mathcal{R}'} t' \text{ for some } \mathcal{R}' \sim_l \mathcal{R} \text{ and } t' \in \mathrm{NF}.$$

(3) An orthogonal term rewriting system is *left sequential* if every $t \in \mathrm{NF}_\Omega$ has an index with respect to Inf.

EXAMPLE 7.3. The term t in Example 7.1 does not have an index with respect to strong sequentiality, but $I_{\mathit{Inf}}(t) = \{3\}$ because $t_1 \ge t$ and $t_1/3 \equiv \Omega$ imply that there does not exist a term rewriting system $\mathcal{R}_1 \sim_l \mathcal{R}$ such that $t_1 \twoheadrightarrow_{\mathcal{R}'} t_2$ for some normal form t_2. Notice that $\mathcal{R}$ is not left sequential: $I_{\mathit{Inf}}(F(\Omega,\Omega,\Omega)) = \varnothing$.

PROPOSITION 7.4.

(1) *Every strongly sequential term rewriting system is left sequential.*

(2) *Every left sequential term rewriting system is sequential.*

PROOF.

(1) Suppose $\mathcal{R}$ is strongly sequential. Take $t \in \mathrm{NF}_\Omega$ and $u \in I_{nf_?}(t)$. We will show that $u \in I_{lnf}(t)$. Let $t' \geq t$ such that $lnf(t')$ holds. Then $nf_?(t')$ also holds and we obtain $t'/u \not\equiv \Omega$ from the assumption $u \in I_{nf_?}(t)$.

(2) Similar to (1), using the implication $nf(t') \Rightarrow lnf(t')$.

□

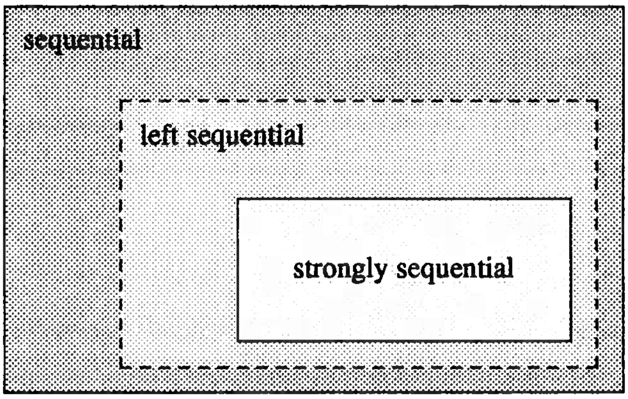


FIGURE 7.1.

PROPOSITION 7.5. *Let $\mathcal{R}$ be a constructor system. If $\mathcal{R}$ is left sequential then $\mathcal{R}$ is strongly sequential.*

PROOF. According to Corollary 6.7 we have to show that every proper preredex of $\mathcal{R}$ has an index with respect to strong sequentiality. Let t be a proper preredex of $\mathcal{R}$ and take some $u \in I_{lnf}(t)$. Suppose u is not an index with respect to strong sequentiality. Then $t[u \leftarrow \bullet]$ is redex compatible by Proposition 6.8 and hence there exists a redex $t' \geq t[u \leftarrow \bullet]$. Clearly $t'' \equiv t[u \leftarrow \Omega]$ also is a redex. Let $l \rightarrow r$ be the rewriting rule of $\mathcal{R}$ such that t'' is an instance of l. Choose some ground normal form r' and let $\mathcal{R}' = \mathcal{R} - \{l \rightarrow r\} \cup \{l \rightarrow r'\}$. Now we have $t'' \rightarrow_{\mathcal{R}'} r'$, $t'' \geq t$ and $t''/u \equiv \Omega$ which contradicts the assumption $u \in I_{lnf}(t)$. We conclude that $\mathcal{R}$ is strongly sequential. □

Thatte writes: "It is less obvious that our results apply to the full class of orthogonal systems." We conjecture that left sequentiality does not coincide with strong sequentiality: the non-constructor system

$$\mathcal{R} = \begin{cases} F(G(A,x),F(A,A)) & \rightarrow \; x, \\ F(G(x,A),F(B,B)) & \rightarrow \; x, \\ F(C_1,F(D_1,G(A,x))) & \rightarrow \; x, \\ F(C_2,F(D_2,G(x,A))) & \rightarrow \; x, \\ G(E,E) & \rightarrow \; E, \end{cases}$$

is not strongly sequential (the term $F(G(\Omega,\Omega),F(G(\Omega,\Omega),G(\Omega,\Omega)))$ does not have an index with respect to $nf_?$) but we think that $\mathcal{R}$ is left sequential.

This concludes our discussion of left sequentiality. We now turn our attention to sufficient sequentiality.

DEFINITION 7.6.

(1) The reduction relation $\rightarrow_!$ is defined as follows:

$$t_1 \rightarrow_! t_2$$

if there exists a context $C[\,]$, a reduction rule $l \rightarrow r$ and a substitution σ such that $t_1 \equiv C[l^\sigma]$, $t_2 \equiv C[t]$ for some term $t \geq r_\Omega$ where $r_\Omega \equiv r[u \leftarrow \Omega \mid r/u \in \mathcal{V}]$.

(2) The predicate $term_!$ is defined on $\mathcal{T}_\Omega$ as follows:

$$term_!(t) \text{ holds} \iff t \twoheadrightarrow_! t' \text{ for some } t' \in \mathcal{T}(\mathcal{F},\mathcal{V}).$$

(3) An orthogonal term rewriting system is *sufficiently sequential* if every $t \in \mathrm{NF}_\Omega$ has an index with respect to $term_!$.

It would be more natural to define sufficient sequentiality in terms of a predicate $nf_!$:

$$nf_!(t) \text{ holds} \iff t \twoheadrightarrow_! t' \text{ for some normal form } t',$$

but Oyamaguchi argued that it will be very difficult to obtain an (efficient) algorithm for finding indices with respect to $nf_!$. Oyamaguchi showed that the computation of indices with respect to $term_!$ can be done in polynomial time.

PROPOSITION 7.7.

(1) *Every strongly sequential term rewriting system is sufficiently sequential.*

(2) *Every sufficiently sequential term rewriting system is sequential.*

PROOF. Similar to the proof of Proposition 7.4, using the implications $term_!(t) \Rightarrow nf_?(t)$ and $nf(t) \Rightarrow term_!(t)$. □

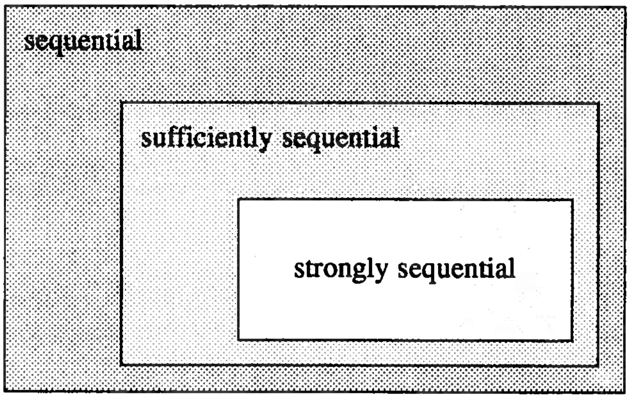


FIGURE 7.2.

Oyamaguchi showed that the first inclusion of Proposition 7.7 is proper by means of the following term rewriting system:

$$\mathcal{R} = \begin{cases} F(F(A,x),F(B,y)) & \rightarrow & F(E,E), \\ F(F(x,A),F(C,y)) & \rightarrow & F(E,E), \\ F(D,D) & \rightarrow & F(E,E). \end{cases}$$

Because $F(F(\Omega,\Omega),F(F(\Omega,\Omega),\Omega))$ does not have an index with respect to $nf_?$, $\mathcal{R}$ is not strongly sequential. The proof that $\mathcal{R}$ is sufficiently sequential can be found in [16], where also the decidability of sufficient sequentiality is shown.

THEOREM 7.8 (Oyamaguchi [16]). *Sufficient sequentiality is a decidable property of orthogonal term rewriting systems.* □

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