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Confluence of the Disjoint Union of Conditional Term Rewriting Systems

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ABSTRACT

Toyama proved that confluence is a modular property of term rewriting systems. This means that the disjoint union of two confluent term rewriting systems is again confluent. In this paper we extend his result to the class of conditional term rewriting systems. In view of the important role of conditional rewriting in equational logic programming, this result may be of relevance in integrating functional programming and logic programming.

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Introduction

Two directions can be distinguished in the use of conditional term rewriting systems. Bergstra and Klop [1], Kaplan [11] and Zhang and Rémy [20] studied conditional term rewriting as a means of enhancing the expressiveness in the algebraic specification of abstract data types. Recently, serious efforts have been initiated for integrating functional and logic programming. It has been recognized that conditional term rewriting systems provide a natural computational mechanism for this integration, see Dershowitz and Plaisted [6,7], Fribourg [8] and Goguen and Meseguer [9].

For ordinary term rewriting systems a sizeable amount of theory has been developed. Only a small part has been extended to conditional term rewriting systems, notably sufficient conditions for confluence and termination ([1], [4], [5], [10], [11]). In this paper we extend a result of Toyama [17], which states that if a term rewriting system can be partitioned into two confluent systems with disjoint alphabets then the original system is confluent, to conditional term rewriting systems.

Conditional term rewriting is introduced in the next section. Section 2 contains a short overview of previous work on disjoint unions of term rewriting systems. In Section 3 we prove that confluence is a modular property of join systems, a particular form of conditional term rewriting introduced in the next section. Section 4 contains the necessary changes in order to make the proof of Section 3 also applicable to the so-called semi-equational and normal systems. We conclude in Section 5 with suggestions for further research.

1. Conditional Term Rewriting Systems: Preliminaries

Before we introduce conditional term rewriting, we review the basic notions of unconditional term rewriting. Term rewriting is surveyed in Klop [12] and Dershowitz and Jouannaud [2].

Let $\mathcal{V}$ be a countably infinite set of *variables*. An *unconditional term rewriting system* (TRS for short) is a pair $(\mathcal{F},\mathcal{R})$. The set $\mathcal{F}$ consists of *function symbols*; associated to every $f\in\mathcal{F}$ is its arity $n\geq 0$. Function symbols of arity 0 are called *constants*. The set of terms built from $\mathcal{F}$ and $\mathcal{V}$, notation $\mathcal{T}(\mathcal{F},\mathcal{V})$, is the smallest set such that:

- $\mathcal{V}\subset\mathcal{T}(\mathcal{F},\mathcal{V})$,
- if $f\in\mathcal{F}$ has arity n and $t_1,\ldots,t_n\in\mathcal{T}(\mathcal{F},\mathcal{V})$ then $f(t_1,\ldots,t_n)\in\mathcal{T}(\mathcal{F},\mathcal{V})$.

Terms not containing variables are *ground* terms. The set $\mathcal{R}$ consists of pairs (l,r) with $l,r\in\mathcal{T}(\mathcal{F},\mathcal{V})$ subject to two constraints:

(1) the left-hand side l is not a variable,

(2) the variables which occur in the right-hand side r also occur in l.

Pairs (l,r) are called *rewrite rules* or *reduction rules* and will henceforth be written as $l\to r$. We usually present a TRS as a set of rewrite rules, without making explicit the set of function symbols. A rewrite rule $l\to r$ is *left-linear* if l does not contain multiple occurrences of the same variable. The rule $l\to r$ is *collapsing* if r is a single variable and it is *duplicating* if r contains more occurrences of some variable than l does.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that the set $\{x\in\mathcal{V}\mid\sigma(x)\not\equiv x\}$ is finite (the symbol $\equiv$ stands for syntactic equality). This set is called the *domain* of σ and will be denoted by $\mathcal{D}(\sigma)$. Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F},\mathcal{V})$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$, i.e. $\sigma(f(t_1,\ldots,t_n))\equiv f(\sigma(t_1),\ldots,\sigma(t_n))$ for every n-ary function symbol f and terms $t_1,\ldots,t_n$. We call $\sigma(t)$ an *instance* of t. An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression). Occasionally we present a substitution σ as $\sigma=\{x\to\sigma(x)\mid x\in\mathcal{D}(\sigma)\}$. The *empty* substitution will be denoted by ε (here $\mathcal{D}(\varepsilon)=\varnothing$).

A *context* $C[\,,\ldots,\,]$ is a 'term' which contains at least one occurrence of a special symbol $\Box$. If $C[\,,\ldots,\,]$ is a context with n occurrences of $\Box$ and $t_1,\ldots,t_n$ are terms then $C[t_1,\ldots,t_n]$ is the result of replacing from left to right the occurrences of □ by $t_1,\ldots,t_n$. A context containing precisely one occurrence of □ is denoted by $C[\,]$. A term s is a *subterm* of a term t if there exists a context $C[\,]$ such that $t \equiv C[s]$.

The *rewrite relation* $\rightarrow_{\mathcal{R}} \subset \mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$ is defined as follows: $s \rightarrow_{\mathcal{R}} t$ if there exists a rewrite rule $l \rightarrow r$ in $\mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$ and $t \equiv C[\sigma(r)]$. The transitive-reflexive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$; if $s \twoheadrightarrow_{\mathcal{R}} t$ we say that s *reduces* to t. We write $s \leftarrow_{\mathcal{R}} t$ if $t \rightarrow_{\mathcal{R}} s$; likewise for $s \twoheadleftarrow_{\mathcal{R}} t$. The symmetric closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\leftrightarrow_{\mathcal{R}}$ (so $\leftrightarrow_{\mathcal{R}} = \rightarrow_{\mathcal{R}} \cup \leftarrow_{\mathcal{R}}$). The transitive-reflexive closure of $\leftrightarrow_{\mathcal{R}}$ is called *conversion* and denoted by $=_{\mathcal{R}}$. If $s =_{\mathcal{R}} t$ then s and t are *convertible*. Two terms t_1,t_2 are *joinable*, notation $t_1 \downarrow_{\mathcal{R}} t_2$, if there exists a term t_3 such that $t_1 \twoheadrightarrow_{\mathcal{R}} t_3 \twoheadleftarrow_{\mathcal{R}} t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2. The relation $\downarrow_{\mathcal{R}}$ is called *joinability*. We often omit the subscript $\mathcal{R}$.

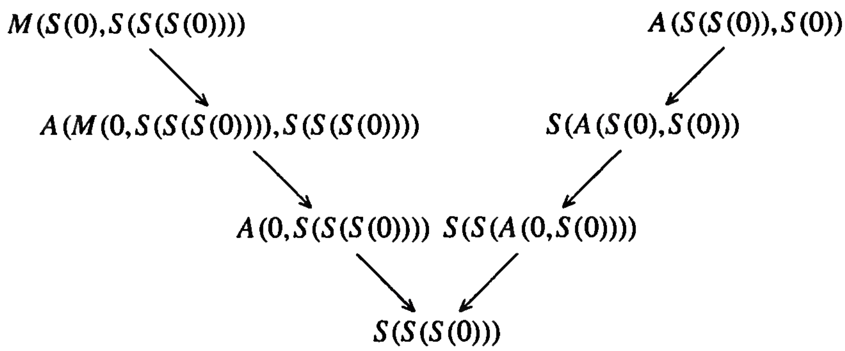
A term s is a *normal form* if there are no terms t with $s \rightarrow t$. A TRS is *terminating* or *strongly normalizing* if there are no infinite reduction sequences $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \ldots$. In other words, every reduction sequence eventually ends in a normal form. A TRS is *confluent* or has the *Church-Rosser* property if for all terms s,t_1,t_2 with $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$ we have $t_1 \downarrow t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable ($t_1 = t_2 \Rightarrow t_1 \downarrow t_2$).

EXAMPLE 1.1. Consider the TRS $\mathcal{R}$ of Table 1. The terms $M(S(0),S(S(S(0))))$ and $A(S(S(0)),S(0))$

$A(0,x)$	$\rightarrow$	x
$A(S(x),y)$	$\rightarrow$	$S(A(x,y))$
$M(0,x)$	$\rightarrow$	0
$M(S(x),y)$	$\rightarrow$	$A(M(x,y),y)$

TABLE 1.

are joinable (and hence convertible):



One easily proves that $\mathcal{R}$ is terminating and confluent.

The rewrite rules of a *conditional term rewriting system* (CTRS) have the form

$$l \rightarrow r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$$

with $s_1,\ldots,s_n,t_1,\ldots,t_n,l,r \in \mathcal{T}(\mathcal{F},\mathcal{V})$. The equations $s_1 = t_1,\ldots,s_n = t_n$ are the *conditions* of the rewrite rule. Depending on the interpretation of the =-sign in the conditions, different rewrite relations can be associated to a given CTRS. In this paper we restrict ourselves to the three most common interpretations.

(1) *Join systems.*
In a join CTRS the =–sign in the conditions is interpreted as *joinability*. Formally: $s \to t$ if there exists a conditional rewrite rule $l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow \sigma(t_i)$ for all $i \in \{1, \ldots, n\}$. Rewrite rules of a join CTRS will henceforth be written as

$$l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n.$$

(2) *Semi-equational systems.*
Semi-equational CTRS's are obtained by interpreting the =–sign in the conditions as *conversion*.

(3) *Normal systems.*
In a normal CTRS the rewrite rules are subject to the constraint that every t_i is a ground normal form with respect to the rewrite relation obtained by interpreting the =–sign in the conditions as *reduction* ($\twoheadrightarrow$). Rewrite rules of a normal CTRS will be presented as

$$l \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_n \twoheadrightarrow t_n.$$

This classification originates essentially from Bergstra and Klop [1]. The nomenclature stems from Dershowitz, Okada and Sivakumar [5].

The restrictions we impose on CTRS's $\mathcal{R}$ in any of the three formulations are the same as for unconditional TRS's: if $l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$ is a rewrite rule of $\mathcal{R}$ then l is not a single variable and variables occurring in r also occur in l. In particular, extra variables in the conditions like in the last rule of the normal CTRS of Table 2 cause no problems whatsoever. In Section 5 we will

$x \le x$	$\to$	*true*			
$x \le S(x)$	$\to$	*true*			
$x \le z$	$\to$	*true*	$\Leftarrow$	$x \le y \twoheadrightarrow$ *true*,	$y \le z \twoheadrightarrow$ *true*

TABLE 2.

discuss the technical problems associated to a possible relaxation of this requirement.

Sufficient conditions for the termination of CTRS's were given by Kaplan [11], Jouannaud and Waldmann [10] and Dershowitz, Okada and Sivakumar [5]. Sufficient conditions for confluence can be found in Bergstra and Klop [1] and Dershowitz, Okada and Sivakumar [4].

EXAMPLE 1.2. The semi-equational CTRS of Table 3(i) is easily shown to be confluent. So conversion in that system coincides with joinability. However, the corresponding join CTRS of Table 3(ii) is not confluent: the reduction step from b to c is no longer allowed.

$a \to b$		
$a \to c$		
$b \to c$	$\Leftarrow$	$b = c$

(i)

$a \to b$		
$a \to c$		
$b \to c$	$\Leftarrow$	$b \downarrow c$

(ii)

TABLE 3.

The following definition is fundamental ([1], [4], [5]) for analyzing the behaviour of CTRS's.

DEFINITION 1.3. Suppose $s \to t$ by application of a rewrite rule $A: l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$ with substitution σ. The *depth* of $s \to t$ is inductively defined by the following clauses:

(1) If A is an unconditional rule (i.e. $n=0$) then the depth of $s \to t$ equals zero.
(2) If A is a conditional rule then the depth of $s \to t$ is one more than the maximum depth of the conversions $\sigma(s_i)=\sigma(t_i)$ $(i=1,\ldots,n)$.
(3) The depth of a conversion $s=t$ is the maximum depth of the individual rewrite steps in $s=t$.

This definition also applies to join systems and normal systems since joinability and reduction are special cases of conversion.

Notice that Definition 1.3 is slightly inaccurate because in general there are several ways to prove that two terms are convertible, each with its own depths. The disturbed reader can take the minimum of all depths that can be associated to the single rewrite step $s \to t$.

EXAMPLE 1.4. Consider the semi-equational CTRS of Table 4. The depth of $a \to b$ is 1, the depth of $b \to c$ is 2 and the depth of $c \to d$ is 3.

$a \to b$	$\Leftarrow$	$a=a$
$b \to c$	$\Leftarrow$	$a=b$
$c \to d$	$\Leftarrow$	$a=c$

TABLE 4.

2. Modular Properties

In this section we review some of the results that have been obtained concerning the disjoint union of TRS's. We will also give the necessary technical definitions and notations for dealing with disjoint unions. These are consistent with [17, 19, 14].

DEFINITION 2.1. Let $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ be CTRS's. The *disjoint union* of $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ is denoted by $\mathcal{R}_1 \oplus \mathcal{R}_2$. That is, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint $(\mathcal{F}_1 \cap \mathcal{F}_2 = \varnothing)$ then $\mathcal{R}_1 \oplus \mathcal{R}_2 = (\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$; otherwise we rename the function symbols of $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ such that the resulting copies $(\mathcal{F}_1',\mathcal{R}_1')$ and $(\mathcal{F}_2',\mathcal{R}_2')$ have disjoint sets of function symbols, and define $\mathcal{R}_1 \oplus \mathcal{R}_2 = (\mathcal{F}_1' \cup \mathcal{F}_2', \mathcal{R}_1' \cup \mathcal{R}_2')$.

DEFINITION 2.2. A property $\mathcal{P}$ of CTRS's is called *modular* if for all CTRS's $\mathcal{R}_1$, $\mathcal{R}_2$ the following equivalence holds:

$$\mathcal{R}_1 \oplus \mathcal{R}_2 \text{ has the property } \mathcal{P} \iff \text{both } \mathcal{R}_1 \text{ and } \mathcal{R}_2 \text{ have the property } \mathcal{P}.$$

All previous research on modularity has been carried out in the world of unconditional TRS's. This research can be characterized by the phrase "simple statements, complicated proofs". Confluence was the first property for which the modularity has been established.

THEOREM 2.3 (Toyama [17]). *Confluence is a modular property of* TRS'*s.* □

Toyama also gave the following simple example showing that termination is not modular.

EXAMPLE 2.4 (Toyama [18]). Let $\mathcal{R}_1 = \{F(0,1,x) \to F(x,x,x)\}$ and

$$\mathcal{R}_2 = \begin{cases} or(x,y) \to x, \\ or(x,y) \to y. \end{cases}$$

Both systems are terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits the following cyclic reduction:

$$\begin{aligned} F(or(0,1),or(0,1),or(0,1)) &\to F(0,or(0,1),or(0,1)) \\ &\to F(0,1,or(0,1)) \\ &\to F(or(0,1),or(0,1),or(0,1)). \end{aligned}$$

Notice that $\mathcal{R}_2$ is not confluent.

In view of this example, Toyama conjectured the modularity of the combination of confluence and termination, but Barendregt and Klop constructed a counterexample involving a non-left-linear TRS (see [18]). Recently, Toyama, Klop and Barendregt [19] gave an extremely complicated proof showing the modularity of the combination of confluence and termination for the restricted class of left-linear TRS's. Modular aspects of properties dealing with the uniqueness of normal forms have been studied by the present author [14]. Rusinowitch [16] and Middeldorp [15] gave sufficient conditions for the termination of $\mathcal{R}_1 \oplus \mathcal{R}_2$ in terms of the distribution of collapsing and duplicating rules among $\mathcal{R}_1$ and $\mathcal{R}_2$. An interesting alternative approach to modularity in term rewriting is explored in Kurihara and Kaji [13].

Let $(\mathcal{F}_1,\mathcal{R}_1)$ and $(\mathcal{F}_2,\mathcal{R}_2)$ be disjoint CTRS's. Every term $t \in \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2,\mathcal{V})$ can be viewed as an alternation of $\mathcal{F}_1$-parts and $\mathcal{F}_2$-parts. This structure is formalized in Definition 2.5, see Figure 1.

NOTATION. We abbreviate $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2,\mathcal{V})$ to $\mathcal{T}$ and we will use $\mathcal{T}_i$ as a shorthand for $\mathcal{T}(\mathcal{F}_i,\mathcal{V})$ $(i=1,2)$.

DEFINITION 2.5.

(1) The *root symbol* of a term t, notation $root(t)$, is defined by

$$root(t) = \begin{cases} F & \text{if } t \equiv F(t_1,\ldots,t_n), \\ t & \text{otherwise.} \end{cases}$$

(2) Let $t \equiv C[t_1,\ldots,t_n] \in \mathcal{T}$ with $C[\,,\ldots,] \not\equiv \Box$. We write $t \equiv C[\![t_1,\ldots,t_n]\!]$ if $C[\,,\ldots,]$ is a $\mathcal{F}_a$-context and $root(t_i) \in \mathcal{F}_{3-a}$ for $i=1,\ldots,n$ $(a=1,2)$. The t_i's are the *principal* subterms of t.

(3) If $t \in \mathcal{T}$ then

$$rank(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T}_1 \cup \mathcal{T}_2, \\ 1+\max\{rank(t_i) \mid 1 \le i \le n\} & \text{if } t \equiv C[\![t_1,\ldots,t_n]\!]. \end{cases}$$

(4) The set $S(t)$ of *special* subterms of a term t is inductively defined by

$$S(t) = \begin{cases} \{t\} & \text{if } rank(t)=1, \\ \{t\} \cup S(t_1) \cup \ldots \cup S(t_n) & \text{if } t \equiv C[\![t_1,\ldots,t_n]\!]. \end{cases}$$

To achieve better readability we will call the function symbols of $\mathcal{F}_1$ *black* and those of $\mathcal{F}_2$ *white*. Variables have no colour. A black (white) term does not contain white (black) function symbols, but may contain variables. In examples, black symbols will be printed as capitals and white symbols in lower case. (This convention has already been used in Example 2.4.)

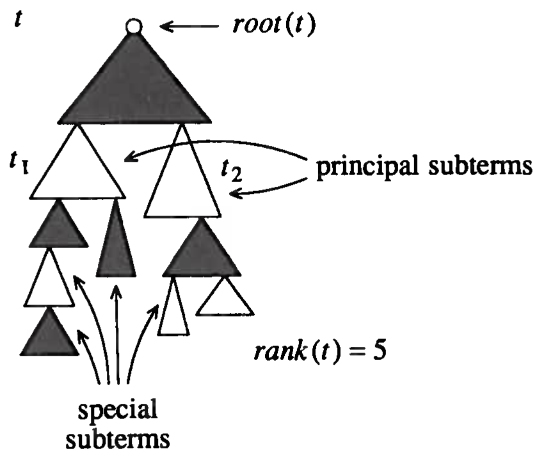


FIGURE 1.

PROPOSITION 2.6. *If $s \twoheadrightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ then $rank(s) \geq rank(t)$.*
PROOF. Straightforward. □

DEFINITION 2.7. Let $s \rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ by application of a rewrite rule A. We write $s \rightarrow^i t$ if $s \equiv C[\![s_1, \ldots, s_n]\!]$ and A is being applied in one of the s_j's. We write $s \rightarrow^o t$ if not $s \rightarrow^i t$. The relation $\rightarrow^i$ is called *inner* reduction and $\rightarrow^o$ is called *outer* reduction.

DEFINITION 2.8. Suppose σ and τ are substitutions. We write $\sigma \propto \tau$ if $\sigma(x) \equiv \sigma(y)$ implies $\tau(x) \equiv \tau(y)$ for all $x, y \in \mathcal{V}$. Notice that $\sigma \propto \varepsilon$ if and only if σ is injective. We write $\sigma \twoheadrightarrow \tau$ if $\sigma(x) \twoheadrightarrow \tau(x)$ for all $x \in \mathcal{V}$. Clearly $\sigma(t) \twoheadrightarrow \tau(t)$ whenever $\sigma \twoheadrightarrow \tau$.

DEFINITION 2.9. A substitution σ is called *black* (*white*) if $\sigma(x)$ is a black (white) term for every $x \in \mathcal{D}(\sigma)$. We call σ *top black* (*top white*) if the root symbol of $\sigma(x)$ is black (white) for every $x \in \mathcal{D}(\sigma)$.

Notice the subtle difference in handling variables: the substitution $\sigma = \{x \rightarrow F(y), y \rightarrow x\}$ is black but not top black. The following proposition is frequently used in the next section.

PROPOSITION 2.10. *Every substitution σ can be decomposed into $\sigma_2 \circ \sigma_1$ such that σ_1 is black (white), σ_2 is top white (top black) and $\sigma_2 \propto \varepsilon$.*
PROOF. Let $\{t_1, \ldots, t_n\}$ be the set of all maximal subterms of $\sigma(x)$ for $x \in \mathcal{D}(\sigma)$ with white root. Choose distinct fresh variables $z_1, \ldots, z_n$ and define the substitution σ_2 by $\sigma_2 = \{z_i \rightarrow t_i \mid 1 \leq i \leq n\}$. Let $x \in \mathcal{D}(\sigma)$. We define $\sigma_1(x)$ by case analysis (see Figure 2).
(1) If the root symbol of $\sigma(x)$ is white then $\sigma(x) \equiv t_i$ for some $i \in \{1, \ldots, n\}$. In this case we define $\sigma_1(x) \equiv z_i$.
(2) If $\sigma(x)$ is a black term then we take $\sigma_1(x) \equiv \sigma(x)$.
(3) In the remaining case we can write $\sigma(x) \equiv C[\![t_{i_1}, \ldots, t_{i_k}]\!]$ for some $1 \leq i_1, \ldots, i_k \leq n$ and we define $\sigma_1(x) \equiv C[z_{i_1}, \ldots, z_{i_k}]$.

By construction we have $\sigma_2 \propto \varepsilon$, σ_1 is black and σ_2 is top white. □

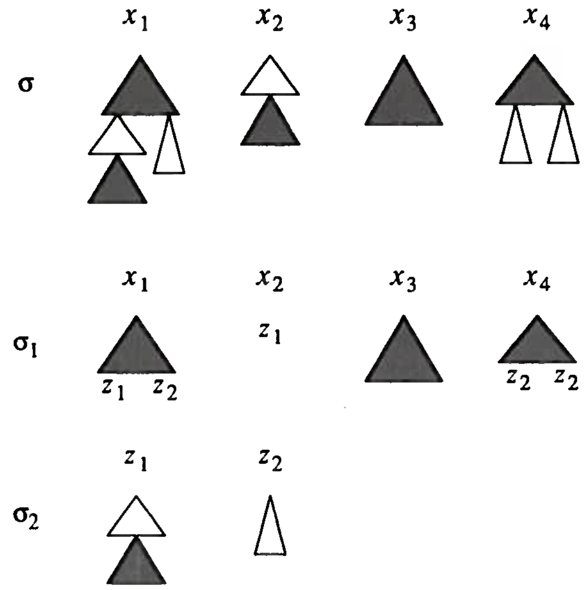


FIGURE 2.

3. Join Systems

In this section we show that confluence is a modular property of join CTRS's. To this end, we assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint confluent join CTRS's. We assume furthermore that all rewrite relations introduced in this section are defined on $\mathcal{T}$, unless stated otherwise. The same assumption is made for terms.

The fundamental property of the disjoint union of two unconditional TRS's $\mathcal{R}_1$ and $\mathcal{R}_2$, that is to say $s \rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ implies either $s \rightarrow_{\mathcal{R}_1} t$ or $s \rightarrow_{\mathcal{R}_2} t$, does no longer hold for CTRS's, as can be seen from the next example.

EXAMPLE 3.1. Let

$$\mathcal{R}_1 = \{ F(x,y) \rightarrow G(x) \Leftarrow x \downarrow y \},$$
$$\mathcal{R}_2 = \{ a \rightarrow b \}.$$

We have $F(a,b) \rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} G(a)$ because $a \downarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} b$, but neither $F(a,b) \rightarrow_{\mathcal{R}_1} G(a)$ nor $F(a,b) \rightarrow_{\mathcal{R}_2} G(a)$.

The problem is that when a rule of one of the CTRS's is being applied, rules of the other CTRS may be needed in order to satisfy the conditions. So the question arises how the rewrite relation $\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ is related to $\rightarrow_{\mathcal{R}_1}$ and $\rightarrow_{\mathcal{R}_2}$. In the example above we have

$$F(a,b) \rightarrow_{\mathcal{R}_2} F(b,b) \rightarrow_{\mathcal{R}_1} G(b) \leftarrow_{\mathcal{R}_2} G(a).$$

This suggests that $\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ corresponds to joinability with respect to the union of $\rightarrow_{\mathcal{R}_1}$ and $\rightarrow_{\mathcal{R}_2}$. However, it turned out that $\rightarrow_{\mathcal{R}_1} \cup \rightarrow_{\mathcal{R}_2}$ is not an entirely satisfactory relation from a technical viewpoint. For instance, confluence of $\rightarrow_{\mathcal{R}_1} \cup \rightarrow_{\mathcal{R}_2}$ is not easily proved (cf. Lemma 3.6). We will define two more manageable rewrite relations $\rightarrow_1$ and $\rightarrow_2$ such that:

(1) their union is confluent (Lemma 3.6),

(2) reduction in $\mathcal{R}_1 \oplus \mathcal{R}_2$ corresponds to joinability with respect to $\rightarrow_1 \cup \rightarrow_2$ (Lemma 3.7).

From these two properties the modularity of confluence for join CTRS's is easily inferred (Theorem 3.19). The proof of the first property is a more or less straightforward reduction to Toyama's confluence result for the disjoint union of TRS's. The proof of the second property is rather technical but we believe that the underlying ideas are simple. The reader familiar with Toyama [17] will notice that some of our proof techniques are imported from his paper. Contrary to usual mathematical practice we present certain parts of our proof in a top-down fashion in order to facilitate the accessibility to its structure. Figure 3 exhibits the dependencies between the various propositions, lemma's and theorems.

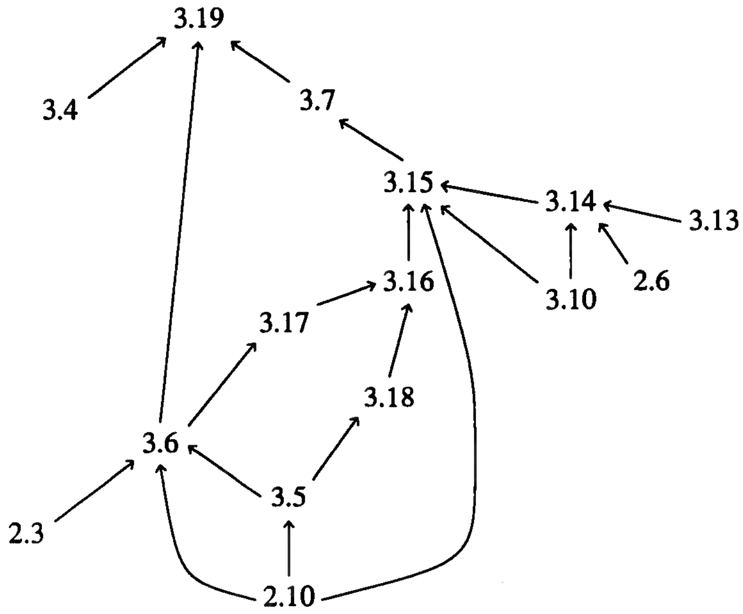


FIGURE 3.

DEFINITION 3.2. The rewrite relation $\rightarrow_\mathrm{I}$ is defined as follows: $s \rightarrow_\mathrm{I} t$ if there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a context $C[\,]$ and a substitution σ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow_\mathrm{I}^o \sigma(t_i)$ for $i = 1, \ldots, n$, where the superscript o in $\sigma(s_i) \downarrow_\mathrm{I}^o \sigma(t_i)$ means that $\sigma(s_i)$ and $\sigma(t_i)$ are joinable using only *outer* $\rightarrow_\mathrm{I}$-reduction steps. Notice that the restrictions of $\rightarrow_\mathrm{I}$ and $\rightarrow_{\mathcal{R}_1}$ to $\mathcal{T}_1 \times \mathcal{T}_1$ coincide. The relation $\rightarrow_2$ is defined similarly.

EXAMPLE 3.3. Let

$$\mathcal{R}_1 = \begin{cases} F(x,y) & \rightarrow & G(x) & \Leftarrow & x \downarrow y \\ A & \rightarrow & B \end{cases}$$

and suppose $\mathcal{R}_2$ contains an unary function symbol g. We have $F(g(A),g(B)) \rightarrow_{\mathcal{R}_1} G(g(A))$ but not $F(g(A),g(B)) \rightarrow_\mathrm{I} G(g(A))$ because $g(A)$ and $g(B)$ are different normal forms with respect to $\rightarrow_\mathrm{I}^o$. The terms $F(g(A),g(B))$ and $G(g(A))$ are joinable with respect to $\rightarrow_\mathrm{I}$:

$$F(g(A),g(B)) \rightarrow_\mathrm{I} F(g(B),g(B)) \rightarrow_\mathrm{I} G(g(B)) \leftarrow_\mathrm{I} G(g(A)).$$

NOTATION. The union of $\rightarrow_1$ and $\rightarrow_2$ is denoted by $\rightarrow_{1,2}$ and we abbreviate $\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ to $\rightarrow$.

PROPOSITION 3.4. *If* $s \rightarrow_{1,2} t$ *then* $s \rightarrow t$.
PROOF. Trivial. □

The next proposition states a desirable property of $\to_1^o$-reduction. This very intuitive property is easily shown to hold for unconditional TRS's (cf. Lemma 3.2 in [17]), but the proof for CTRS's is a bit harder to write down.

PROPOSITION 3.5. *Let s,t be black terms and suppose σ is a top white substitution such that $\sigma(s) \to_1^o \sigma(t)$. If τ is a substitution with $\sigma \propto \tau$ then $\tau(s) \to_1^o \tau(t)$.*

PROOF. We prove the statement by induction on the depth of $\sigma(s) \to_1^o \sigma(t)$. The case of zero depth is straightforward. If the depth of $\sigma(s) \to_1^o \sigma(t)$ equals $n+1$ ($n \geq 0$) then there exist a context $C[\,]$, a substitution ρ and a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_m \downarrow t_m$ in $\mathcal{R}_1$ such that $\sigma(s) \equiv C[\rho(l)]$, $\sigma(t) \equiv C[\rho(r)]$ and $\rho(s_i) \downarrow_1^o \rho(t_i)$ for $i = 1, \ldots, m$ with depth less than or equal to n. Proposition 2.10 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black, ρ_2 is top white and $\rho_2 \propto \varepsilon$. It is not difficult to see that for every variable $x \in \mathcal{D}(\rho_2)$ there exists a variable $y \in \mathcal{D}(\sigma)$ with $\rho_2(x) \equiv \sigma(y)$, see Figure 4.

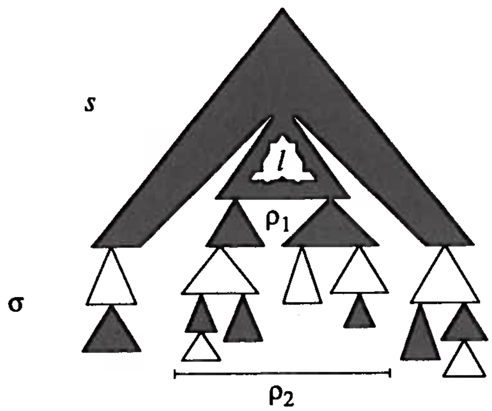
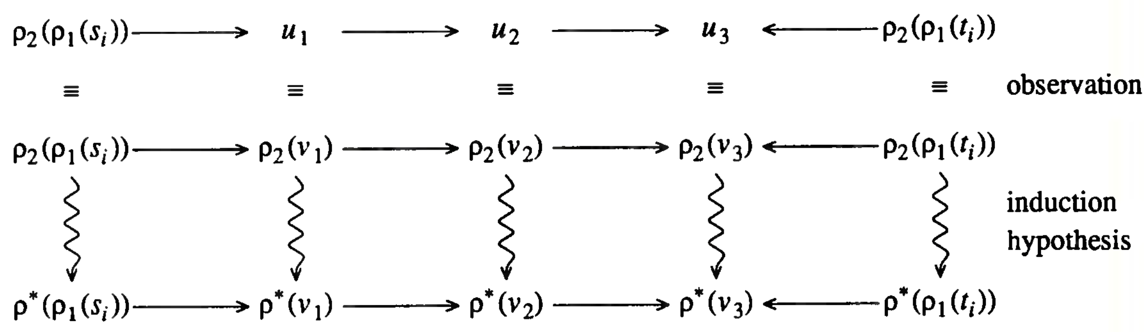


FIGURE 4.

We define the substitution ρ^* by $\rho^*(x) \equiv \tau(y)$ for every $x \in \mathcal{D}(\rho_2)$ and $y \in \mathcal{D}(\sigma)$ satisfying the above identity. Notice that ρ^* is well-defined by the assumption $\sigma \propto \tau$. We have $\rho_2 \propto \rho^*$ since $\rho_2 \propto \varepsilon$ and $\varepsilon \propto \rho^*$. Combined with $\rho_2(\rho_1(s_i)) \downarrow_1^o \rho_2(\rho_1(t_i))$, the induction hypothesis and the observation that if $\rho_2(u_1) \to_1^o u_2$ and u_1 is a black term then $u_2 \equiv \rho_2(u_3)$ for some black term u_3, we obtain $\rho^*(\rho_1(s_i)) \downarrow_1^o \rho^*(\rho_1(t_i))$ by a straightforward induction on the length of the conversion $\rho_2(\rho_1(s_i)) \downarrow_1^o \rho_2(\rho_1(t_i))$ for $i = 1, \ldots, m$ (see Figure 5). Hence $\rho^*(\rho_1(l)) \to_1^o \rho^*(\rho_1(r))$. Let $C^*[\,]$ be



In this picture $\to$ means $\to_1^o$.

FIGURE 5.

the context obtained from $C[\,]$ by replacing every principal subterm, which has the form $\sigma(x)$ for some variable $x \in \mathcal{D}(\sigma)$, by the corresponding $\tau(x)$. We leave it to the motivated reader to show that $\tau(s) \equiv C^*[\rho^*(\rho_1(l))]$ and $\tau(t) \equiv C^*[\rho^*(\rho_1(r))]$. We conclude that $\tau(s) \to_1^o \tau(t)$. □

LEMMA 3.6. *The rewrite relation $\rightarrow_{1,2}$ is confluent.*

PROOF. Define the unconditional TRS's $\mathcal{S}_1$ and $\mathcal{S}_2$ by $(i=1,2)$

$$\mathcal{S}_i = \{ s \rightarrow t \mid s,t \in \mathcal{T}_i \text{ and } s \rightarrow_i t \}.$$

The restrictions of $\rightarrow_{\mathcal{S}_i}$, $\rightarrow_i$ and $\rightarrow_{\mathcal{R}_i}$ to $\mathcal{T}_i \times \mathcal{T}_i$ are clearly the same. Therefore $\mathcal{S}_1$ and $\mathcal{S}_2$ are confluent TRS's. Theorem 2.3 yields the confluence of $\mathcal{S}_1 \oplus \mathcal{S}_2$. We will show that $\rightarrow_{\mathcal{S}_i}$ and $\rightarrow_i$ coincide (on $\mathcal{T} \times \mathcal{T}$). Without loss of generality, we only consider the case $i=1$.

$\subseteq$ If $s \rightarrow_{\mathcal{S}_1} t$ then there exists a rewrite rule $l \rightarrow r$ in $\mathcal{S}_1$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$ and $t \equiv C[\sigma(r)]$. By definition $l \rightarrow_1 r$, from which we immediately obtain $s \rightarrow_1 t$.

$\supseteq$ If $s \rightarrow_1 t$ then there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ for $i=1,\ldots,n$. According to Proposition 2.10 we can decompose σ into $\sigma_2 \circ \sigma_1$ such that σ_1 is black, σ_2 is top white and $\sigma_2 \propto \varepsilon$. Induction on the number of rewrite steps in $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ together with Proposition 3.5 and the observation made in the proof of Proposition 3.5 yields $\sigma_1(s_i) \downarrow_1^o \sigma_1(t_i)$ $(i=1,\ldots,n)$. Hence $\sigma_1(l) \rightarrow_1 \sigma_1(r)$. Because $\sigma_1(l)$ and $\sigma_1(r)$ are black terms, $\sigma_1(l) \rightarrow \sigma_1(r)$ is a rewrite rule of $\mathcal{S}_1$. Therefore $s \equiv C[\sigma_2(\sigma_1(l))] \rightarrow_{\mathcal{S}_1} C[\sigma_2(\sigma_1(r))] \equiv t$.

Now we have $\rightarrow_{\mathcal{S}_1 \oplus \mathcal{S}_2} = \rightarrow_{\mathcal{S}_1} \cup \rightarrow_{\mathcal{S}_2} = \rightarrow_1 \cup \rightarrow_2 = \rightarrow_{1,2}$. We conclude that $\rightarrow_{1,2}$ is confluent. □

LEMMA 3.7. *If $s \rightarrow t$ then $s \downarrow_{1,2} t$.*

PROOF. We use induction on the depth of $s \rightarrow t$. The case of zero depth is trivial. Suppose the depth of $s \rightarrow t$ equals $n+1$ $(n \geq 0)$. By definition there exist a context $C[\,]$, a substitution σ and a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_m \downarrow t_m$ in $\mathcal{R}_1 \oplus \mathcal{R}_2$ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow \sigma(t_i)$ $(i=1,\ldots,m)$ with depth less than or equal to n. Using the induction hypothesis and Lemma 3.6 we obtain $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ $(i=1,\ldots,m)$, see Figure 6 where (1) is obtained from the induction hypothesis

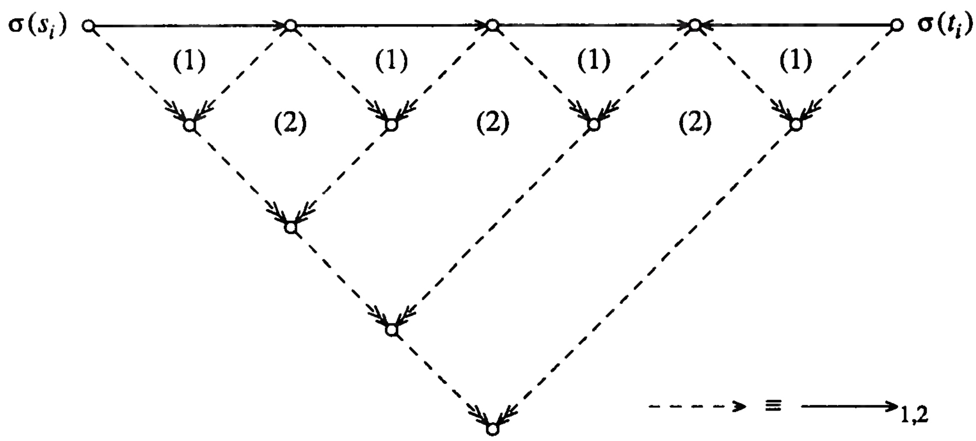


FIGURE 6.

and (2) signals an application of Lemma 3.6). Without loss of generality we assume that the applied rewrite rule stems from $\mathcal{R}_1$. Proposition 3.15 yields a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $\tau(s_i) \downarrow_1^o \tau(t_i)$ $(i=1,\ldots,m)$. The next conversion shows that $s \downarrow_{1,2} t$:

$$s \equiv C[\sigma(l)] \twoheadrightarrow_{1,2} C[\tau(l)] \rightarrow_1 C[\tau(r)] \twoheadleftarrow_{1,2} C[\sigma(r)] \equiv t.$$

□

Assume $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ is a rewrite rule of $\mathcal{R}_1$ and suppose σ is a substitution such that $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ for $i=1,\ldots,n$. We have to show the existence of a substitution τ with the properties $\sigma \twoheadrightarrow_{1,2} \tau$ and $\tau(s_i) \downarrow_1^o \tau(t_i)$ $(i=1,\ldots,n)$. First we show that σ has a 'preserved' normal form σ', meaning that in the layered structure (see Figure 1) of $\sigma'(x)$ $(x \in \mathcal{D}(\sigma'))$ no intermediate layer may disappear as a result of certain reduction steps.

DEFINITION 3.8.
(1) A term t is *root preserved* if the root symbols of s and t have the same colour for every term s with $t \twoheadrightarrow_{1,2} s$.
(2) A term t is *preserved* if t is root preserved and every principal subterm of t is preserved. In other words, t is preserved if all special subterms of t are root preserved.
(3) A substitution σ is *preserved* if $\sigma(x)$ is preserved for every $x \in \mathcal{D}(\sigma)$.

DEFINITION 3.9. We write $s \to_c t$ if there exists a context $C[\ ]$ and terms s_1, t_1 such that $s \equiv C[s_1]$, $t \equiv C[t_1]$, s_1 is a special subterm of s, $s_1 \twoheadrightarrow_{1,2} t_1$ and the root symbols of s_1 and t_1 have different colours. This relation $\to_c$ is called *collapsing reduction* and s_1 is a *collapsing redex*.

Collapsing reduction differs slightly from the notion of *deletion reduction* $(\to_d)$ introduced by Toyama [17], in that he only allows *innermost* collapsing redexes to be deleted.

PROPOSITION 3.10.
(1) *If $s \to_c t$ then $s \twoheadrightarrow_{1,2} t$.*
(2) *A term t is preserved if and only if t contains no collapsing redexes.*
PROOF. Immediate consequence of the definition. □

EXAMPLE 3.11. Let

$$\mathcal{R}_1 = \begin{cases} F(x,y) \to y \Leftarrow x \downarrow G(y) \\ G(x) \to C \end{cases}$$

and $\mathcal{R}_2 = \{e(x) \to x\}$. Starting from $t \equiv F(C, e(F(e(C), G(e(C)))))$ we have the following collapsing reduction sequence:

$$\begin{aligned} t &\to_c F(C, e(F(C, G(e(C))))) \\ &\to_c e(F(C, G(e(C)))) \\ &\to_c F(C, G(e(C))) \\ &\to_c F(C, G(C)). \end{aligned}$$

The use of multiset orderings (Dershowitz and Manna [3]) enables an elegant proof of the termination of $\to_c$.

NOTATION. The set of all finite multisets over the natural numbers is denoted by $\mathcal{M}(\mathbb{N})$. To distinguish between sets and multisets, we use brackets instead of braces for the latter.

DEFINITION 3.12. The ordering $\gg$ on $\mathcal{M}(\mathbb{N})$ is defined as follows: $M_1 \gg M_2$ if there exist multisets $X, Y \in \mathcal{M}(\mathbb{N})$ such that:
(1) $[\,] \neq X \subseteq M_1$,

(2) $M_2 = (M_1 - X) + Y$,

(3) $\forall y \in Y\ \exists x \in X$ with $x > y$.

THEOREM 3.13 (Dershowitz and Manna [3]). *The relation $\gg$ is a well-founded ordering on $\mathcal{M}(\mathbb{N})$.* □

PROPOSITION 3.14. *Collapsing reduction is a terminating relation.*

PROOF. Assign to every term t the multiset $\| t \| = [\, rank(s) \mid s$ is a special subterm of $t\,]$. Suppose that $t \rightarrow_c t'$. Using Proposition 2.6, one easily shows that $\| t \| \gg \| t' \|$. Theorem 3.13 yields the desired result. □

The relation $\rightarrow_c$ is extended to substitutions in the obvious way (i.e. $\sigma \rightarrow_c \tau$ if $\sigma(x) \rightarrow_c \tau(x)$ for some $x \in \mathcal{V}$). Notice that the previous proposition immediately extends to substitutions.

PROPOSITION 3.15. *Let $s_1,\ldots,s_n,t_1,\ldots,t_n$ be black terms. For every substitution σ with $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ $(i=1,\ldots,n)$ there exists a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $\tau(s_i) \downarrow_1^o \tau(t_i)$ $(i=1,\ldots,n)$.*

PROOF. Let σ' be a normal form of σ with respect to $\rightarrow_c$. From Proposition 3.10(1) and Lemma 3.6 we obtain $\sigma'(s_i) \downarrow_{1,2} \sigma'(t_i)$ $(i=1,\ldots,n)$. Proposition 2.10 yields a decomposition of σ' into $\sigma_2 \circ \sigma_1$ such that σ_1 is black and σ_2 is top white. Notice that σ_2 is preserved. Using Proposition 3.16 we obtain a substitution σ^* with $\sigma_2 \twoheadrightarrow_{1,2} \sigma^*$ such that $\sigma^*(\sigma_1(s_i)) \downarrow_1^o \sigma^*(\sigma_1(t_i))$ $(i=1,\ldots,n)$. Let τ be the restriction of $\sigma^* \circ \sigma_1$ to $\mathcal{D}(\sigma_1)$. It is easy to show that $\sigma \twoheadrightarrow_{1,2} \tau$. Hence τ satisfies the requirements. □

PROPOSITION 3.16. *Let $s_1,\ldots,s_n,t_1,\ldots,t_n$ be black terms. If σ is a top white and preserved substitution with $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ $(i=1,\ldots,n)$ then there exists a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $\tau(s_i) \downarrow_1^o \tau(t_i)$ $(i=1,\ldots,n)$.*

PROOF. According to Proposition 3.17 we can construct a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $\sigma(x) \downarrow_{1,2} \sigma(y)$ implies $\tau(x) \equiv \tau(y)$ for all $x,y \in \mathcal{D}(\sigma)$. We will show that $\tau(s_i) \downarrow_1^o \tau(t_i)$ for $i=1,\ldots,n$. Fix i. By definition there exists a term u_i such that $\sigma(s_i) \twoheadrightarrow_{1,2} u_i \twoheadleftarrow_{1,2} \sigma(t_i)$. Let $A = \{a_1,\ldots,a_m\}$ be the set of all maximal special subterms with white root symbol occurring in this conversion. We define a mapping ϕ from A to $\{\tau(x) \mid x \in \mathcal{D}(\sigma)\}$ as follows:

> Let $a \in A$. From Proposition 3.18 we know that there is a variable $x \in \mathcal{D}(\sigma)$ such that $\sigma(x) \twoheadrightarrow_{1,2} a$. We put $\phi(a) \equiv \tau(x)$.

We remark that ϕ is well-defined because if there exists another variable $y \in \mathcal{D}(\sigma)$ with $\sigma(y) \twoheadrightarrow_{1,2} a$, then $\sigma(x) \downarrow_{1,2} \sigma(y)$ and hence $\tau(x) \equiv \tau(y)$. The result of replacing in a term t all maximal special subterms $a \in A$ by the corresponding $\phi(a)$ is denoted by $\Phi(t)$. Let t be any term such that $\sigma(s_i) \twoheadrightarrow_{1,2} t$. We will prove by induction on the length of the reduction from $\sigma(s_i)$ to t that $\Phi(\sigma(s_i)) \twoheadrightarrow_1^o \Phi(t)$. If the length is zero then $t \equiv \sigma(s_i)$ and we have nothing to prove. Suppose $\sigma(s_i) \twoheadrightarrow_{1,2} t' \rightarrow_{1,2} t$. From the induction hypothesis we learn $\Phi(\sigma(s_i)) \twoheadrightarrow_1^o \Phi(t')$. By case analysis we will show that either $\Phi(t') \equiv \Phi(t)$ or $\Phi(t') \rightarrow_1^o \Phi(t)$.

(1) If the rewritten redex in the step $t' \rightarrow_{1,2} t$ occurs in a maximal special subterm v of t' with white root symbol, then we can write $t' \equiv C[v]$ and $t \equiv C[v']$ for some context $C[\,]$ and term v' with $v \rightarrow_{1,2} v'$. Clearly v and v' (because σ is preserved) are elements of A. Therefore $\phi(v)$ and $\phi(v')$ are defined and since $v \rightarrow_{1,2} v'$, $\phi(v)$ and $\phi(v')$ are identical. We obtain $\Phi(t') \equiv \Phi(t)$.

(2) In the previous case we covered $\rightarrow_1^i$, $\rightarrow_2^i$ and $\rightarrow_2^o$ (when $C[\,] \equiv \Box$). One possibility remains: $t' \rightarrow_1^o t$. If t' is a black term (and hence t also is black) then $\Phi(t') \equiv t' \rightarrow_1^o t \equiv \Phi(t)$. Otherwise

we can write

$$t' \equiv C[\![v_1,\ldots,v_m]\!] \to_1^o C^*[\![v_{i_1},\ldots,v_{i_k}]\!] \equiv t$$

for certain terms $v_1,\ldots,v_m \in A$. Choose mutually different fresh variables $x_1,\ldots,x_m$ and define terms $w' \equiv C[x_1,\ldots,x_m]$, $w \equiv C^*[x_{i_1},\ldots,x_{i_k}]$ and substitutions $\rho = \{x_i \to v_i \mid 1 \le i \le m\}$, $\rho' = \{x_i \to \phi(v_i) \mid 1 \le i \le m\}$. From the construction of ϕ we learn that $\rho \propto \rho'$. Notice also that ρ and ρ' are top white. We have $\rho(w') \equiv t' \to_1^o t \equiv \rho(w)$. Proposition 3.5 yields $\rho'(w') \to_1^o \rho'(w)$ and since $\Phi(t') \equiv \rho'(w')$ and $\Phi(t) \equiv \rho'(w)$ we are done.

By the same argument we also have $\Phi(\sigma(t_i)) \twoheadrightarrow_1^o \Phi(t)$ whenever $\sigma(t_i) \twoheadrightarrow_{1,2} t$. Putting everything together, we obtain $\tau(s_i) \equiv \Phi(\sigma(s_i)) \downarrow_1^o \Phi(\sigma(t_i)) \equiv \tau(t_i)$. □

PROPOSITION 3.17. *For every substitution σ there exists a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and if $\sigma(x) \downarrow_{1,2} \sigma(y)$ then $\tau(x) \equiv \tau(y)$ for all $x,y \in \mathcal{D}(\sigma)$.*

PROOF. Partition the set $\{\sigma(x) \mid x \in \mathcal{D}(\sigma)\}$ into equivalence classes $C_1,\ldots,C_n$ of $\to_{1,2}$-convertible terms. With every class C_i we associate a 'common reduct' u_i as suggested in Figure 7. The

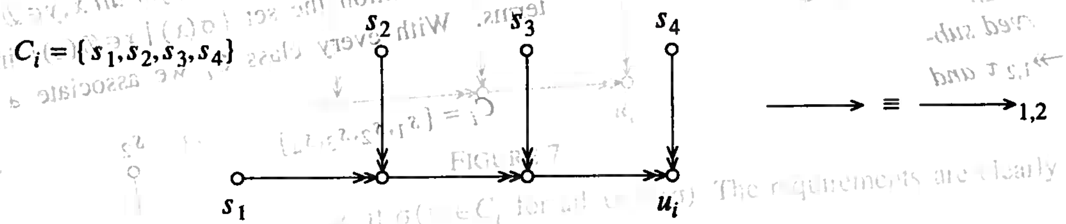


FIGURE 7.

substitution τ is defined by $\tau(x) \equiv u_i$ if $\sigma(x) \in C_i$ for all $x \in \mathcal{D}(\sigma)$. The requirements are clearly fulfilled. □

PROPOSITION 3.18. *Let t be a black term and suppose σ is a top white and preserved substitution. If $\sigma(t) \twoheadrightarrow_{1,2} t'$ and s is a maximal special subterm of t' with white root symbol, then there exists a variable $x \in \mathcal{D}(\sigma)$ such that $\sigma(x) \twoheadrightarrow_{1,2} s$.*

PROOF. Straightforward induction on the length of the reduction from $\sigma(t)$ to t'. □

THEOREM 3.19. *Confluence is a modular property of join* CTRS's.

PROOF. Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint join CTRS's. We must prove the following equivalence: $\mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent $\Leftrightarrow$ both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent.

$\Rightarrow$ Trivial.

$\Leftarrow$ Easy consequence of Lemma 3.6, Lemma 3.7 and Proposition 3.4.

□

4. Semi-Equational and Normal Systems

We now turn our attention to semi-equational and normal CTRS's. The proof of the modularity of confluence for normal CTRS's is a straightforward adaptation of Section 3. The main difference is the change of joinability into reduction at several places. We only mention the changed definitions and (statements of the) propositions. The number of the corresponding definition or proposition in Section 3 is given in parentheses.

DEFINITION 4.1 (3.2). We write $s \to_1 t$ if there exists a rewrite rule $l \to r \Leftarrow s_1 \twoheadrightarrow t_1,\ldots,s_n \twoheadrightarrow t_n$ in $\mathcal{R}_1$, a context $C[\,]$ and a substitution σ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \twoheadrightarrow_1^o t_i$ for

$i=1,\ldots,n$. (Notice that $\sigma(t_i) \equiv t_i$ because t_i is a ground term.) The relation $\rightarrow_2$ is defined similarly.

PROPOSITION 4.2 (3.15). *Let* $s_1,\ldots,s_n,t_1,\ldots,t_n$ *be black terms. For every substitution* σ *with* $\sigma(s_i) \twoheadrightarrow_{1,2} t_i$ *for* $i=1,\ldots,n$ *there exists a substitution* τ *such that* $\sigma \twoheadrightarrow_{1,2} \tau$ *and* $\tau(s_i) \twoheadrightarrow_1^o t_i$ *for* $i=1,\ldots,n$. □

PROPOSITION 4.3 (3.16). *Let* $s_1,\ldots,s_n,t_1,\ldots,t_n$ *be black terms. If* σ *is a top white and preserved substitution with* $\sigma(s_i) \twoheadrightarrow_{1,2} t_i$ $(i=1,\ldots,n)$ *then there exists a substitution* τ *such that* $\sigma \twoheadrightarrow_{1,2} \tau$ *and* $\tau(s_i) \twoheadrightarrow_1^o t_i$ $(i=1,\ldots,n)$. □

THEOREM 4.4 (3.19). *Confluence is a modular property of normal* CTRS's. □

The proof of the modularity of confluence for semi-equational CTRS's has exactly the same structure, apart from the proof of Proposition 3.5, which is more complicated because the observation made in order to make the second induction hypothesis applicable is no longer sufficient. So in addition to the changed definitions and propositions, we will also give the modified proof of Proposition 3.5 (4.6).

DEFINITION 4.5 (3.2). We write $s \rightarrow_1 t$ if there exists a rewrite rule $l \rightarrow r \Leftarrow s_1 = t_1,\ldots,s_n = t_n$ in $\mathcal{R}_1$, a context $C[\;]$ and a substitution σ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) =_1^o \sigma(t_i)$ for $i=1,\ldots,n$. The relation $\rightarrow_2$ is defined similarly.

PROPOSITION 4.6 (3.5). *Let* s,t *be black terms and suppose* σ *is a top white substitution such that* $\sigma(s) \rightarrow_1^o \sigma(t)$. *If* τ *is a substitution with* $\sigma \propto \tau$ *then* $\tau(s) \rightarrow_1^o \tau(t)$.

PROOF. We prove the statement by induction (①) on the depth of $\sigma(s) \rightarrow_1^o \sigma(t)$. The case of zero depth is straightforward. If the depth of $\sigma(s) \rightarrow_1^o \sigma(t)$ equals $n+1$ $(n \geq 0)$ then there exist a context $C[\;]$, a substitution ρ and a rewrite rule $l \rightarrow r \Leftarrow s_1 = t_1,\ldots,s_m = t_m$ in $\mathcal{R}_1$ such that $\sigma(s) \equiv C[\rho(l)]$, $\sigma(t) \equiv C[\rho(r)]$ and $\rho(s_i) =_1^o \rho(t_i)$ for $i=1,\ldots,m$ with depth less than or equal to n. Proposition 2.10 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black, ρ_2 is top white and $\rho_2 \propto \varepsilon$. It is not difficult to see that for every variable $x \in \mathcal{D}(\rho_2)$ there exists a variable $y \in \mathcal{D}(\sigma)$ with $\rho_2(x) \equiv \sigma(y)$, see Figure 4. We define the substitution ρ^* by $\rho^*(x) \equiv \tau(y)$ for every $x \in \mathcal{D}(\rho_2)$ and $y \in \mathcal{D}(\sigma)$ satisfying the above identity. Notice that ρ^* is well-defined by the assumption $\sigma \propto \tau$. We have $\rho_2 \propto \rho^*$ since $\rho_2 \propto \varepsilon$ and $\varepsilon \propto \rho^*$. By induction (②) on the length of the conversion $\rho_2(\rho_1(s_i)) =_1^o \rho_2(\rho_1(t_i))$ we will show that $\rho^*(\rho_1(s_i)) =_1^o \rho^*(\rho_1(t_i))$ for $i=1,\ldots,m$. The basis of the induction being trivial, we consider two cases for the induction step:

(1) If $\rho_2(\rho_1(s_i)) \rightarrow_1^o s' =_1^o \rho_2(\rho_1(t_i))$ then we may write $\rho_2(\rho_1(s_i)) \equiv C_1[\![u_1,\ldots,u_p]\!]^\dagger$ and $s' \equiv C_2[\![u_{j_1},\ldots,u_{j_q}]\!]^\dagger$. For every term $u' \in \{u_1,\ldots,u_p\}$ there is a unique variable $\psi(u') \in \mathcal{D}(\rho_2)$ such that $\rho_2(\psi(u')) \equiv u'$. We have $s' \equiv \rho_2(s'')$ with $s'' \equiv C_2[\psi(u_{j_1}),\ldots,\psi(u_{j_q})]$ a black term. We obtain $\rho^*(\rho_1(s_i)) \rightarrow_1^o \rho^*(s'')$ from induction hypothesis ① and induction hypothesis ② yields $\rho^*(s'') =_1^o \rho^*(\rho_1(t_i))$.

(2) If $\rho_2(\rho_1(s_i)) \leftarrow_1^o s' =_1^o \rho_2(\rho_1(t_i))$ then we may write $s' \equiv C_1[\![u_1,\ldots,u_p]\!]^\dagger$ and $\rho_2(\rho_1(s_i)) \equiv C_2[\![u_{j_1},\ldots,u_{j_q}]\!]^\dagger$. Let $\{v_1,\ldots,v_r\}$ be the difference between the sets(!) $\{u_1,\ldots,u_p\}$ and $\{u_{j_1},\ldots,u_{j_q}\}$. Choose distinct fresh variables $x_1,\ldots,x_r$ and define a mapping ψ from

† In order to avoid an explosion of cases to be considered, we allow for $p=0$ and $q=0$.

$\{u_1,\ldots,u_p\}$ to $\mathcal{D}(\rho_2)\cup\{x_1,\ldots,x_r\}$ as follows: if $u'\in\{u_1,\ldots,u_p\}$ is an element of $\{u_{j_1},\ldots,u_{j_q}\}$ then there exists an unique variable $\psi(u')\in\mathcal{D}(\rho_2)$ such that $\rho_2(\psi(u'))\equiv u'$, otherwise $u'\equiv v_k$ for some $k\in\{1,\ldots,r\}$ and we put $\psi(u')\equiv x_k$. We define the substitution ρ_3 by $\rho_3=\rho_2\cup\{x_i\to v_i \mid 1\le i\le r\}$. By construction we have $\rho_2(\rho_1(s_i))\equiv\rho_3(\rho_1(s_i))$, $s'\equiv\rho_3(s'')$ with $s''\equiv C_1[\psi(u_1),\ldots,\psi(u_p)]$ a black term and $\rho_2(\rho_1(t_i))\equiv\rho_3(\rho_1(t_i))$. Notice that ρ_3 is top white and $\rho_3\propto\rho^*$. Just as in the preceding case, we obtain $\rho^*(\rho_1(s_i))=^o_1\rho^*(\rho_1(t_i))$ from both induction hypotheses.

Hence $\rho^*(\rho_1(l))\to^o_1\rho^*(\rho_1(r))$. Let $C^*[\ ]$ be the context obtained from $C[\ ]$ by replacing every principal subterm, which has the form $\sigma(x)$ for some variable $x\in\mathcal{D}(\sigma)$, by the corresponding $\tau(x)$. A routine argument shows that $\tau(s)\equiv C^*[\rho^*(\rho_1(l))]$ and $\tau(t)\equiv C^*[\rho^*(\rho_1(r))]$. We conclude that $\tau(s)\to^o_1\tau(t)$. □

PROPOSITION 4.7 (3.15). *Let $s_1,\ldots,s_n,t_1,\ldots,t_n$ be black terms. For every substitution σ with $\sigma(s_i)=_{1,2}\sigma(t_i)$ $(i=1,\ldots,n)$ there exists a substitution τ such that $\sigma\twoheadrightarrow_{1,2}\tau$ and $\tau(s_i)=^o_1\tau(t_i)$ $(i=1,\ldots,n)$.* □

PROPOSITION 4.8 (3.16). *Let $s_1,\ldots,s_n,t_1,\ldots,t_n$ be black terms. If σ is a top white and preserved substitution with $\sigma(s_i)=_{1,2}\sigma(t_i)$ $(i=1,\ldots,n)$ then there exists a substitution τ such that $\sigma\twoheadrightarrow_{1,2}\tau$ and $\tau(s_i)=^o_1\tau(t_i)$ $(i=1,\ldots,n)$.* □

THEOREM 4.9 (3.19). *Confluence is a modular property of semi-equational* CTRS's. □

5. Concluding Remarks

We have shown that confluence is modular property of three types of conditional term rewriting systems: semi-equational CTRS's, join CTRS's and normal CTRS's. The proofs were very much alike, suggesting that we might prove a more general theorem from which we not only immediately obtain the above results, but also

(1) the modularity of confluence for other kinds of CTRS's like *normal-join* systems or *meta-conditional* systems (see [5]), and

(2) confluence results for the disjoint union of two different kinds of CTRS's.

This matter clearly has to be further pursued.

Another point which needs investigation is the syntactic restrictions imposed on the rewrite rules. From a programming point of view the assumption of a rewrite rule $l\to r\Leftarrow s_1=t_1,\ldots,s_n=t_n$ satisfying the requirement that r only contains variables occurring in l, is too restrictive. A semi-equational CTRS like ([4])

$$\mathcal{R}=\begin{cases} Fib(0) \to \langle 0,S(0)\rangle & \\ Fib(S(x)) \to \langle z,A(y,z)\rangle \Leftarrow Fib(x)=\langle y,z\rangle \end{cases}$$

should be perfectly legitimate. The CTRS's $\mathcal{R}$ we are interested in, can be characterized by the phrase "if $s\to_{\mathcal{R}}t$ then $s\to t$ is a legal unconditional rewrite rule". However, the proofs in the preceding sections cannot easily be modified to cover these systems. For instance, Proposition 2.6 is no longer true and the proofs of Proposition 3.5 and 3.16 seem insufficient.

Finally, it is worthwhile to extend the other results on modularity mentioned in Section 2 to CTRS's.

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