Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

H.J.A.M. Heijmans

Grey-level morphology

Department of Analysis, Algebra and Geometry Report AM-R9003 January

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).

Grey-level Morphology

H.J.A.M. Heijmans
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

This paper presents a detailed study of morphological operators on the space of grey-level functions. It is shown how "classical" morphological operators for binary images can be extended to the space of grey-level images. Particular attention is given to the class of so-called flat operators, i.e., operators which commute with thresholding. It is also shown how to define dilations and erosions with non–flat structuring elements if the grey-level set is finite.

AMS 1980 Mathematics Subject Classification: 68U10, 06A23.

Keywords: mathematical morphology, image analysis, complete lattice, dilation, erosion, translation-invariance, increasing operator, grey-level function, set operator, function operator, threshold set, flat operator, threshold superposition principle, umbra, upper-semi-continuous function, truncation.

1. Introduction

Originally, the theory of mathematical morphology was developed for binary images, mainly by Matheron and Serra [9,12]. Afterwards, the theory has been extended to grey-level images by Sternberg [15,16] and Serra [12, Chapter XII]. Binary dilations, erosions, closings and openings can be naturally extended to grey-level images by the use of *min* and *max* operations. Such extensions can be visualized geometrically with the aid of so-called *umbras*, the points on and below the graph of a function. In the literature the extension of dilations and erosions by means of the umbra transform has received disproportionally much attention [2,3,8,15,16]. We refer to Section 8 for some further discussion on this issue.

The recent extension of mathematical morphology to arbitrary complete lattices by Serra [13] has also resulted in a conceptually different view point with respect to grey-level morphology: see in particular [4, Subsection 4.3] and [10].

In this paper the emphasis lies on the question how to extend (increasing) binary morphological operators to grey-level images. A central notion is formed by the concept of a *flat operator*, which is defined as the extension of a binary (or set) operator to grey-level functions using threshold sets. Such operators are called *FSP filters* by Maragos [7], *flat filters* by Janowitz [6] and *stack filters* by Wendt, Coyle and Gallagher [18].

In Section 2 we present a brief discussion of the extension of mathematical morphology to complete lattices as described in [4,11,13]. In Section 3 we consider the space of grey-level functions and introduce the concept of a threshold set. In Section 4 we show how morphological grey-level operators can be

Report AM-R9003
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

constructed by application of a different binary morphological operator at any threshold level. If the same binary operator is used at every level, then the resulting grey-level operator is called flat. Flat operators are discussed in Section 5. Section 6 and 7 respectively deal with *T-operators* and *H-operators*: a T-operator is defined to be a grey-level operator which is invariant under spatial and grey-level translations, whereas an H-operator is required to be invariant under spatial (horizontal) translations only. In the literature [3,10,12,16] grey-level morphology is mostly discussed in terms of umbras. In Section 8 we point out some connections between both approaches, and devote a few words to u.s.c. functions. Throughout Sections 3–8 the grey-level set was the extended real line. In Sections 9–11 we consider the case where the grey-level set is discrete. It turns out that many of the statements and proofs simplify considerably in this case. In Sections 9,10 we respectively examine the infinite and finite discrete case. Finally, in Section 11 we explain how to define dilations and erosions with non-flat structuring elements if the grey-level set is finite: it turns out that merely truncation leads to wrong results.

2. Morphology on complete lattices

In this section we briefly discuss some of the basic theory on complete lattices, adjunctions and their relations to mathematical morphology. A space $\mathcal{L}$ with a partial order $\leq$ is called a *complete lattice* if for every subset $\mathcal{H}$ of $\mathcal{L}$ its supremum $\bigvee \mathcal{H}$ or $\sup \mathcal{H}$ and infimum $\bigwedge \mathcal{H}$ or $\inf \mathcal{H}$ exist. For a comprehensive exposition on the theory of complete lattices we refer to the monograph of Birkhoff [1].

Let $\mathcal{L}_1, \mathcal{L}_2$ be two complete lattices. A mapping or operator $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ is called *increasing* (or *monotone*) if $X \leq Y$ implies that $\psi(X) \leq \psi(Y)$, for any pair $X, Y \in \mathcal{L}_1$. The operator $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ is called a dilation if δ distributes over suprema, that is, $\delta(\bigvee_{i\in I} X_i) = \bigvee_{i\in I} \delta(X_i)$ for any collection $X_i \in \mathcal{L}_1$, $i \in I$. The operator $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$ is called an erosion if ε distributes over infima, that is, $\varepsilon(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \varepsilon(X_i)$ for any collection $X_i \in \mathcal{L}_1$, $i \in I$. It is clear that dilations and erosions define increasing operators. Furthermore they are dual notions in the sense that that a dilation becomes an erosion (and vice versa) if one reverses the ordering of both $\mathcal{L}_1$ and $\mathcal{L}_2$. It is well-known [4,13] that to any dilation $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ there corresponds a unique erosion $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$ such that

$$\delta(X_1) \leq X_2 \iff X_1 \leq \varepsilon(X_2), \tag{2.1}$$

for $X_1 \in \mathcal{L}_1$, $X_2 \in \mathcal{L}_2$. Conversely, to every erosion $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$ there corresponds a unique dilation $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ such that (2.1) holds. Furthermore if $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ and $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$ are mappings such that (2.1) holds, then δ is a dilation and ε an erosion. In this case we call (ε, δ) an adjunction between $\mathcal{L}_1$ and $\mathcal{L}_2$. If (ε, δ) is an adjunction then

$$\varepsilon\delta\varepsilon = \varepsilon \quad \text{and} \quad \delta\varepsilon\delta = \delta. \tag{2.2}$$

Example 2.1. *Adjunctions on* $\overline{\mathbb{R}}$.
Obviously, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the usual ordering is a complete lattice (even more, it is a complete chain). By definition, (e, d) is an adjunction on $\overline{\mathbb{R}}$ if and only if for $s, t \in \overline{\mathbb{R}}$,

$$d(s) \leq t \iff s \leq e(t).$$

Then d is a dilation. It is quite obvious that a mapping $d : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a dilation if and only if the following are satisfied:

- $d(-\infty) = -\infty$

- d is increasing
- d is continuous from the left.

In a similar way, erosions are characterized. If the dilation d is invertible then the adjoint erosion is obtained by taking its inverse. For example, if $d(t) = At + G$, where $A > 0$ and $G \in \mathbb{R}$, then $e(t) = (t - G)/A$. For some further results concerning adjunctions on a finite set we refer to Section 11.

Let $\mathcal{L}$ be any complete lattice. The identity operator which maps any element of $\mathcal{L}$ onto itself is denoted by id. An operator $\psi : \mathcal{L} \to \mathcal{L}$ is called *extensive* if $\psi \geq \text{id}$, and *anti-extensive* if $\psi \leq \text{id}$. If $\psi^2 = \psi$ then ψ is said to be *idempotent*. An increasing, idempotent, extensive (resp. anti-extensive) operator is called a *closing* (resp. *opening*). From (2.1) it is easily deduced that if (ε, δ) is an adjunction between $\mathcal{L}_1$ and $\mathcal{L}_2$, then $\varepsilon\delta$ is a closing on $\mathcal{L}_1$ and $\delta\varepsilon$ is an opening on $\mathcal{L}_2$.

Let $X_n \in \mathcal{L}$ for $n \in \mathbb{N}$. We say that $X_n \downarrow X$ if X_n is decreasing and $X = \bigwedge_{n \geq 1} X_n$. Similarly $X_n \uparrow X$ is defined. An increasing operator $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ is called $\downarrow$-continuous if $X_n \downarrow X$ in $\mathcal{L}_1$ implies that $\psi(X_n) \downarrow \psi(X)$ in $\mathcal{L}_2$. In a similar way, $\uparrow$-continuity of an operator is defined. In [5] it is explained how these notions of $\uparrow$- and $\downarrow$-continuity can be extended to general operators.

Originally, mathematical morphology has been developed as a toolbox of image transformations and functionals for binary images: see [12]. A binary image can be represented mathematically as a set, usually a subset of $\mathbb{R}^d$ or $\mathbb{Z}^d$. Morphological operators (or transformations) can be considered as operators on the complete lattice $\mathcal{P}(E)$, where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$, ordered by inclusion. Its building blocks are formed by such elementary operations as set union, set intersection, and set difference. Often one restricts to translation invariant operators: an operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is called translation invariant if it satisfies

$$\psi(X_h) = (\psi(X))_h,$$

for any $X \subset E$ and $h \in E$. Here X_h denotes the translate of the set X along the vector h, that is $X_h = \{x + h \mid x \in X\}$. The two basic translation invariant operations of binary morphology are dilation and erosion with a given structuring element A. The dilation by A is defined as

$$\delta_A(X) = X \oplus A = \bigcup_{h \in A} X_h,$$

and the erosion by A is given by the equivalent expressions

$$\varepsilon_A(X) = X \ominus A = \bigcap_{h \in A} X_{-h} = \{h \mid A_h \subseteq X\}.$$

It is easy to verify that $(\varepsilon_A, \delta_A)$ forms an adjunction on $\mathcal{P}(E)$. In fact, every adjunction on $\mathcal{P}(E)$ which is invariant under translations is of this form.

The lattice $\mathcal{L}$ is called *Boolean* if every element X has a complement which is then denoted by X^*. The operator $X \to X^*$ is an example of a so-called *dual automorphism*, a bijection which reverses the ordering. The lattice $\mathcal{P}(E)$ is Boolean, X^* being the ordinary set complement, also denoted by X^c. If ψ is an operator on the complete Boolean lattice $\mathcal{L}$, then ψ^* given by

$$\psi^*(X) = (\psi(X^*))^*$$

is also an operator, and

- ψ^* is increasing if and only ψ is;
- ψ^* is a dilation if and only if ψ is an erosion and vice versa;

– ψ^* is a closing if and only if ψ is an opening and vice versa.

There exists an important theorem due to Matheron [9] which says that every increasing translation invariant operator on $\mathcal{P}(E)$ can be written as a union of erosions ε_A, or, alternatively, as an intersection of dilations δ_A. To state this theorem we need the notion of a kernel. Let ψ be an operator on $\mathcal{P}(E)$, where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$, then the *kernel* $\mathcal{V}(\psi)$ of ψ is defined as

$$\mathcal{V}(\psi) = \{A \in \mathcal{P}(E) \mid 0 \in \psi(A)\}.$$

For $A \subset E$ we define the *reflected* set $\check{A}$ by $\check{A} = \{-a \mid a \in A\}$.

Proposition 2.2. (Matheron)
Let ψ be an increasing translation invariant operator on $\mathcal{P}(E)$. Then

$$\psi(X) = \bigcup_{A \in \mathcal{V}(\psi)} X \ominus A = \bigcap_{A \in \mathcal{V}(\psi^*)} X \oplus \check{A}.$$

An extension of this result to arbitrary complete lattices can be found in [4].

3. Functions, threshold sets and function operators

Let E be the continuous space $\mathbb{R}^d$ or $\mathbb{Z}^d$. We denote by Fun(E) the space of all functions $F : E \to \overline{\mathbb{R}}$. It is easy to check that Fun(E) is a complete lattice with respect to the pointwise ordering, i.e., $F \le G$ if and only if $F(x) \le G(x)$ for every $x \in E$. For every function F we define F^* by

$$F^*(x) = -F(x). \tag{3.1}$$

It is obvious that $F \to F^*$ defines a dual automorphism on Fun(E). In general, however, F^* is not the lattice complement of F (which requires that $\min\{F(x), F^*(x)\} = -\infty$ and $\max\{F(x), F^*(x)\} = \infty$ for all $x \in E$). In particular, Fun(E) is not a Boolean lattice.

Remark 3.1. Notice that $\overline{\mathbb{R}}$ may be replaced by other grey-level sets such as $\overline{\mathbb{R}}_+ = [0, \infty]$, the discrete set $\overline{\mathbb{Z}}$, or the finite set $\{0, 1, \ldots, N\}$. In the first case we define $F^*(x) = 1/F(x)$, in the second case $F^*(x) = -F(x)$, and in the third case $F^*(x) = N - F(x)$: see also Sections 9–10. However, a dual automorphism cannot be defined if the grey-level set is chosen to be $\overline{\mathbb{Z}}_+ = \{0, 1, 2, \ldots, \infty\}$.

We define the *threshold sets* of a function F by

$$\mathcal{X}_t(F) = \{x \in E \mid F(x) \ge t\},\ t \in \overline{\mathbb{R}}. \tag{3.2}$$

Definition 3.2. A family $\{X_t\}_{t \in \mathbb{R}}$ in $\mathcal{P}(E)$ is said to be *continuously decreasing* if $X_t \subseteq X_s$, $t \ge s$ and

$$\bigcap_{s<t} X_s = X_t. \tag{3.3}$$

We denote this as $X_s \downarrow X_t$ if $s \uparrow t$.

The following result expresses that a function is uniquely specified by its threshold sets. Its proof is easy and therefore omitted.

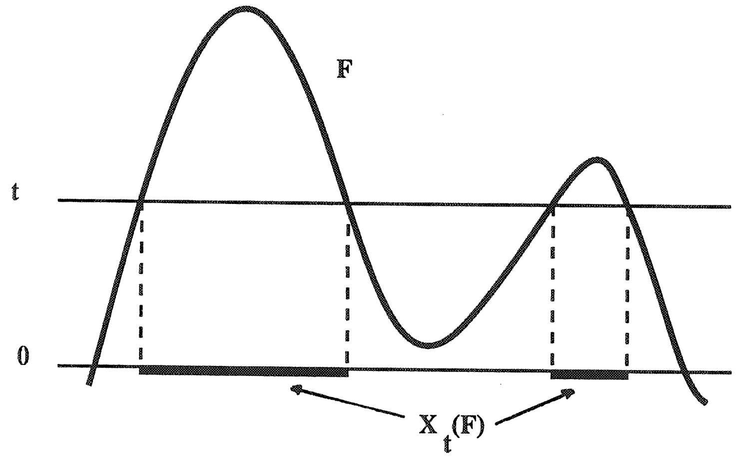

FIGURE 1. *Threshold set of a function.*

Proposition 3.3.
(a) For any function $F \in \mathrm{Fun}(E)$, the family $\{\mathcal{X}_t(F)\}_{t\in\mathbb{R}}$ is continuously decreasing.
(b) Let $\{X_t\}_{t\in\mathbb{R}}$ be a continuously decreasing family in $\mathcal{P}(E)$ and define

$$F(x) := \sup\{t \in \mathbb{R} \mid x \in X_t\}, \tag{3.4}$$

then $\mathcal{X}_t(F) = X_t$ for $t \in \mathbb{R}$.

If $F \in \mathrm{Fun}(E)$ and $h \in E$, $v \in \mathbb{R}$, then we define its horizontal respectively vertical translate F_h and $F + v$ by

$$F_h(x) = F(x-h), \quad x \in E \tag{3.5}$$

$$(F+v)(x) = F(x) + v \quad x \in E. \tag{3.6}$$

The following result is straightforward.

Lemma 3.4. *Let I be some index set, let $F, F_i \in \mathrm{Fun}(E)$ for $i \in I$, and let $h \in E$, $v \in \mathbb{R}$. Then*
(a) $\mathcal{X}_t(F_h) = \mathcal{X}_t(F)_h$
(b) $\mathcal{X}_t(F+v) = \mathcal{X}_{t-v}(F)$
(c) $\mathcal{X}_t(\bigwedge_{i\in I} F_i) = \bigcap_{i\in I} \mathcal{X}_t(F_i)$
(d) $\mathcal{X}_t(\bigvee_{i\in I} F_i) \supseteq \bigcup_{i\in I} \mathcal{X}_t(F_i)$
(e) $\mathcal{X}_t(\bigvee_{i\in I} F_i) = \bigcap_{s<t} \bigcup_{i\in I} \mathcal{X}_s(F_i)$.

The reason that equality in (d) can not be proved is that $\bigvee_{i\in I} \mathcal{X}_t(F_i)$ needs not be continuously decreasing. (e) shows how one can overcome this deficiency.

In fact, Lemma 3.4(c) states that the operator $\mathcal{X}_t : \mathrm{Fun}(E) \to \mathcal{P}(E)$ is an erosion. The adjoint dilation, which we denote by $\mathcal{F}_t$, is a mapping from $\mathcal{P}(E)$ to $\mathrm{Fun}(E)$ given by

$$\mathcal{F}_t(X) = \begin{cases} t, & x \in X \\ -\infty, & x \notin X. \end{cases} \tag{3.7}$$

In this paper we shall denote mappings (or operators) on $\mathcal{P}(E)$ by lower case Greek letters such as ψ, ϕ, etc, and we call them *set operators*. On the other hand, mappings on $\mathrm{Fun}(E)$ are denoted by uppercase Greek letters such as Ψ, Φ, etc, and they are called *function operators*.

Definition 3.5. Let Ψ be a function operator. We call Ψ an *H-operator* if Ψ is invariant under *horizontal* translations, i.e., $\Psi(F_h) = \Psi(F)_h$. If, in addition, Ψ is invariant under *vertical* translations, i.e., $\Psi(F + v) = \Psi(F) + v$, then Ψ is called a *T-operator.*

We define the *dual operator* Ψ^* by

$$\Psi^*(F) := (\Psi(F^*))^*. \tag{3.8}$$

It is clear that Ψ^* is a dilation if Ψ is an erosion, etc.

> *Throughout the rest of this paper we shall deal exclusively with increasing operators without mentioning this explicitly. Unless otherwise stated "operator" will always mean "increasing operator".*

4. From set operators to function operators

We denote by $\{\psi_t\}_{t\in\mathbb{R}}$ denote a family of increasing set operators which is decreasing with respect to t, that is,

$$\psi_t \leq \psi_s \quad \text{if } s \leq t.$$

It is obvious that by means of the definition $X_t := \bigcap_{s<t} \psi_s(\mathcal{X}_s(F))$, $t \in \mathbb{R}$, we get a continuously decreasing family in $\mathcal{P}(E)$ (see Definition 3.2) to which, by Proposition 3.3, there corresponds a unique function. This function, which we call $\Psi(F)$, is characterized by either of the following two relations:

$$\Psi(F)(x) = \sup\{t \mid x \in \psi_t(\mathcal{X}_t(F))\} \tag{4.1}$$

$$\mathcal{X}_t(\Psi(F)) = \bigcap_{s<t} \psi_s(\mathcal{X}_s(F)). \tag{4.2}$$

Repeating this procedure for any F in $\text{Fun}(E)$ we obtain a function operator Ψ, and we say that Ψ is *generated* by $\{\psi_t\}_{t\in\mathbb{R}}$. It is easily seen that Ψ is increasing.

One can easily construct counterexamples which show that the identity $\mathcal{X}_t(\Psi(F)) = \psi_t(\mathcal{X}_t(F))$ is in general false. But it is true under some extra assumptions.

Definition 4.1. Let $\{\psi_t\}_{t\in\mathbb{R}}$ be a decreasing family of set operators. We say that $\psi_s \downarrow \psi_t$ for $s \uparrow t$ if

$$\bigcap_{s<t} \psi_s(X) = \psi_t(X)$$

for every $X \in \mathcal{P}(E)$. If this relation holds for every $t \in \mathbb{R}$ then we call the family $\{\psi_t\}_{t\in\mathbb{R}}$ *continuously decreasing.*

Proposition 4.2. *Let $\{\psi_t\}_{t\in\mathbb{R}}$ be a continuously decreasing family of $\downarrow$-continuous set operators, and let Ψ be the generated function operator. Then*

$$\mathcal{X}_t(\Psi(F)) = \psi_t(\mathcal{X}_t(F)), \tag{4.3}$$

and Ψ is a $\downarrow$-continuous operator on $\text{Fun}(E)$.

PROOF. Let the assumptions of the proposition be satisfied, let $t \in \mathbb{R}$ and $F \in \mathrm{Fun}(E)$. Then

$$\begin{aligned}\mathcal{X}_t(\Psi(F)) &= \bigcap_{s<t} \psi_s(\bigcap_{r<s} \mathcal{X}_r(F)) \\ &= \bigcap_{s<t}\bigcap_{r<s} \psi_s(\mathcal{X}_r(F)) \\ &= \bigcap_{r<t}\bigcap_{r<s<t} \psi_s(\mathcal{X}_r(F)) \\ &= \bigcap_{r<t} \psi_t(\mathcal{X}_r(F)) = \psi_t(\mathcal{X}_t(F)),\end{aligned}$$

which proves the first assertion.

To prove the second assertion assume that $F_n \downarrow F$. We must show that $\Psi(F_n) \downarrow \Psi(F)$, or equivalently that

$$\bigwedge_{n\geq 1} \Psi(F_n) = \Psi(F).$$

Let $t \in \mathbb{R}$. Then by Lemma 3.4(c) we have

$$\begin{aligned}\mathcal{X}_t(\bigwedge_{n\geq 1} \Psi(F_n)) &= \bigcap_{n\geq 1} \mathcal{X}_t(\Psi(F_n)) \\ &= \bigcap_{n\geq 1}\bigcap_{s<t} \psi_s(\mathcal{X}_s(F_n)) \\ &= \bigcap_{s<t}\bigcap_{n\geq 1} \psi_s(\mathcal{X}_s(F_n)).\end{aligned}$$

Since, for every s, $\mathcal{X}_s(F_n) \downarrow \mathcal{X}_s(F)$ if $n \to \infty$, and since ψ_s is $\downarrow$-continuous we get that

$$\begin{aligned}\mathcal{X}_t(\bigwedge_{n\geq 1} \Psi(F_n)) &= \bigcap_{s<t} \psi_s(\bigcap_{n\geq 1} \mathcal{X}_s(F_n)) \\ &= \bigcap_{s<t} \psi_s(\mathcal{X}_s(F)) \\ &= \mathcal{X}_t(\Psi(F)).\end{aligned}$$

This proves the result.

∎

Proposition 4.3. *If $\{\psi_t\}_{t\in\mathbb{R}}$ is a decreasing family of set operators which generates Ψ, then $\{\psi^*_{-t}\}_{t\in\mathbb{R}}$ is also a decreasing family and it generates Ψ^*.*

PROOF. We define $\mathcal{X}^\circ_t(F) := \{x \mid F(x) > t\}$, and observe that the following identities hold for every function F and every $t \in \mathbb{R}$:

$$\mathcal{X}_t(F) = \bigcap_{s<t} \mathcal{X}^\circ_s(F) \tag{4.4}$$

$$\mathcal{X}_t(-F) = \left(\mathcal{X}^\circ_{-t}(F)\right)^c \tag{4.5}$$

$$\bigcap_{s<t} \psi_s(\mathcal{X}_s(F)) = \bigcap_{s<t} \psi_s(\mathcal{X}^\circ_s(F)). \tag{4.6}$$

With these relations and the identity $\Psi^*(F) = -\Psi(-F)$ we get

$$\begin{aligned}
\mathcal{X}_t(\Psi^*(F)) &= (\mathcal{X}^\circ_{-t}(\Psi(-F)))^c = (\bigcup_{s<t} \mathcal{X}_{-s}(\Psi(-F)))^c \\
&= (\bigcup_{s<t} \bigcap_{r<-s} \psi_r(\mathcal{X}_r(-F)))^c = (\bigcup_{s<t} \bigcap_{r>s} \psi_{-r}(\mathcal{X}_{-r}(-F)))^c \\
&= (\bigcup_{s<t} \bigcap_{r>s} \psi_{-r}((\mathcal{X}^\circ_r(F))^c))^c = (\bigcup_{s<t} \bigcap_{r>s} (\psi^*_{-r}(\mathcal{X}^\circ_r(F)))^c)^c \\
&= \bigcap_{s<t} \bigcup_{r>s} \psi^*_{-r}(\mathcal{X}^\circ_r(F)) = \bigcap_{r<t} \psi^*_{-r}(\mathcal{X}^\circ_r(F)) \\
&= \bigcap_{r<t} \psi^*_{-r}(\mathcal{X}_r(F)),
\end{aligned}$$

by (4.6). This concludes the proof.

∎

Proposition 4.4. *Let $\{\psi_t\}_{t\in\mathbb{R}}$ and $\{\psi'_t\}_{t\in\mathbb{R}}$ be decreasing families of set operators which generate the function operators Ψ and Ψ' respectively. Then the composition $\Psi'\Psi$ is generated by $\{\psi'_t\psi_t\}_{t\in\mathbb{R}}$.*

PROOF. Let $F \in \mathrm{Fun}(E)$ and $t \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{X}_t(\Psi'\Psi(F)) &= \bigcap_{v>0} \psi'_{t-v}(\mathcal{X}_{t-v}(\Psi(F))) \\
&= \bigcap_{v>0} \psi'_{t-v}(\bigcap_{w>0} \psi_{t-v-w}(\mathcal{X}_{t-v-w}(F))) \\
&\subseteq \bigcap_{v>0} \psi'_{t-v}(\psi_{t-2v}(\mathcal{X}_{t-2v}(F))) \\
&\subseteq \bigcap_{v>0} \psi'_{t-2v}\psi_{t-2v}(\mathcal{X}_{t-2v}(F)) \\
&= \bigcap_{v>0} \psi'_{t-v}\psi_{t-v}(\mathcal{X}_{t-v}(F)).
\end{aligned}$$

The reverse inclusion $\supseteq$ is easy.

∎

Proposition 4.5. *Let, for every i in I, $\{\psi_t^{(i)}\}_{t\in\mathbb{R}}$ generate $\Psi^{(i)}$. Then $\{\bigwedge_{i\in I}\psi_t^{(i)}\}_{t\in\mathbb{R}}$ generates $\bigwedge_{i\in I}\Psi^{(i)}$ and $\{\bigvee_{i\in I}\psi_t^{(i)}\}_{t\in\mathbb{R}}$ generates $\bigvee_{i\in I}\Psi^{(i)}$.*

PROOF. The first statement is easy. The second statement follows easily from the first by duality: use Proposition 4.3.

∎

Proposition 4.6. *Let the family $\{\psi_t\}_{t\in\mathbb{R}}$ generate Ψ. If every ψ_t is an erosion (resp. dilation, closing, opening) then Ψ is an erosion (resp. dilation, closing, opening).*

PROOF. Let ψ_t be a decreasing family of set erosions generating the function operator Ψ. To show that Ψ is an erosion we must show that $\Psi(\bigwedge F_i) = \bigwedge \Psi(F_i)$ for any family F_i ($i \in I$), or, equivalently, that $\mathcal{X}_t(\Psi(\bigwedge F_i)) = \mathcal{X}_t(\bigwedge \Psi(F_i))$, for every $t \in \mathbb{R}$. We use (4.2):

$$\begin{aligned}
\mathcal{X}_t(\Psi(\bigwedge F_i)) &= \bigcap_{s<t} \psi_s(\mathcal{X}_s(\bigwedge F_i)) = \bigcap_{s<t} \psi_s(\bigcap \mathcal{X}_s(F_i)) \\
&= \bigcap_{s<t} \bigcap \psi_s(\mathcal{X}_s(F_i)) = \bigcap \bigcap_{s<t} \psi_s(\mathcal{X}_s(F_i)) \\
&= \bigcap \mathcal{X}_t(\Psi(F_i)) = \mathcal{X}_t(\bigwedge \Psi(F_i)).
\end{aligned}$$

To prove the same result for dilations, one proceeds as follows. If $\{\psi_t\}_{t\in\mathbb{R}}$ is a decreasing family of dilations, then $\{\psi^*_{-t}\}_{t\in\mathbb{R}}$ is a decreasing family of erosions. Now apply Proposition 4.3 and the result above. One gets that Ψ^* is an erosion and hence that Ψ is a dilation.

Now suppose that every ψ_t is a closing. Then $\psi_t \geq \mathrm{id}$ which immediately yields that $\Psi \geq \mathrm{id}$. To prove idempotence of Ψ one may apply Proposition 4.4 with $\psi'_t = \psi_t$. The proof for openings is analogous.

∎

Proposition 4.7. *Let the family $\{\psi_t\}_{t\in\mathbb{R}}$ generate Ψ. If every ψ_t is translation-invariant, then Ψ is an H-operator.*

5. Flat operators

In this section we consider a particular class of increasing operators, called *flat operators.* A function operator Ψ is called *flat* if it is generated by a set operator ψ, that is

$$\mathcal{X}_t(\Psi(F)) = \bigcap_{s<t} \psi(\mathcal{X}_s(F)). \tag{5.1}$$

or, equivalently, that

$$\Psi(F)(x) = \sup\{t \mid x \in \psi(\mathcal{X}_t(F))\}. \tag{5.2}$$

We call ψ the *generator* of Ψ. The class of flat operators has been extensively studied in the literature, under many different names. Maragos [7] calls them *FSP filters* (FSP being the abbreviation of "function-set processing"), and refers to relation (5.2) as the *threshold superposition principle.* Following Serra [12, Chapter XII], Maragos restricts to upper-semi-continuous functions, in which case (5.1) reduces to (compare Proposition 5.7 below)

$$\mathcal{X}_t(\Psi(F)) = \psi(\mathcal{X}_t(F)).$$

It follows immediately that any set operator ψ satisfying this latter relation (for some function operator Ψ) must be increasing. This is not necessarily the case for operators satisfying (5.1). In [18], operators satisfying (5.1)-(5.2) are called *stack filters.* In that paper one deals exclusively with a finite grey-level set: see also Section 10.

Note that every flat operator Ψ has a unique generator ψ which is given by

$$\psi = \mathcal{X}_t \circ \Psi \circ \mathcal{F}_t, \quad t \in \mathbb{R} \cup \{\infty\}. \tag{5.3}$$

The latter expression does not depend on t. It is easily understood that Ψ is an H-operator if and only if ψ is translation invariant.

Proposition 5.1. *Any flat operator Ψ is invariant under vertical translations, i.e.,*

$$\Psi(F+v) = \Psi(F) + v,$$

for $F \in \mathrm{Fun}(E)$ and $v \in \mathbb{R}$. Conversely, if Ψ is generated by the set operators $\{\psi_t\}_{t\in\mathbb{R}}$, and if Ψ is invariant under vertical translations, then all ψ_t are the same and Ψ is a flat operator.

Proof. Let $F \in \mathrm{Fun}(E)$ and $v \in \mathbb{R}$. Then, for $t \in \mathbb{R}$,

$$\begin{aligned}\mathcal{X}_t(\Psi(F+v)) &= \bigcap_{s<t} \psi(\mathcal{X}_s(F+v)) = \bigcap_{s<t} \psi(\mathcal{X}_{s-v}(F)) \\ &= \bigcap_{s<t-v} \psi(\mathcal{X}_s(F)) = \mathcal{X}_{t-v}(\Psi(F)) \\ &= \mathcal{X}_t(\Psi(F)+v).\end{aligned}$$

To prove the second result, assume that $\mathcal{X}_t(\Psi(F)) = \bigcap_{s<t} \psi_s(\mathcal{X}_s(F))$ and that Ψ is invariant under vertical translations. This yields that for $v \in \mathbb{R}$,

$$\mathcal{X}_t(\Psi(F)+v) = \mathcal{X}_{t-v}(\Psi(F)) = \bigcap_{s<t-v} \psi_s(\mathcal{X}_s(F)).$$

On the other hand,

$$\mathcal{X}_t(\Psi(F+v)) = \bigcap_{s<t} \psi_s(\mathcal{X}_s(F+v)) = \bigcap_{s<t} \psi_s(\mathcal{X}_{s-v}(F)).$$

Substituting $F = \mathcal{F}_\infty(X)$, where $X \subset E$, in both expressions and using that $\mathcal{X}_s(\mathcal{F}_\infty(X)) = X$ we find that $\bigcap_{s<t-v} \psi_s(X) = \bigcap_{s<t} \psi_s(X)$. From this equality, which holds for every $t, v \in \mathbb{R}$, the assertion follows.

∎

This result implies in particular that every flat H-operator is a T-operator.

Before we continue our exposition we shall justify the adjective "flat" by showing that a flat operator maps flat functions (the functions $\mathcal{F}_t(X)$ to be precise) onto flat functions.

Propositon 5.2. *Let Ψ be a flat function operator generated by the set operator ψ, and assume that $\psi(\emptyset) = \emptyset$. Then*

$$\Psi(\mathcal{F}_t(X)) = \mathcal{F}_t(\psi(X)),$$

for $t \in \mathbb{R} \cup \{\infty\}$ and $X \in \mathcal{P}(E)$.

Proof. We use (5.2). Let $t \in \mathbb{R}$ and $X \subseteq E$. Then, since $\mathcal{X}_s(\mathcal{F}_t(X)) = X$ if $s \le t$,

$$\begin{aligned}\Psi(\mathcal{F}_t(X))(x) &= \sup\{s \in \mathbb{R} \mid x \in \psi(\mathcal{X}_s(\mathcal{F}_t(X)))\} \\ &= \sup\{s \le t \mid x \in \psi(X)\} \\ &= \mathcal{F}_t(\psi(X)).\end{aligned}$$

The proof for $t = \infty$ is left to the reader.

∎

Remarks 5.3.

(a) From (5.3) it can easily be shown that $\Psi(\mathcal{O}) = \mathcal{O}$ iff $\psi(\emptyset) = \emptyset$ and that $\Psi(\mathcal{I}) = \mathcal{I}$ iff $\psi(E) = E$.

(b) It is easy to adapt this result for the case that $\psi(\emptyset) \neq \emptyset$. In this case $\Psi(F)$ takes the value ∞ on the set $\psi(\emptyset)$, no matter what F is. Namely, $\psi(\emptyset) \subseteq \mathcal{X}_t(\Psi(F))$ for every t and F. Note that the only translation-invariant set operator ψ for which $\psi(\emptyset) \neq \emptyset$ is the trivial operator given by $\psi(X) = E$ for every X.

(c) The converse of Proposition 5.2 does not hold: there exist function operators which map any flat function onto a flat function but which are not flat. The following may serve as an example. Define, for a function F, $M_F := \sup\{F(x) \mid x \in E\}$. Define the function operator Ψ as follows:

$$\Psi(F)(x) := \begin{cases} M_F, & \text{if } F(x) > -\infty \\ -\infty, & \text{if } F(x) = -\infty. \end{cases}$$

Notice that Ψ is a T-operator and a closing. One sees immediately that $\Psi \circ \mathcal{F}_t = \mathcal{F}_t \circ \mathrm{id}$, but that (5.1) does not hold with $\psi = \mathrm{id}$.

We present an alternative way to characterize flat operators. Thereto we define for $t \in \mathbb{R} \cup \{\infty\}$ the thresholding operators $e_t : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by

$$e_t(s) = \begin{cases} \infty, & s \geq t \\ -\infty, & s < t. \end{cases} \tag{5.4}$$

It is obvious that e_t is an erosion on the complete lattice $\overline{\mathbb{R}}$. We can extend e_t to Fun(E) by means of the definition $e_t(F)(x) := e_t(F(x))$. Then

$$\mathcal{X}_s \circ e_t = \mathcal{X}_t, \quad s, t \in \mathbb{R} \cup \{\infty\}. \tag{5.5}$$

Proposition 5.4. *The function operator Ψ is flat if and only if the identity*

$$e_t \circ \Psi = \bigwedge_{s<t} \Psi \circ e_s \tag{5.6}$$

holds for every $t \in \mathbb{R} \cup \{\infty\}$.

PROOF.
"if": Assume that (5.6) holds. We define the set operator ψ by (5.3) with $t = \infty$, that is $\psi = \mathcal{X}_\infty \circ \Psi \circ \mathcal{F}_\infty$. Applying $\mathcal{X}_\infty$ to (5.6) yields that

$$\mathcal{X}_\infty \circ e_t \circ \Psi = \mathcal{X}_\infty \circ \bigwedge_{s<t} \Psi \circ e_s.$$

Using that $\mathcal{X}_\infty \circ e_t = \mathcal{X}_t$ and Lemma 3.4(c) we get that

$$\mathcal{X}_t \circ \Psi = \bigwedge_{s<t} \mathcal{X}_\infty \circ \Psi \circ e_s.$$

Now we use that $e_s = \mathcal{F}_\infty \circ \mathcal{X}_s$ as an operator on Fun(E), and get

$$\mathcal{X}_t \circ \Psi = \bigwedge_{s<t} \mathcal{X}_\infty \circ \Psi \circ \mathcal{F}_\infty \circ \mathcal{X}_s = \bigwedge_{s<t} \psi \circ \mathcal{X}_s,$$

which is (5.2). This shows that Ψ is flat.
"only if": Assume that (5.2) holds, so

$$\mathcal{X}_t \circ \Psi = \bigwedge_{s<t} \psi \circ \mathcal{X}_s.$$

To conclude that (5.6) holds, it suffices to show that

$$\mathcal{X}_\tau \circ e_t \circ \Psi = \mathcal{X}_\tau \circ \bigwedge_{s<t} \Psi \circ e_s,$$

for every $\tau \in \mathbb{R}$. Now

$$\begin{aligned} \mathcal{X}_\tau \circ \bigwedge_{s<t} \Psi \circ e_s &= \bigwedge_{s<t} \mathcal{X}_\tau \circ \Psi \circ e_s = \bigwedge_{s<t} \Big(\bigwedge_{\sigma<\tau} \psi \circ \mathcal{X}_\sigma \Big) \circ e_s \\ &= \bigwedge_{s<t} \bigwedge_{\sigma<\tau} \psi \circ \mathcal{X}_\sigma \circ e_s = \bigwedge_{s<t} \bigwedge_{\sigma<\tau} \psi \circ \mathcal{X}_s \\ &= \bigwedge_{s<t} \psi \circ \mathcal{X}_s = \mathcal{X}_t \circ \Psi = \mathcal{X}_\tau \circ e_t \circ \Psi, \end{aligned}$$

which proves the result.

∎

An important feature of flat operators is their compatibility under increasing mappings on the set of grey-levels (or *anamorphoses* as Serra [12] calls them). The following result was inspired by [6].

Proposition 5.5. *Let Ψ be a flat operator and assume that $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a function such that the following are satisfied*

(i) h is continuous and strictly monotone increasing

(ii) $\Psi(\mathcal{I}) = \mathcal{I}$ if $h(-\infty) > -\infty$

(iii) $\Psi(\mathcal{O}) = \mathcal{O}$ if $h(\infty) < \infty$.

Then

$$\Psi \circ h = h \circ \Psi. \tag{5.7}$$

In particular, this compatibility condition holds for every order automorphism h on $\overline{\mathbb{R}}$.

PROOF.

(1): since h is continuous and strictly increasing it has a well-defined inverse h^{-1} on the interval $(h(-\infty), h(\infty))$. Let $t \in (h(-\infty), h(\infty))$. Then

$$\begin{aligned} \mathcal{X}_t(h \circ \Psi(F)) &= \mathcal{X}_{h^{-1}(t)}(\Psi(F)) = \bigcap_{s<h^{-1}(t)} \psi(\mathcal{X}_s(F)) \\ &= \bigcap_{s<t} \psi(\mathcal{X}_{h^{-1}(s)}(F)) = \bigcap_{s<t} \psi(\mathcal{X}_s(h \circ F)) \\ &= \mathcal{X}_t(\Psi \circ h(F)). \end{aligned}$$

(2): suppose $h(-\infty) > -\infty$ and $t \in (-\infty, h(-\infty))$. Then $\mathcal{X}_t(h \circ \Psi(F)) = E$. On the other hand, by (iii),

$$\mathcal{X}_t(\Psi \circ h(F)) = \bigcap_{s<t} \psi(\mathcal{X}_s(h(F)) = \bigcap_{s<t} \psi(E) = E.$$

(3): the proof for $t \in (h(\infty), \infty)$ in case $h(\infty) < \infty$ follows by a similar argument as in (2).

∎

The following is an immediate consequence of Propositions 4.4 and 4.5.

Corollary 5.6. *The class of flat operators is closed under arbitrary suprema and infima and finite compositions.*

Proposition 5.7. *Let the flat operator Ψ be generated by the set operator ψ. If ψ is $\downarrow$-continuous then Ψ is $\downarrow$-continuous as well, and*

$$\mathcal{X}_t(\Psi(F)) = \psi(\mathcal{X}_t(F)), \tag{5.8}$$

for every $t \in \mathbb{R}$. Conversely, if (5.8) holds then both ψ and Ψ are $\downarrow$-continuous.

PROOF. The first statement is an immediate consequence of Proposition 4.2. To prove the converse result, assume that (5.8) is satisfied, and that $Y_n \downarrow Y$. Choose $\tau \in \mathbb{R}$ arbitrary and let $\{t_n\}$ be a sequence in $\mathbb{R}$ which is strictly increasing and converges towards τ. Let the function F be defined through its threshold sets $\mathcal{X}_t(F)$ given by

$$\mathcal{X}_t(F) = \begin{cases} E, & \text{if } t \le t_1 \\ Y_n, & \text{if } t_n < t \le t_{n+1},\ n \ge 1 \\ Y, & \text{if } t = \tau \\ \emptyset, & \text{if } t > \tau. \end{cases}$$

Note that the family $\mathcal{X}_t(F)$ thus defined is continuously decreasing. Now,

$$\begin{aligned} \psi(Y) = \psi(\mathcal{X}_\tau(F)) &= \mathcal{X}_\tau(\Psi(F)) = \bigcap_{t<\tau} \mathcal{X}_t(\Psi(F)) \\ &= \bigcap_{t<\tau} \psi(\mathcal{X}_t(F)) = \bigcap_{n\ge 1} \psi(Y_n). \end{aligned}$$

Therefore ψ is $\downarrow$-continuous, and it follows from the first assertion that Ψ is $\downarrow$-continuous as well.

∎

As a special case of Proposition 4.3 we get the following result.

Proposition 5.8. *If Ψ is a flat operator generated by the set operator ψ, then Ψ^* is a flat operator which is generated by ψ^*.*

Similarly, Proposition 4.6 can be specialized to the case of flat operators. The function operator generated by a set dilation is again a dilation and we call it a flat dilation. Similarly, flat erosions, closings, and openings are defined. The following result states that if one of the operators in an adjunction on Fun(E) is flat then both are.

Proposition 5.9. *Let $(\mathcal{E}, \Delta)$ be an adjunction on* Fun(E) *and suppose that at least one of the two operators is flat. Then both operators are flat. If ε and δ are the respective generators of $\mathcal{E}$ and Δ, then (ε, δ) defines an adjunction on $\mathcal{P}(E)$.*

PROOF. Suppose e.g. that $\mathcal{E}$ is a flat erosion and let ε be the set erosion which generates $\mathcal{E}$. Let δ be the corresponding set dilation and, finally, let Δ be the function dilation generated by δ. We show that $(\mathcal{E}, \Delta)$ is an adjunction. Since Δ is uniquely determined by $\mathcal{E}$, this implies the result.
(i) Let $F, G \in$ Fun(E) be such that $\Delta(F) \leq G$. We show that $F \leq \mathcal{E}(G)$. For $t \in \mathbb{R}$ we have $\mathcal{X}_t(\Delta(F)) = \bigcap_{s<t} \delta(\mathcal{X}_s(F)) \subseteq \mathcal{X}_t(G)$, hence $\delta(\mathcal{X}_t(F)) \subseteq \mathcal{X}_t(G)$. Since (ε, δ) is an adjunction we get that $\mathcal{X}_t(F) \subseteq \varepsilon(\mathcal{X}_t(G)) = \mathcal{X}_t(\mathcal{E}(G))$. Here we have used Proposition 5.7 and the fact that any erosion is $\downarrow$-continuous. We conclude that $F \leq \mathcal{E}(G)$.
(ii) Let $F \leq \mathcal{E}(G)$. We show that $\Delta(F) \leq G$. Let $s \in \mathbb{R}$. Then $\mathcal{X}_s(F) \subseteq \mathcal{X}_s(\mathcal{E}(G)) = \varepsilon(\mathcal{X}_s(G))$. Since (ε, δ) is an adjunction, $\delta(\mathcal{X}_s(F)) \subseteq \mathcal{X}_s(G)$. Let $t \in \mathbb{R}$, the taking the intersection for all $s < t$ yields $\bigcap_{s<t} \delta(\mathcal{X}_s(F)) \subseteq \bigcap_{s<t} \mathcal{X}_s(G) = \mathcal{X}_t(G)$, or equivalently, $\mathcal{X}_t(\Delta(F)) \subseteq \mathcal{X}_t(G)$. This implies that $\Delta(F) \leq G$.

∎

Proposition 5.10. (Matheron's theorem for flat H-operators)
Every flat H-operator can be decomposed as a supremum of flat H-erosions, or, alternatively, as an infimum of flat H-dilations.

PROOF. Let Ψ be a flat H-operator generated by ψ. Then ψ is a translation invariant operator on $\mathcal{P}(E)$, hence Proposition 2.2 applies. Now the result follows easily from Corollary 5.6.

∎

Example 5.11. Let δ_B be the set dilation given by $\delta_B(X) = X \oplus B$, where B is some subset of E. Then the corresponding flat dilation Δ_B on Fun(E) is given by

$$\Delta_B(F) = \bigvee_{b \in B} F_b,$$

or, alternatively,

$$\Delta_B(F)(x) = \bigvee_{b \in B} F(x - b).$$

Namely,

$$\begin{aligned} \mathcal{X}_t(\Delta_B(F)) &= \bigcap_{s<t} \delta_B(\mathcal{X}_s(F)) = \bigcap_{s<t} \bigcup_{b \in B} (\mathcal{X}_s(F))_b \\ &= \bigcap_{s<t} \bigcup_{b \in B} \mathcal{X}_s(F_b) = \mathcal{X}_t(\bigvee_{b \in B} F_b). \end{aligned}$$

Here we have used Lemma 3.4(e). The adjoint erosion $\mathcal{E}_B$ is given by

$$\mathcal{E}_B(F)(x) = \bigwedge_{b \in B} F(x+b).$$

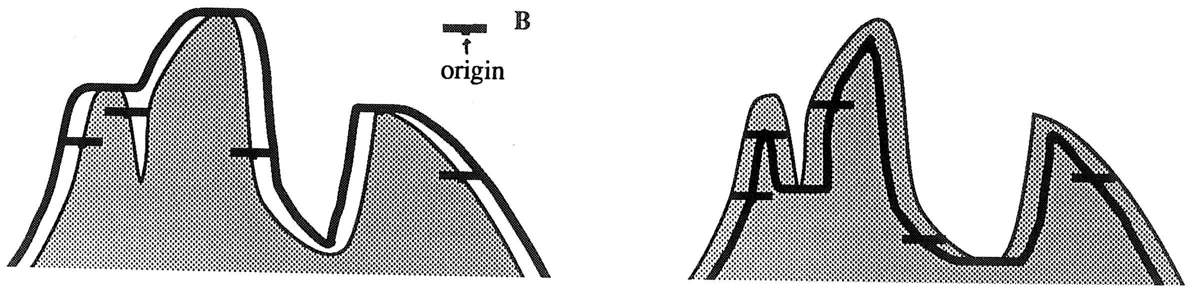

FIGURE 2. *Dilation and erosion by a flat structuring element.*

6. T-operators

There exist a number of nice results concerning increasing translation-invariant operators on the Boolean lattice $\mathcal{P}(E)$. A number of them have been extended to the abstract framework of complete lattices: here translation invariance has been generalized to invariance under some abelian group of automorphisms: see [4]. The abstract results can be applied to the case where the object space is the complete lattice of all grey-level functions Fun(E), and where the automorphism group consists of all horizontal $(F \to F_h)$ and vertical $(F \to F + v)$ translations. In Definition 3.5 the operators which are invariant under this translation group have been called T-operators. In [4] it is shown that every T-dilation is of the form $\Delta = \Delta_G$ given by

$$\Delta_G(F)(x) = (F \oplus G)(x) = \bigvee_{h \in E} [F(x-h) + G(h)], \tag{6.1}$$

for some $G \in$ Fun(E). The adjoint erosion $\mathcal{E}_G$ is given by

$$\mathcal{E}_G(F)(x) = (F \ominus G)(x) = \bigwedge_{h \in E} [F(x+h) - G(h)]. \tag{6.2}$$

It is easy to deduce the following duality relations:

$$(F^* \oplus G)^* = F \ominus \overline{G}, \tag{6.3}$$

or alternatively

$$(F^* \ominus G)^* = F \oplus \overline{G}. \tag{6.4}$$

Here $\overline{G}(x) := G(-x)$. There also exists formulas for $F \oplus G$ and $F \ominus G$ which have a nice geometric interpretation:

$$(F \oplus G)(x) = \inf\{v \in \mathbb{R} \mid -(\overline{G})_x + v \geq F\} \qquad (6.5),$$

$$(F \ominus G)(x) = \sup\{v \in \mathbb{R} \mid G_x + v \leq F\}. \tag{6.6}$$

Here we shall derive the second identity. The first one can be proved analogously.

$$\begin{aligned}(F \ominus G)(x) &= \inf_{h \in E}[F(x+h) - G(h)] \\ &= \sup\{v \in \mathbb{R} \mid \forall h \in E: \; v \leq F(x+h) - G(h)\} \\ &= \sup\{v \in \mathbb{R} \mid \forall y \in E: \; v \leq F(y) - G(y-x)\} \\ &= \sup\{v \in \mathbb{R} \mid G_x + v \leq F\}.\end{aligned}$$

One can also prove Matheron's theorem for T-operators. Thereto we define the kernel $\mathcal{V}(\Psi)$ of a function operator Ψ by

$$\mathcal{V}(\Psi) = \{G \in \mathrm{Fun}(E) \mid \Psi(G)(0) \geq 0\}.$$

For a proof of the following result we refer to [4].

Proposition 6.1. (Matheron's theorem for T-operators)
For any T-operator Ψ on Fun(E) *we have*

$$\Psi(F) = \bigvee_{G \in \mathcal{V}(\Psi)} F \ominus G,$$

as well as

$$\Psi(F) = \bigwedge_{G \in \mathcal{V}(\Psi^*)} F \oplus \overline{G}.$$

7. H-operators

In this section we shall investigate function operators which are invariant under horizontal translations, the so-called H-operators. Although we do not have an explicit characterization of dilations and erosions like in the T-invariant case, one can still establish some general properties.

First we note that any composition, supremum, and infimum of H-operators yields again an H-operator. In [4], we have derived an implicit characterization of H-dilations and H-erosions in terms of adjunctions on $\overline{\mathbb{R}}$. These objects have been treated in Example 2.1.

Now suppose that for every $h \in E$, d_h is a dilation on $\overline{\mathbb{R}}$, and e_h is the adjoint erosion. Then the function operator Δ given by

$$\Delta(F)(x) = \bigvee_{h \in E} d_h(F(x - h)) \tag{7.1}$$

defines an H-dilation. The adjoint erosion $\mathcal{E}$ is given by

$$\mathcal{E}(F)(x) = \bigwedge_{h \in E} e_h(F(x+h)). \tag{7.2}$$

In [4] we have shown that all H-dilations and H-erosions are of the form (7.1) respectively (7.2). The following duality relations exist. Defining $d_h^*(t) = -d_h(-t)$ and $e_h^*(t) = -e_h(-t)$ we get that d_h^* is an erosion and e_h^* a dilation on $\overline{\mathbb{R}}$. Furthermore, the function erosion Δ^* (that is, the dual operator of Δ) is given by

$$\Delta^*(F)(x) = \inf_{h \in E} d_h^*(F(x-h)),$$

and the function dilation $\mathcal{E}^*$ is given by

$$\mathcal{E}^*(F)(x) = \sup_{h \in E} e_h^*(F(x+h)).$$

If we choose $d_h(t) = A(h)t + G(h)$, where $A(h) > 0$, then $e_h(t) = (t - G(h))/A(h)$ (see Example 2.1), and

$$\Delta(F)(x) = \bigvee_{h \in E} [A(h)F(x-h) + G(h)]$$

$$\mathcal{E}(F)(x) = \bigwedge_{h \in E} [\frac{1}{A(h)}(F(x+h) - G(h))].$$

Note that for $A \equiv 1$ we are back in the T-invariant case. If $G \equiv 0$ then Δ is a dilation by the so-called *multiplicative structuring function* A: see [4]. Such dilations are invariant under vertical multiplication, i.e.,

$$\Delta(v \cdot F) = v \cdot \Delta(F),$$

for $v \in \mathbb{R}$ and $F \in \text{Fun}(E)$.

We now state Matheron's theorem for H-operators.

Proposition 7.1. (Matheron's theorem for H-operators)
(a) Every H-operator Ψ satisfying $\Psi(\mathcal{O}) = \mathcal{O}$ can be written as an infimum of H-dilations.
(b) Every H-operator Ψ satisfying $\Psi(\mathcal{I}) = \mathcal{I}$ can be written as an supremum of H-erosions.

PROOF. We only prove here the first statement. Then the second follows by duality. Let $\Psi(\mathcal{O}) = \mathcal{O}$, and let $\mathcal{D}$ be the set of all H-dilations Δ which dominate Ψ, that is $\Delta \geq \Psi$. It is clear that $\bigwedge \mathcal{D} \geq \Psi$. To prove the reverse inequality it sufices to show that for every $F \in \text{Fun}(E)$ there is a $\Delta \in \mathcal{D}$ such that

$$\Psi(F)(0) \geq \Delta(F)(0). \tag{$*$}$$

Namely, this yields that

$$\Psi(F)(0) \geq (\bigwedge \mathcal{D})(F)(0),$$

for every $F \in \text{Fun}(E)$, and hence that

$$\Psi(F)(x) = \Psi(F_{-x})(0) \geq (\bigwedge \mathcal{D})(F_{-x})(0) = (\bigwedge \mathcal{D})(F)(x),$$

whence we finally obtain that $\Psi \geq \bigwedge \mathcal{D}$.

To prove ($*$), take $F \in \text{Fun}(E)$. For $h \in E$ we define the mapping $d_h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by:

$$d_h(t) = \begin{cases} -\infty, & \text{if } t = -\infty \\ \Psi(F)(0), & \text{if } -\infty < t \leq F(-h) \\ \infty, & \text{if } t > F(-h). \end{cases}$$

Obviously, d_h is a dilation. Let Δ be the H-dilation given by (7.1). It follows immediately that

$$\Delta(F)(0) = \sup_{h \in E} d_h(F(-h)) = \Psi(F)(0).$$

So we are done if we can show that $\Delta \in \mathcal{D}$, i.e., $\Psi \leq \Delta$. First we note that $\Psi \leq \Delta$ if and only if $\Psi(F')(0) \leq \Delta(F')(0)$ for every function F'. We distinguish three cases.

(i) $F' = \mathcal{O}$: trivial.

(ii) $\mathcal{O} \neq F' \leq F$. Then

$$\Delta(F')(0) = \sup_{h \in E} d_h(F'(-h)) = \Psi(F)(0) \geq \Psi(F')(0).$$

(iii) $F' \not\leq F$. Then $\Delta(F')(0) = \infty \geq \Psi(F')(0)$.

This concludes the proof. ∎

We now give characterizations of H-dilations and H-erosions which have a straightforward geometrical interpretation: compare (6.5)–(6.6).

Proposition 7.2.

(a) *The function operator* $\mathcal{E}$ *is an H-erosion if and only if there exists a continuously increasing family* $\{G^{(t)} \mid t \in \mathbb{R}\}$ *in* $\mathrm{Fun}(E)$ *such that*

$$\mathcal{E}(F)(x) = \sup\{t \in \mathbb{R} \mid G_x^{(t)} \leq F\}. \tag{7.3}$$

(b) *The function operator* Δ *is an H-dilation if and only if there exists a continuously decreasing family* $\{H^{(t)} \mid t \in \mathbb{R}\}$ *in* $\mathrm{Fun}(E)$ *such that*

$$\Delta(F)(x) = \inf\{t \in \mathbb{R} \mid F \leq H_x^{(t)}\}. \tag{7.4}$$

PROOF. We only prove (a).
"only if": let $\mathcal{E}$ be an H-erosion, then $\mathcal{E}$ is of the form (7.2), that is,

$$\mathcal{E}(F)(x) = \bigwedge_{h \in E} e_h(F(x+h)),$$

where every e_h is an erosion on $\overline{\mathbb{R}}$. Let d_h be its adjoint dilation and define $G^{(t)}(x) := d_x(t)$. From the fact that d_x is a dilation it follows that $\{G^{(t)} \mid t \in \mathbb{R}\}$ is a continuously increasing family in $\mathrm{Fun}(E)$. Furthermore

$$\begin{aligned} \mathcal{E}(F)(x) &= \bigwedge_{h \in E} e_h(F(x+h)) \\ &= \sup\{s \in \mathbb{R} \mid s \leq \bigwedge_{h \in E} e_h(F(x+h))\} \\ &= \sup\{s \in \mathbb{R} \mid \forall h: \; d_h(s) \leq F(x+h)\} \\ &= \sup\{s \in \mathbb{R} \mid \forall h: \; G^{(s)}(h) \leq F(x+h)\} \\ &= \sup\{s \in \mathbb{R} \mid G_x^{(s)} \leq F\}. \end{aligned}$$

"if": ley $\mathcal{E}$ be given by (7.3) and define the mapping d_x on $\overline{\mathbb{R}}$, for any $x \in E$, by

$$\begin{aligned} d_x(-\infty) &= -\infty \\ d_x(t) &= G^{(t)}(x), \quad t \in \mathbb{R} \\ d_x(\infty) &= \bigvee_{t \in \mathbb{R}} G^{(t)}(x). \end{aligned}$$

It is obvious that d_x is a dilation. To prove that $\mathcal{E}$ obeys (7.2), one may read the proof given above backwards.

∎

If $G^{(t)} = G + t$ and $H^{(t)} = H + t$ for some function G and H then we are back in the T-invariant case described in the previous section: see (6.5)–(6.6).

8. Umbra's and u.s.c. functions

In this section we assume that $\mathcal{G} = \overline{\mathbb{R}}$ unless explicitly stated otherwise. Some authors [2,3,12,15,16] prefer to describe morphological operators for grey-level functions in terms of umbras instead of working with the numerical functions explicitly. Although such an approach may help to obtain a geometrical picture of the operations, it is in fact an unnecessary intermediate step and a source of many mistakes [10, Section 1]. For completeness we briefly describe the interrelation between umbras and grey-level functions.

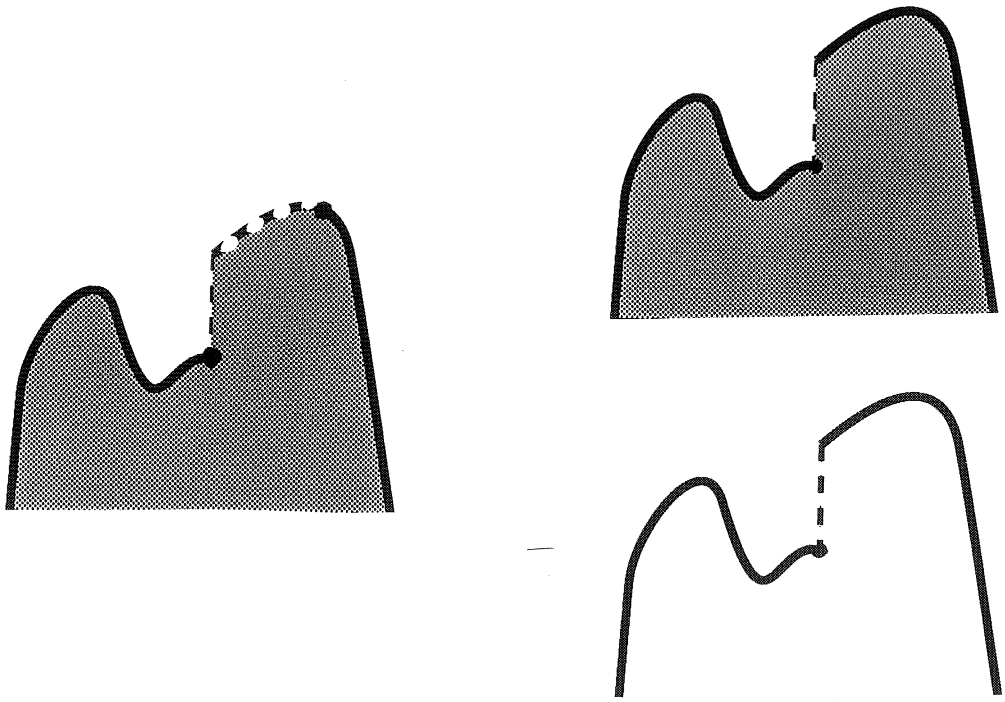

FIGURE 3. *A pseudo-umbra (left), the umbra obtained by completion (top-right), and the corresponding function (bottom-right).*

Definition 8.1. A subset $U \subset E \times \overline{\mathbb{R}}$ is called a *pseudo-umbra* if

$$(x,t) \in U \text{ implies that } (x,s) \in U \text{ for } s < t.$$

and an *umbra* if

$$(x,t) \in U \text{ if and only if } (x,s) \in U \text{ for } s < t.$$

We denote the collection of all umbras by Umbra(E).

Clearly, umbras differ from pseudo-umbras in the sense that all their vertical cross-sections (given by $x = constant$) are closed.

To every pseudo-umbra V there corrseponds a unique minimal umbra $\mathcal{U}_u(V)$ dominating V. It is given by

$$\mathcal{U}_u(V) := \bigcap\{U \subset E \times \overline{\mathbb{R}} \mid U \text{ is an umbra and } V \subseteq U\}.$$

We call $\mathcal{U}_u(V)$ the completion of V. Note that $\mathcal{U}_u$ defines a mapping from the set of pseudo-umbras into the set of umbras.

If U is an umbra we define the level set U_t by $U_t = \{x \in E \mid (x,t) \in U\}$. Now the family $\{U_t\}_{t\in\mathbb{R}}$ is continuously decreasing and hence corresponds with a unique function $F \in$ Fun(E) with threshold sets $\mathcal{X}_t(F) = U_t$. Thus F is given by

$$F(x) = \sup\{t \in \mathbb{R} \mid x \in U_t\}.$$

Conversely, the umbra $U = \mathcal{U}_f(F)$ corresponding with an arbitrary function F is given by

$$\mathcal{U}_f(F) = \{(x,t) \mid t \leq F(x)\}.$$

It can easily be seen that the family Umbra(E) ordered by inclusion defines a complete lattice which is isomorphic to Fun(E), and that the mapping $\mathcal{U}_f$: Fun(E) $\rightarrow$ Umbra(E) defines a lattice isomorphism between both spaces. The infimum and supremum of the collection of umbras U_i, $i \in I$ are respectively given by

$$\bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i$$
$$\bigvee_{i \in I} U_i = \mathcal{U}_u(\bigcup_{i \in I} U_i),$$

where $\bigcap$ and $\bigcup$ denote set intersection and union respectively. Note that we must take the completion in the latter expression since, in general, the union of a collection of umbras is only a pseudo-umbra: take e.g. $U_i = \{(0,t) \mid -\infty \leq t \leq 1 - 1/i]\}$ for $i \geq 1$, then $\bigcup_{i \geq 1} U_i = \{(0,t) \mid -\infty \leq t < 1\}$. It is for this reason that the Minkowski set addition of two umbras is not necessarily an umbra again, and as Ronse points out in [10], the identity $\mathcal{U}_f(F \oplus G) = \mathcal{U}_f(F) \oplus \mathcal{U}_f(G)$ is false in general. Here $F \oplus G$ denotes the dilation of F by G as defined in (6.1). The expression becomes true if we take the completion of the pseudo-umbra at the right-hand-side:

$$\mathcal{U}_f(F \oplus G) = \mathcal{U}_u(\mathcal{U}_f(F) \oplus \mathcal{U}_f(G)).$$

for the erosion these problems do not exist and we may write

$$\mathcal{U}_f(F \ominus G) = \mathcal{U}_f(F) \ominus \mathcal{U}_f(G).$$

Summarizing one can say that many of the mistakes found in the literature arise from the fact that one doesn't work with umbras but with pseudo-umbras, and that with one and the same function F there may be associated many pseudo-umbras whose completion is $\mathcal{U}_f(F)$.

Remark 8.2. A different representation of umbras can be obtained by supplying E with the trivial topology (all subsets being open and closed) and $\overline{\mathbb{R}}$ with the half-line topology, i.e., the closed sets are given by $[-\infty, t]$ where $t \in \overline{\mathbb{R}}$. Then Umbra($E$) is isomorphic with the complete lattice of all closed subsets of the product space $E \times \overline{\mathbb{R}}$. See also [17].

Both Serra [12] and Maragos and Schafer [8] restrict to upper semi-continuous (u.s.c.) functions in their discussion of the extension of set operators to function operators. If a function is u.s.c. then its umbra and all its threshold sets are closed sets. In our opinion, this restriction to u.s.c. functions, apart from being superfluous, makes the analysis more tedious. We believe that the choice of the space of u.s.c. functions as the underlying image space is not required unless one wishes to deal with topological or probabilistic aspects of grey-level morphology.

9. Infinite discrete grey-level set

So far we have assumed that the set of grey-levels is $\overline{\mathbb{R}}$. The remainder of this paper will be concerned with the case that the grey-level set is discrete. In this section we consider the infinite case whereas the final two sections are devoted to the finite case.

So assume that the set of grey-levels is $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$. As before, the dual of a function F is given by $F^*(x) = -F(x)$. All the results of the previous sections carry over to the present situation almost wordly; the major difference consists hereof that Definitions 3.2 and 4.1 become abundant and that in the statements of Proposition 3.3 and 4.2 the phrase "continuously decreasing" may be replaced by "decreasing". Furthermore, (4.2) has to be replaced by the simpler expression

$$\mathcal{X}_t(\Psi(F)) = \psi_t(\mathcal{X}_t(F)), \tag{9.1}$$

for $t \in \mathbb{Z}$, and a similar remark applies to (5.1). A function operator Ψ is flat if there is a set operator ψ such that

$$\mathcal{X}_t(\Psi(F)) = \psi(\mathcal{X}_t(F)), \tag{9.2}$$

for $t \in \mathbb{Z}$. As in Proposition 5.7, Ψ is $\downarrow$-continuous if ψ is. It can be shown that this is actually the case if and only if (9.2) holds also for $t = \infty$. Let e_t be the thresholding operator on $\overline{\mathbb{Z}}$ defined similarly as in (5.4). Then the function operator Ψ is flat if and only if

$$e_t \circ \Psi = \Psi \circ e_t.$$

Also the results of Sections 6 and 7 carry over to the present situation immediately. The definition of a pseudo–umbra is the same as in Section 8, but the definition of an umbra has to be adapted in the following way: a subset $U \subseteq E \times \overline{\mathbb{Z}}$ is an umbra if

$$(x,t) \in U \text{ implies that } (x,s) \in U \text{ for } s < t$$
$$(x,t) \in U \text{ for } t < \infty \text{ implies that } (x,\infty) \in U.$$

10. Finite grey-level set

In this section we consider the case where the grey-level set is finite, say $\{0,1,\ldots,N\}$.The dual F^* of a function F is given by $F^*(x) = N - F(x)$. Although essentially all the results of the previous sections remain valid, some of the formulations have to be adapted slightly, often they become simpler or can be extended.

First we note that the inclusion in Lemma 3.4(d) becomes an equality, that is,

$$\mathcal{X}_t(\bigvee_{i\in I} F_i) = \bigcup_{i\in I} \mathcal{X}_t(F_i), \tag{10.1}$$

for $t = 0,1,\ldots,N$. From Section 4 we know that any decreasing family $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_N$ of increasing set operators generates a function operator Ψ which is characterized by either of the following relations:

$$\Psi(F)(x) = \max\{1 \leq t \leq N \mid x \in \psi_t(\mathcal{X}_t(F))\} \tag{10.2}$$

$$\mathcal{X}_t(\Psi(F)) = \psi_t(\mathcal{X}_t(F)), \quad t \geq 1. \tag{10.3}$$

If the right-hand-side of (10.2) is the empty set then $\Psi(F)(x) = 0$.

As in Proposition 4.2 it follows that Ψ is $\downarrow$-continuous if every ψ_t, $t = 1,2,\ldots,N$, is $\downarrow$-continuous. We modify Proposition 4.3 to the case of a finite grey-level set.

Proposition 10.1. *Let the set operators $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_N$ generate the function operator Ψ. Then the dual operator Ψ^* defined by $\Psi^*(F) = (\Psi(F^*))^*$ is generated by the set operators $\psi_N^* \geq \psi_{N-1}^* \geq \cdots \geq \psi_1^*$.*

PROOF. We use twice the fact that

$$\mathcal{X}_t(F^*) = \big(\mathcal{X}_{N-t+1}(F)\big)^*.$$

For $t = 1, \ldots, N$ and $F \in \mathrm{Fun}(E)$ we have

$$\begin{aligned}\mathcal{X}_t(\Psi^*(F)) &= \big(\mathcal{X}_{N-t+1}(\Psi(F^*))\big)^c \\ &= \big(\psi_{N-t+1}(\mathcal{X}_{N-t+1}(F^*))\big)^c \\ &= \big(\psi_{N-t+1}(\mathcal{X}_t(F)^c)\big)^c \\ &= \psi^*_{N-t+1}(\mathcal{X}_t(F)),\end{aligned}$$

which proves the result.

∎

Propositions 4.4–4.7 also hold for discrete grey-level sets. In fact, the proofs become much simpler in this case.

A function operator Ψ is flat if there exists a set operator ψ such that

$$\mathcal{X}_t(\psi(F)) = \psi(\mathcal{X}_t(F)), \tag{10.4}$$

for $t = 1, 2, \ldots, N$ and $F \in \mathrm{Fun}(E)$. Flat operators are invariant under vertical translations (Proposition 5.1) and map flat functions onto flat functions (Proposition 5.2). If Ψ is flat, then ψ is uniquely determined and given by (compare (5.3))

$$\psi = \mathcal{X}_t \circ \Psi \circ \mathcal{F}_t \tag{10.5}$$

for any $t = 1, 2, \ldots, N$.

Proposition 10.2. *Let Ψ be flat, and assume that $h : \{0, 1, \ldots, N\} \to \{0, 1, \ldots, N\}$ is increasing, and that*
(i) $\Psi(\mathcal{I}) = \mathcal{I}$ if $h(0) > 0$
(ii) $\Psi(\mathcal{O}) = \mathcal{O}$ if $h(N) < N$.
Then

$$\Psi \circ h = h \circ \Psi.$$

PROOF. Let Ψ be generated by ψ. Recall from Remark 5.3(a) that $\Psi(\mathcal{I}) = \mathcal{I}$ (respectively $\Psi(\mathcal{O}) = \mathcal{O}$) is equivalent with $\psi(E) = E$ (respectively $\psi(\emptyset) = \emptyset$). We show that $\mathcal{X}_t(h \circ \Psi) = \mathcal{X}_t(\Psi \circ h)$ for $t = 1, 2, \ldots, N$.
(1) Let $h(0) > 0$ and $t \le h(0)$, hence $\psi(E) = E$. Then $\mathcal{X}_t(h \circ \Psi(F)) = E$ and $\mathcal{X}_t(\Psi \circ h(F)) = \psi(\mathcal{X}_t(h(F))) = \psi(E)$.
(2) Let $h(N) < N$ and $t > h(N)$, hence $\psi(\emptyset) = \emptyset$. Then $\mathcal{X}_t(h \circ \Psi(F)) = \emptyset$, whereas $\mathcal{X}_t(\Psi \circ h(F)) = \psi(\mathcal{X}_t(h(F))) = \psi(\emptyset)$.
(3) Let $h(0) < t \le h(N)$. For such t we define $l(t) := \min\{s \mid t \le h(s)\}$. The following assertion is obvious. If $h(0) \le t \le h(N)$ then $t \le h(s) \iff l(t) \le s$. Now

$$\begin{aligned}\mathcal{X}_t(h \circ \Psi(F)) &= \mathcal{X}_{l(t)}(\Psi(F)) = \psi(\mathcal{X}_{l(t)}(F) \\ &= \psi(\mathcal{X}_t(h(F))) = \mathcal{X}_t(\Psi \circ h(F)).\end{aligned}$$

This concludes the proof.

∎

Analogous to (5.4) we define the thresholding operators e_t, $t = 1, 2, \ldots, N$ by

$$e_t(s) = \begin{cases} N, & s \geq t \\ 0, & s < t. \end{cases} \tag{10.6}$$

Then $e_t(0) = 0$ and $e_t(N) = N$, for every $t = 1, 2, \ldots, N$. From the previous result it follows that a flat operator commutes with every thresholding operator e_t. But the converse also holds.

Proposition 10.3. *The function operator Ψ is flat if and only if*

$$\Psi \circ e_t = e_t \circ \Psi$$

for every $t = 1, 2, \ldots, N$.

The proof of this result proceeds along the same lines as the proof of Proposition 5.4. The following result was first proved by Janowitz [6, Lemma 1].

Corollary 10.4. *Let Ψ be a flat operator with $\Psi(\mathcal{O}) = \mathcal{O}$ and $\Psi(\mathcal{I}) = \mathcal{I}$. Then*

$$\mathrm{Ran}(\Psi(F)) \subseteq \mathrm{Ran}(F), \tag{10.7}$$

for any $F \in \mathrm{Fun}(E)$

PROOF. Let $F \in \mathrm{Fun}(E)$. it is easy to find an increasing mapping $h : \{0, 1, \ldots, N\} \to \{0, 1, \ldots, N\}$ such that $\mathrm{Ran}(h) = \mathrm{Ran}(F)$ and $h(t) = t$ for $t \in \mathrm{Ran}(F)$. Define e.g., $h(t) = \min(\mathrm{Ran}(F) \cap \{t, t+1, \ldots, N\})$. By Proposition 10.2, Ψ commutes with h, so in particular

$$\Psi \circ h(F) = h \circ \Psi(F).$$

But $h(F) = F$ so $\Psi(F) = h \circ \Psi(F)$, whence we conclude that $\mathrm{Ran}(\Psi(F)) \subseteq \mathrm{Ran}(h) = \mathrm{Ran}(F)$. ∎

Note that for any operator Ψ satisfying (10.7) one has $\Psi(\mathcal{O}) = \mathcal{O}$ and $\Psi(\mathcal{I}) = \mathcal{I}$.

The converse of Corollary 10.4 does not hold as the following example shows.

Example 10.5. Let $p \in \{0, 1, \ldots, N-2\}$ (where $N \geq 2$). For a function F we define M_F as the maximum grey-level of F. Let the increasing function operator Ψ be defined as follows:

$$\Psi(F)(x) = \begin{cases} F(x), & \text{if } F(x) \leq p \\ M_F, & \text{if } F(x) > p. \end{cases}$$

Then $\mathrm{Ran}(\Psi(F)) \subseteq \mathrm{Ran}(F)$. Now suppose that Ψ is flat. Then its generator is the set operator

$$\psi = \mathcal{X}_N \circ \Psi \circ \mathcal{F}_N = \mathrm{id}.$$

But this yields that $\Psi = \mathrm{id}$ which is a contradiction.

We also present an example which shows that the analogue for infinite grey-level set does not hold.

Example 10.6. Assume the grey-level set to be $\mathbb{Z}$. Let ψ be the set operator given by

$$\psi(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset \\ E, & \text{if } X \neq \emptyset. \end{cases}$$

Let Ψ be the corresponding flat function operator. It is easily seen that $\Psi(F) = \mathcal{I}$ if there exists a sequence $\{x_n\}$ in E with $F(x_n) \to \infty$. So there are many functions F with $\mathrm{Ran}(F) \subseteq \mathbb{Z}$ but $\infty \in \mathrm{Ran}(\Psi(F))$.

11. Finite grey-level sets and truncation

Every flat H-dilation Δ on Fun(E) is of the form

$$\Delta(F) = \bigvee_{h \in A} F_h,$$

where F_h is the translate of F and A is a structuring element in E. The adjoint erosion $\mathcal{E}$ is given by

$$\mathcal{E}(F) = \bigwedge_{h \in A} F_{-h}.$$

This holds for the grey-level sets $\overline{\mathbb{R}}$, $\overline{\mathbb{Z}}$ as well as $\{0, 1, \ldots, N\}$. In the first two cases this adjunction is a T-adjunction, that is, Δ and $\mathcal{E}$ are also invariant under vertical translations. If however the grey-level set $\mathcal{G}$ is finite then we cannot speak of vertical translations since $\mathcal{G}$ is not closed under addition. In particular the Minkowski addition and subtraction of two functions F and G is not well-defined. A possible solution to this problem might be to truncate $F(x-h) + G(h)$ below 0 and above N. So let us define for $s, t \in \overline{\mathbb{Z}}$,

$$\lfloor s+t \rfloor := \begin{cases} 0, & \text{if } s+t < 0 \\ s+t, & \text{if } s+t \in \{0, 1, \ldots, N\} \\ N, & \text{if } s+t > N, \end{cases}$$

and let $\lfloor s-t \rfloor := \lfloor s + (-t) \rfloor$. Indeed, if $F \in$ Fun(E) and if G is a function with domain dom(G) $\subseteq E$ with values in $\overline{\mathbb{Z}}$ then

$$\Delta_G(F)(x) := \bigvee_{h \in \text{dom}(G)} \lfloor F(x-h) + G(h) \rfloor$$

defines a dilation, and we would expect that the erosion $\mathcal{E}_G$ given by

$$\mathcal{E}_G(F)(x) := \bigwedge_{h \in \text{dom}(G)} \lfloor F(x+h) - G(h) \rfloor$$

is the adjoint of Δ_G. But surprisingly the pair $(\mathcal{E}_G, \Delta_G)$ does **not** define an adjunction in general as the example below shows. Note that instead of restricting G to its domain dom(G) we can also put $G(h) = -\infty$ outside dom(G).

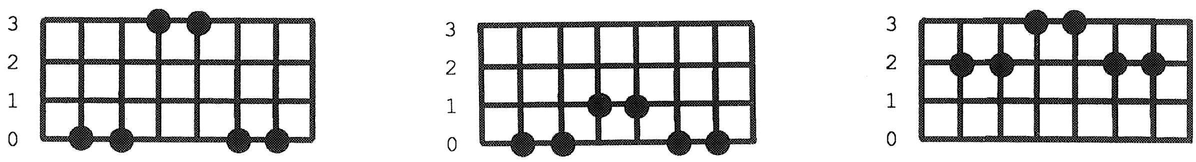

FIGURE 4. *From left to right: a function F and its transforms $\mathcal{E}_G(F)$ and $\Delta_G\mathcal{E}_G(F)$ with $\mathcal{E}_G$ and Δ_G as in Example 11.1. The pair $(\mathcal{E}_G, \Delta_G)$ does not form an adjunction since $\Delta_G\mathcal{E}_G(F) \not\leq F$.*

Example 11.1. Let $E = \mathbb{Z}$, $\mathcal{G} = \{0, 1, 2, 3\}$ and let G be a structuring function with dom(G) $= \{0\}$ and $G(0) = 2$. Let $\mathcal{E}_G(F)(x) = \lfloor F(x) - G(0) \rfloor = \lfloor F(x) - 2 \rfloor$ and $\Delta_G(F)(x) = \lfloor F(x) + 2 \rfloor$. As Figure 4 shows, $\Delta_G \mathcal{E}_G(F) \not\leq F$ in general, and hence the pair $(\mathcal{E}_g, \Delta_G)$ does not define an adjunction. In this example the erosion adjoint to Δ_G is given by

$$\mathcal{E}(F) = \begin{cases} \mathcal{O}, & \text{if } F(x) < 2 \text{ for some } x \in E \\ F - 2, & \text{if } F(x) \geq 2 \text{ for every } x \in E. \end{cases}$$

It follows from this example that truncation by itself is not adequate to handle grey-level dilations and erosions with non-flat structuring elements. It follows also that the smallest and largest grey-level take in an exceptional position. To explain this in detail we need some results which we stated in Section 7.

A mapping $d : \{0, 1, \ldots, N\} \to \{0, 1, \ldots, N\}$ is a dilation if $d(0) = 0$ and d is non-decreasing. The adjoint erosion can be computed from

$$e(t) = \max\{s \mid d(s) \leq t\},$$

where the maximum of the empty set is defined to be zero. Now every H-adjunction $(\mathcal{E}, \Delta)$ on Fun(E) is of the form

$$\Delta(F)(x) = \bigvee_{h \in E} d_h(F(x-h))$$
$$E(F)(x) = \bigwedge_{h \in E} e_h(F(x+h)),$$

where (e_h, d_h) forms an adjunction on $\{0, 1, \ldots, N\}$ for every $h \in E$.

We now consider a special class of adjunctions. For $v \in \mathbb{Z}$ we define the operation $t \to t \dot{+} v$ on $\mathcal{G}$ as follows:

$$\begin{cases} 0 \dot{+} v = 0, & \text{for every } v \\ t \dot{+} v = 0, & \text{if } t > 0 \text{ and } t + v \leq 0 \\ t \dot{+} v = t + v, & \text{if } t > 0 \text{ and } 0 \leq t + v \leq N \\ t \dot{+} v = N, & \text{if } t > 0 \text{ and } t + v > N. \end{cases}$$

Similarly we define the operation $t \to t \dot{-} v$ on $\mathcal{G}$ by

$$\begin{cases} t \dot{-} v = 0, & \text{if } t < N \text{ and } t - v \leq 0 \\ t \dot{-} v = t - v, & \text{if } t < N \text{ and } 0 \leq t - v \leq N \\ t \dot{-} v = N, & \text{if } t < N \text{ and } t - v > N \\ N \dot{-} v = N, & \text{for every } v. \end{cases}$$

The following lemma (the proof of which is straightforward) forms the basis for the definiton of dilations and erosions on Fun(E) using non-flat structuring functions.

Lemma 11.2. *Let $v \in Z$, $d(t) = t \dot{+} v$ and $e(t) = t \dot{-} v$. Then (e, d) is an adjunction on $\mathcal{G}$.*

In Section 7 we have given a characterization of H-dilations and H-erosions: see (7.1)-(7.2). Let G be a function with domain dom(G) which takes values in $\mathbb{Z}$. We take $d_h(t) = t \dot{+} G(h)$ if $h \in$ dom(G) and $d_h \equiv 0$ otherwise. Let e_h be the adjoint, i.e., $e_h(t) = t \dot{-} G(h)$ for $h \in$ dom(G) and $e_h \equiv N$ otherwise. Substitution in (7.1)-(7.2) gives the H-dilation and H-erosion

$$\Delta(F)(x) = (F \dot{\oplus} G)(x) := \bigvee_{h \in \mathrm{dom}(G)} (F(x-h) \dot{+} G(h)) \tag{11.1}$$

$$\mathcal{E}(F)(x) = (F \dot{\ominus} G)(x) := \bigwedge_{h \in \mathrm{dom}(G)} (F(x+h) \dot{-} G(h)) \tag{11.2}$$

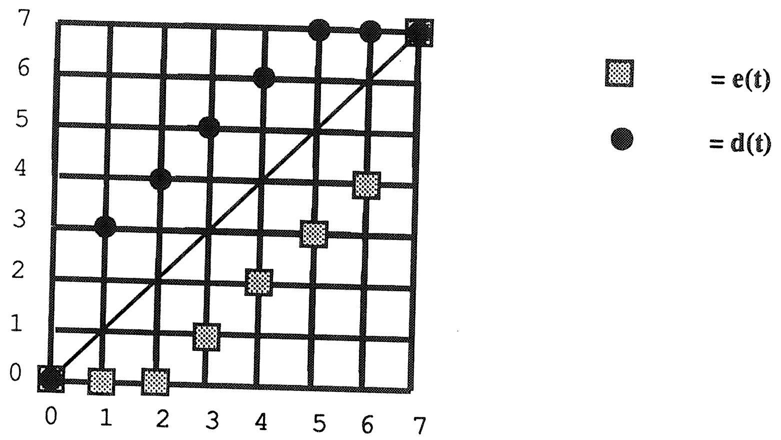

FIGURE 5. *The pair* (e, d) *given by* $d(t) = t \dot{+} 2$, $e(t) = t \dot{-} 2$ *forms an adjunction on* $\mathcal{G} = \{0, 1, 2 \ldots, 7\}$.

The pair $(\mathcal{E}, \Delta)$ forms an adjunction on $\mathrm{Fun}(E)$. In practical cases G will only take values inside $\{-N, \ldots, -1, 0, 1, \ldots, N\}$. If $G \leq 0$ on E then the dilation given by (11.1) amounts to truncation at 0, i.e.,

$$\Delta(F)(x) = \bigvee_{h \in \mathrm{dom}(G)} \lfloor F(x-h) + G(h) \rfloor .$$

See also [10, Section 4]. If G is nonnegative on E then the dilation given by (11.1) satisfies

$$\Delta(F \dot{+} v) = \Delta(F) \dot{+} v \tag{11.3}$$

for $F \in \mathrm{Fun}(E)$ and $v \geq 0$. Here $(F \dot{+} v)(x) := F(x) \dot{+} v$. Similarly

$$\mathcal{E}(F \dot{-} v) = \mathcal{E}(F) \dot{-} v \tag{11.4}$$

for $F \in \mathrm{Fun}(E)$ and $v \geq 0$. In fact we can prove the following result.

Proposition 11.3. *Let* Δ *be a H-dilation on* $\mathrm{Fun}(E)$ *satisfying* $\Delta(F \dot{+} v) = \Delta(F) \dot{+} v$ *for* $F \in \mathrm{Fun}(E)$ *and* $v \geq 0$, *then there exists a nonnegative function* G *with domain* $\mathrm{dom}(G) \subseteq E$ *such that* Δ *is given by (11.1). Analogously, if* $\mathcal{E}$ *is a H-erosion on* $\mathrm{Fun}(E)$ *satisfying* $\mathcal{E}(F \dot{-} v) = \mathcal{E}(F) \dot{-} v$ *for* $F \in \mathrm{Fun}(E)$ *and* $v \geq 0$, *then there exists a nonnegative function* G *with domain* $\mathrm{dom}(G) \subseteq E$ *such that* $\mathcal{E}$ *is given by (11.2).*

PROOF. Let $f_{0,1}$ be the pulse function with altitude 1 in $x = 0$, i.e.,

$$f_{0,1}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

We define G as follows:

$$\mathrm{dom}(G) = \{x \mid \Delta(f_{0,1})(x) \geq 1\}$$
$$G(x) = \Delta(f_{0,1})(x) - 1, \quad \text{for } x \in \mathrm{dom}(G).$$

Since any function F can be written as $F = \bigvee_{h \in E} f_{h,F(h)}$ it suffices to show that $\Delta(f_{h,v}) = f_{h,v} \oplus G$ for $h \in E$ and $v \in \mathcal{G}$. Because of the horizontal translation invariance we may restrict to the case

$h = 0$. The result is trivial for $v = 0$ since $f_{0,0} = \mathcal{O}$. So it remains to show that $\Delta(f_{0,v}) = f_{0,v} \dot{\oplus} G$ for $v = 1, \ldots, N$.

$$\Delta(f_{0,v}) = \Delta(f_{0,1} \dot{+} (v-1)) = \Delta(f_{0,1}) \dot{+} (v-1)$$

whence we obtain that

$$\Delta(f_{0,v})(x) = \Delta(f_{0,1})(x) \dot{+} (v-1) = \begin{cases} (G(x)+1) \dot{+} (v-1), & \text{if } x \in \mathrm{dom}(G) \\ 0, & \text{if } x \notin \mathrm{dom}(G). \end{cases}$$

On the other hand

$$(f_{0,v} \dot{\oplus} G)(x) = \bigvee_{h \in \mathrm{dom}(G)} f_{0,v}(x-h) \dot{+} G(h) = \begin{cases} v \dot{+} G(x), & \text{if } x \in \mathrm{dom}(G) \\ 0, & \text{if } x \notin \mathrm{dom}(G). \end{cases}$$

Since $(G(x)+1) \dot{+} (v-1) = v \dot{+} G(x)$ if $v \geq 1$ and $G(x) \geq 0$ the two expressions are equal, and the result has been proved.

∎

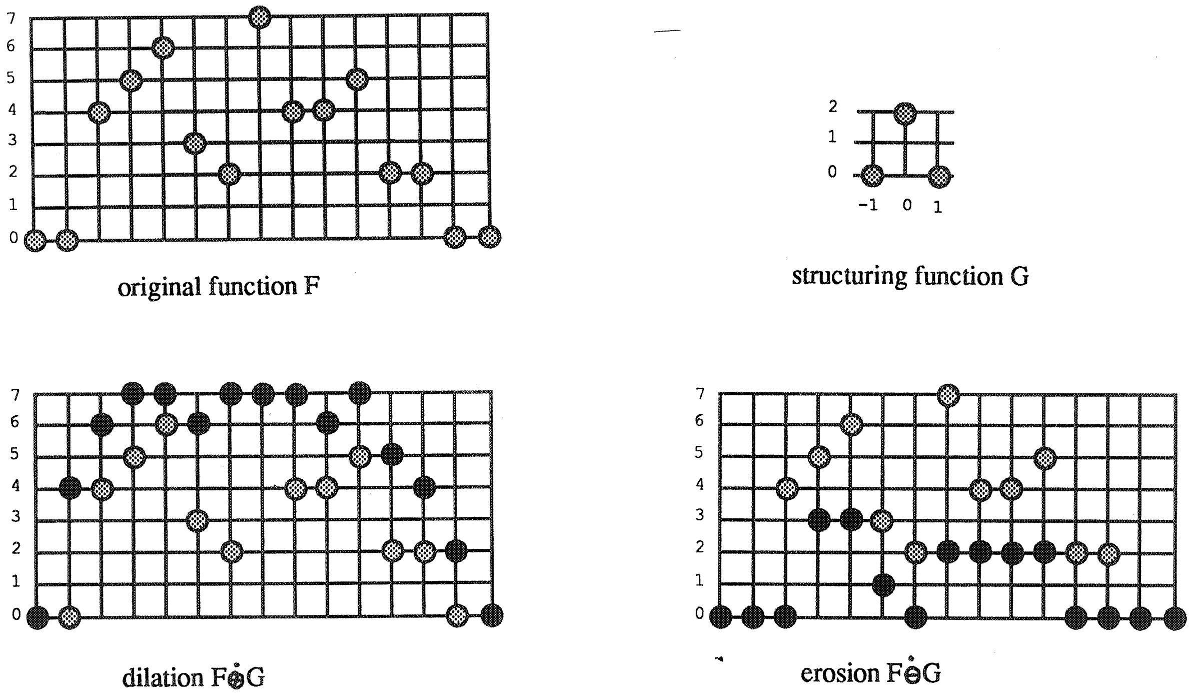

FIGURE 6. *Dilation $F \dot{\oplus} G$ and erosion $F \dot{\ominus} G$ of a function $F \in \mathrm{Fun}(\mathbb{Z})$. Here G is the structuring function with $\mathrm{dom}(G) = \{-1, 0, 1\}$, $G(-1) = G(1) = 0$ and $G(0) = 2$. The original function F is represented by grey dots and its dilation and erosion by black dots.*

It is clear that Δ and $\mathcal{E}$ given by (11.1)-(11.2) are flat if and only if $G = 0$ on its domain: in that case we have $F \dot{\oplus} G = F \oplus \mathrm{dom}(G)$ and $F \dot{\ominus} G = F \ominus \mathrm{dom}(G)$.

Proposition 11.4. Matheron's theorem
Let $\Psi : \mathrm{Fun}(E) \to \mathrm{Fun}(E)$ be an H-operator which satisfies $\Psi(\mathcal{I}) = \mathcal{I}$ and

$$\Psi(F \dot{-} v) = \Psi(F) \dot{-} v \tag{11.5}$$

for $F \in \mathrm{Fun}(E)$ and $v \geq 0$. Then Ψ can be written as a supremum of erosions of the form (11.2). Similarly, if $\Psi : \mathrm{Fun}(E) \to \mathrm{Fun}(E)$ is an H-operator which satisfies $\Psi(\mathcal{O}) = \mathcal{O}$ and

$$\Psi(F \dot{+} v) = \Psi(F) \dot{+} v \tag{11.6}$$

for $F \in \mathrm{Fun}(E)$ and $v \geq 0$. Then Ψ can be written as a infimum of dilations of the form (11.1).

PROOF. We only prove the first statement. We use the following notation. If G is a nonnegative function with domain $\mathrm{dom}(G)$, then $\widehat{G} \in \mathrm{Fun}(E)$ is defined as follows

$$\widehat{G}(x) = \begin{cases} 0, & \text{if } x \notin \mathrm{dom}(G) \\ \min\{G(x) + 1, N\}, & \text{if } x \in \mathrm{dom}(G). \end{cases}$$

Let Ψ be an H-operator satisfying $\Psi(\mathcal{I}) = \mathcal{I}$. The kernel $\mathcal{V}(\Psi)$ is defined as follows: the function G with domain $\mathrm{dom}(G)$ is contained in $\mathcal{V}(\Psi)$ if and only if $\Psi(\widehat{G})(0) \geq 1$. We prove that

$$\Psi(F) = \bigvee_{G \in \mathcal{V}(\Psi)} F \dot{\ominus} G.$$

"$\leq$": let $\Psi(F)(x) \geq t$ for some $t \geq 1$. We show that $F \dot{\ominus} G(x) \geq t$ for some $G \in \mathcal{V}(\Psi)$. Let G be defined by

$$\begin{aligned} \mathrm{dom}(G) &= \{h \mid F(x+h) \geq t\} \\ G(h) &= F(x+h) \dot{-} t. \end{aligned}$$

Then $\widehat{G} = F_x \dot{-} (t-1)$ as one easily verifies. Now $\Psi(F)(x) \geq t$ implies that $\Psi(F_x \dot{-} (t-1))(0) \geq 1$, hence $G \in \mathcal{V}(\Psi)$. But

$$(F \dot{\ominus} G)(x) = \bigwedge_{h \in \mathrm{dom}(G)} F(x+h) \dot{-} G(h) \geq t$$

as follows immediately from the definition of G.
"$\geq$": let $t \geq 1$ and suppose that $(F \dot{\ominus} G)(x) \geq t$ for some $G \in \mathcal{V}(\Psi)$. Then $F(x+h) \dot{-} G(h) \geq t$ for all $h \in \mathrm{dom}(G)$. This however implies that $F_x(h) \dot{-} t \geq G(h)$ for all $h \in \mathrm{dom}(G)$ (since $t \neq 0$), and hence that $F_x \dot{-} (t-1) \geq \widehat{G}$. But this yields that

$$\Psi(F_x \dot{-} (t-1))(0) \geq \Psi(\widehat{G})(0) \geq 1,$$

or equivalently, that

$$\Psi(F)(x) \dot{-} (t-1) \geq 1.$$

From this we obtain that $\Psi(F)(x) \geq 1 \dot{+} (t-1) = t$.

∎

Acknowledgment. We thank Dr. Christian Ronse for a number of useful comments and suggestions, and Prof. Petros Maragos and Prof. Jean Serra for some stimulating discussions. The author also gratefully acknowledges correspondences with Prof. M.F. Janowitz.

References.

1 G. Birkhoff (1984). *Lattice Theory*, American Mathematical Society Colloquium Publications, Vol. 25, 3rd edition, Providence, RI.

2 C.R. Giardina and E.R. Dougherty (1988). *Morphological Methods in Image and Signal Processing*. Prentice Hall, Englewood Cliffs, NJ.

3 R.M. Haralick, S.R. Sternberg, and X. Zhuang (1987). Image analysis using mathematical morphology, *IEEE Trans. Pattern Analysis & Machine Intelligence* PAMI-9, pp. 532–550.

4 H.J.A.M. Heijmans and C. Ronse (1990). The algebraic basis of mathematical morphology. Part I: dilations and erosions, *to appear in Computer Vision, Graphics and Image Processing*.

5 H.J.A.M. Heijmans and J. Serra (1990). Convergence, continuity and iteration in mathematical morphology, *In preparation*.

6 M.F. Janowitz (1986). A model for ordinal filtering of digital images, In: *Statistical Image Processing and Graphics*, E.J. Wegman and D.J. DePriest eds., Marcel Dekker, New York.

7 P. Maragos (1988). A representation theory for morphological image and signal processing, *to appear*.

8 P. Maragos and R.W. Schafer (1987). Morphological filters – Part I: their set-theoretic analysis and relations to linear shift-invariant filters, *IEEE Trans. Acoustics, Speech and Signal Proc.* 35, pp. 1153–1169.

9 G. Matheron (1975). *Random Sets and Integral Geometry*, J. Wiley & Sons, New York.

10 C. Ronse (1988). Why mathematical morphology needs complete lattices, *preprint*.

11 C. Ronse and H.J.A.M. Heijmans (1988). The algebraic basis of mathematical morphology. Part II: openings and closings, *submitted*.

12 J. Serra (1982). *Image Analysis and Mathematical Morphology*, Academic Press, London.

13 J. Serra, ed. (1988). *Image Analysis and Mathematical Morphology, Vol. 2: Theoretical Advances*, Academic Press, London.

14 F.Y. Shih and O.R. Mitchell (1989). Threshold decomposition of gray-scale morphology into binary morphology, *IEEE Trans. Pattern Anal. Mach. Intell.* 11, pp. 31–42.

15 S.R. Sternberg (1982). Cellular computers and biomedical image processing, *Biomedical Images & Computers*, J. Sklansky & J.C. Bisconte eds., *Lecture Notes in Medical Informatics*, Vol. 17, Springer-Verlag, Berlin, pp. 294–319.

16 S.R. Sternberg (1986). Grayscale morphology, *Computer Vision, Graphics, and Image Processing* 35, pp. 333–355.

17 W. Vervaat (1988). Random upper semicontinuous functions and extremal processes, *CWI Report MS-R8801* .

18 P.D. Wendt, E.J. Coyle and N.C. Gallagher (1986). Stack Filters, *IEEE Trans. Acoustics, Speech and Signal Proc.* 34, pp. 898–911.