Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

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On cutting planes and matrices

Department of Operations Research, Statistics, and System Theory   Report BS-R8930   November

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On Cutting Planes and Matrices

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Continuing the work of Chvátal and Gomory, Schrijver proved that any rational polyhedron $\{x|Ax \leqslant b\}$ has finite Chvátal rank. This was extended by Cook, Gerards, Schrijver and Tardos, who proved that in fact this Chvátal rank can be bounded from above by a number only depending on A, so independent of b. The aim of this note is to show that the latter result can be proved quite easily from the result of Chvátal and Schrijver.

1980 Mathematics Subject Classification: 90C10.
Key Words & Phrases: cutting planes, Chvátal rank.
Note: The research for this paper was carried out when the author was at Tilburg University, Tilburg, The Netherlands.

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INTRODUCTION

Consider a *rational polyhedron* P, i.e. $P = \{ x \in \mathbb{R}^n \mid Ax \leqq b \}$ with $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. A *cutting plane for* P is an inequality

$$c^T x \leqq \lfloor \delta \rfloor,$$
with $c \in \mathbb{Z}^n$,
and $\delta \geqq \max\{ c^T x \mid x \in P\}$.

The set of vectors satisfying all cutting planes for P is denoted by P'. Obviously, P' satisfies

(1) $$P_I \subset P' \subset P,$$

where P_I := convex hull $(P \cap \mathbb{Z}^n)$. Moreover P' is a polyhedron again (Schrijver [1980]) and satisfies

(2) $$P' = P \Leftrightarrow P_I = P.$$

(1) and (2) suggest the following procedure to get a system of inequalities $Mx \leqq d$ such that $P_I = \{ x \in \mathbb{R}^n \mid Mx \leqq d \}$. Namely, define

(3) $$P^{(0)} := P;\ P^{(i)} := (P^{(i-1)})' \text{ for } i = 1, 2, \ldots .$$

Form (1) and (2) we get

(4) $$P = P^{(0)} \supset P^{(1)} \supset P^{(2)} \supset \ldots \supset P^{(i)} \supset \ldots \supset P_I,$$
$$P^{(i)} = P^{(i-1)} \Leftrightarrow P^{(i)} = P_I \ (i = 1, 2, \ldots).$$

Schrijver [1980] proved that

(5) for each rational polyhedron P there exists a $t \in \mathbb{N}$, such that $P^{(t)} = P_I$.

Cook, Gerards, Schrijver and Tardos [1986] extended this result by proving that

(6) for each matrix $A \in \mathbb{Z}^{m\times n}$, there exists a $t \in \mathbb{N}$, such that for each $b \in \mathbb{Z}^m$ we have that $\{ x \in \mathbb{R}^n \mid Ax \leqq b \}^{(t)} = \{ x \in \mathbb{R}^n \mid Ax \leqq b \}_I$.

The aim of this note is to present a short proof of (6) using (5).

REMARKS:

(i) The procedure described above can be considered as a polyhedral version of Gomory's cutting plane method for integer linear programming (Gomory [1963]). Chvátal [1973] proved (5), for the case that P is bounded in $\mathbb{R}^n$.

(ii) As C. Blair observed, (6) is equivalent with the result, due to Blair and Jeroslow [1982], that "each integer programming value function is a Gomory function". For a discussion see Cook, Gerards, Schrijver and Tardos [1986].

(iii) In fact, Cook, Gerards, Schrijver and Tardos [1986], proved that t in (6) can be taken equal to $2^{n^3+1}n^{5n}\Delta(A)^{n+1}$, where $\Delta(A)$ denotes the maximum of the absolute values of the subdeterminants of A. Since the proof of (6) given below relies on (5), it can not be expected to give such an explicit bound.

PROOF OF (6)

Let $A \in \mathbb{Z}^{m\times n}$, and assume that it violates (6). This implies the existence of a sequence

(7) $\{b_i, w_i, \alpha_i\}_{i\in\mathbb{N}}$ with $b_i \in \mathbb{Z}^m$, $w_i \in \mathbb{Z}^n$, $\alpha_i \in \mathbb{Z}$ for $i \in \mathbb{N}$

such that

(8) for each $i \in \mathbb{N}$, $w_i^T x \leqq \alpha_i$ is valid for $(P_i)_I$, but not valid for $(P_i)^{(i)}$, where $P_i := \{ x \in \mathbb{R}^n \mid Ax \leqq b_i \}$.

In the sequel we often use the following fact, which trivially follows from (4).

(9) (8) is invariant under taking subsequences of (7).

By (9), it is obvious that we only need to consider one of the following two cases:

Case 1: $P_i \neq \emptyset = (P_i)_I$ for each $i \in \mathbb{N}$;
Case 2: $(P_i)_I \neq \emptyset$ for each $i \in \mathbb{N}$.
(Indeed, by (8) none of the P_i is empty, so (7) has to have a subsequence satisfying one of the two possibilities above.)

We settle the cases seperately.

Case 1: (8) is invariant under translation of the polyhedra P_i over an integral vector x_i (i.e. replacing b_i by $b_i + Ax_i$). So we may assume that each P_i contains a vector in $\{ x \in \mathbb{R}^n \mid 0 \leqq x \leqq 1 \}$. This means that the "component sequences" $\{(b_i)_j\}_{i \in \mathbb{N}}$ are bounded from below for $j = 1, \ldots, m$. Hence we may assume (by (9) and by renumbering indices j) that there exists a constant vector $c = [c_1, \ldots, c_k]^T$ such that

(10) $(b_i)_j = c_j$ for $i \in \mathbb{N}$ and $j = 1, \ldots, k$, and
(11) $\{(b_i)_j\}$ is strictly increasing for $j = k+1, \ldots, m$.

Split each system $Ax \leqq b_i$ in the two subsytems $A_1x \leqq c$ and $A_2x \leqq d_i$ ($d_i := [(b_i)_{k+1}, \ldots, (b_i)_m]^T$), and set $Q := \{ x \in \mathbb{R}^n \mid A_1x \leqq c \}$. Let $t \in \mathbb{N}$, such that $Q^{(t)} = Q_I$ (t exists by (5)). For $i > t$ we have that $w_i^Tx \leqq \alpha_i$ is not valid for $(P_i)^{(i)} \subset Q^{(i)} = Q_I$. Hence Q_I is not empty, which by (11) implies that $(P_i)_I$ is not empty for some $i \in \mathbb{N}$. Contradiction, Case 1 cannot occur.

Case 2: For each $i \in \mathbb{N}$, let $x_i \in P_i \cap \mathbb{Z}^n$ such that $w_i^Tx_i = \max \{ w_i^Tx \mid x \in P_i \cap \mathbb{Z}^n\}$. By translation, we may assume that, for each $i \in \mathbb{N}$, x_i is the all-zero vector $0 \in P_i$ and that $\alpha_i = 0$. Using the same arguements as used in Case 1 we may assume that $Ax \leqq b_i$ can be split into two subsystems $A_1x \leqq c$ and $A_2x \leqq d_i$, where c and d_i are as in Case 1 and satisfy (10) and (11). Again we define $Q := \{ x \in \mathbb{R}^n \mid A_1x \leqq c \}$.

Before we proceed we construct a finite set L as follows. Choose an integral vector, called y_F, in each minimal face F of Q_I. Moreover, choose a collection $v_1, \ldots, v_k \in \mathbb{Z}^n$ such that $v_1, \ldots,$ and v_k generate the cone $\{ x \in \mathbb{R}^n \mid A_1 x \leqq 0 \}$. Define $L := \{ y_F \mid F$ minimal face of $Q_I \} \cup \{ v_1, \ldots, v_k \}$.

Let $t \in \mathbb{N}$, such that $Q^{(t)} = Q_I$ (t exists by (5)). For $i > t$ we have that $w_i^T x \leqq 0$ is not valid for $(P_i)^{(i)} \subset Q^{(i)} = Q_I$. Hence there exists for each $i \in \mathbb{N}$ a vector $z_i \in Q \cap \mathbb{Z}^n$ with $w_i^T z_i > 0$. By standard linear programming theory, we may assume that $z_i \in L$ for each $i \in \mathbb{N}$. By (10), (11) and the fact that L is bounded, there exists an $i \in \mathbb{N}$, such that $z_i \in P_i$. As $z_i \in \mathbb{Z}^n$, this contradicts our assumption that $\max \{ w_i^T x \mid x \in P_i \cap \mathbb{Z}^n \} = w_i^T x_i = w_i^T 0 = 0$. So also Case 2 is not possible.

As it turned out that both cases do not hold, (6) follows. □.

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