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M. Kuijper, J.M. Schumacher

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Minimality of Descriptor Representations under External Equivalence

M. Kuijper
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

J.M. Schumacher
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands
Department of Economics, Tilburg University
P.O. Box 90153, 5000 LE Tilburg, The Netherlands

Necessary and sufficient conditions are derived for the minimality of descriptor representations under external equivalence. These conditions are stated completely in terms of the matrices *E, A, B, C* and *D*. Use is made of the close connection between the descriptor representation and the so-called pencil representation. This connection is further exploited to derive the transformation group for minimal descriptor representations. It is shown that the transformations coincide with the operations of strong equivalence as introduced by Verghese *et al.*

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1. INTRODUCTION

In this paper we consider time invariant linear systems represented by

$$\sigma E\xi = A\xi + Bu$$
$$y = C\xi + Du. \tag{1.1}$$

Here σ denotes differentiation or shift, depending on whether one works in continuous or discrete time. The representation (1.1) is called a descriptor representation (D representation). The domain of the mappings E and A will be denoted by X_d (descriptor space), the codomain will be written as X_e (equation space). When the matrix E is invertible, (1.1) can be rewritten in standard state space form

$$\sigma x = Ax + Bu$$
$$y = Cx + Du. \tag{1.2}$$

Here A is a linear mapping from X to X where X is the state space. Needless to say, standard state space representations have been studied extensively in the past. In this paper we are primarily concerned with results on so-called 'external equivalence' (see [13]). Systems are called externally equivalent if their induced 'behaviours' are the same. Here the behaviour of a system consists of the time trajectories of the input and output variables (the 'external variables') that arise from the system representation. For a more extensive treatment on external equivalence the reader is referred to [11, 13, 14]. The state space representation (1.2) is called minimal under external equivalence if dim X is minimal among all representations that are externally equivalent. In the following theorem it is stated that observability is a necessary and sufficient condition for minimality under external equivalence.

THEOREM (J.C. Willems [13]) *A state space representation (A,B,C,D) is minimal under external equivalence if and only if $[sI^{\mathrm{T}} - A^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank for all $s \in \mathbb{C}$.*

It should be noted that controllability is not required for minimality under external equivalence. This is due to the fact that the set of time trajectories of the output variables that are not influenced by the input variables (the 'uncontrolled behaviour') remains invariant under external equivalence. This constitutes one of the main differences between external equivalence and so-called transfer equivalence where the

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invariant is the transfer function instead of the behaviour. At this point we remark that the notion of external equivalence as used by Aplevich in [2, 3] is different from the notion used here. His notion bears closer resemblance to the notion of transfer equivalence; for example, the systems $\dot{y} = \dot{u}$ and $y = u$ are equivalent in the sense of [3] but not in the sense of this paper.

Two state space representations (A,B,C,D) and $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ are called isomorphic (or 'Kalman equivalent') if there exists an invertible mapping T such that $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$ and $\tilde{D} = D$. The following theorem can be considered as a state space isomorphism theorem for external equivalence.

THEOREM (J. C. Willems [13]) *Let (A,B,C,D) be a minimal representation in state space form. A minimal state space representation $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ is externally equivalent with (A,B,C,D) if and only if it is isomorphic to (A,B,C,D).*

By this theorem, the transformation group for minimal state space representations, i. e. the group of transformations under which minimal state space representations are externally equivalent, coincides with the group of isomorphisms.

When using a standard state space representation for a linear time-invariant system it is implicitly assumed that a transfer function exists and that this transfer function is proper rational: the polynomial matrix $sI - A$ is invertible and the matrix $C(sI - A)^{-1}B + D$ is proper. However there are situations in which the transfer function either doesn't exist or is non-proper. In these situations the choice of inputs and outputs is fixed for some reason. Examples can be found in circuit models [10], econometric models [8] and system inversion [4]. The previously mentioned descriptor form (1.1) can be used for representing such systems. The existence of a transfer function is then related to the invertibility of $sE - A$ and the matrix $C(sE - A)^{-1}B + D$ is not necessarily proper.

In the literature, D representations have received a lot of attention. Results have been derived that use transfer equivalence, rather than the concept of external equivalence that was introduced into systems theory quite recently. In [12] Verghese, Lévy and Kailath consider D representations for which $sE - A$ is invertible. Defining minimality in terms of the rank of E they find that, under transfer equivalence, a D representation is minimal if and only if it is strongly irreducible. Here a representation is called strongly irreducible if it is reachable and observable at both finite and infinite modes:

- $[sE - A \quad B]$ has full row rank for all $s \in \mathbb{C}$
- $[E \quad B]$ has full row rank
- $[sE^{\mathrm{T}} - A^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank for all $s \in \mathbb{C}$
- $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank.

In [5] Grimm takes D representations into account for which $sE - A$ is square but not necessarily invertible. In that case transfer equivalence is no longer defined and instead he defines a concept of equivalence in terms of the 'input-output relation' of the system. In comparison with the work of Verghese *et al.* a perhaps more obvious notion of minimality is used: a D representation (E, A, B, C, D) is called minimal if the *size* of E is minimal among all equivalent representations. In [5] a D representation is then found to be minimal if and only if it is strongly irreducible and in addition free from so-called 'non-dynamic' variables:

$$A\,[\ker E\,] \subset \operatorname{im} E. \tag{1.3}$$

In an attempt to define a concept of equivalence for D representations that comes close to 'Kalman equivalence' as defined for standard state space representations, Rosenbrock introduced 'restricted system equivalence' in [9]. However, in his definition the non-dynamic variables that can be present in a D representation were not treated in a satisfactory way. Verghese *et al.* introduced a modified version of restricted system equivalence which they termed 'strong equivalence'. The definition of strong equivalence that is given in [12] involves the introduction of certain operations on the system matrix of a D representation which we repeat here (it should be noted that in [1] a more concise definition of strong equivalence is given):

DEFINITION 1.1 The modification of the system matrix

$$\left[\begin{array}{c|c} sE - A & -B \\ \hline C & D \end{array}\right]$$

to the form

$$\left[\begin{array}{cc|c} sE - A & 0 & -B \\ 0 & I & 0 \\ \hline C & 0 & D \end{array}\right]$$

is called a *trivial augmentation*. The reverse process, corresponding to the deletion of trivial variables, is called a *trivial deflation*.

DEFINITION 1.2 The D representations (E, A, B, C, D) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are related by operations of *strong equivalence* if there exist matrices M, N, X and Y with M and N invertible such that

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sE-A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s\tilde{E}-\tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}. \tag{1.4}$$

DEFINITION 1.3 Two D representations are called *strongly equivalent* if one can be obtained from the other by some sequence of operations of strong equivalence and/or trivial augmentations and/or trivial deflations.

It is shown in [12] that for strongly irreducible representations the transfer function is a complete invariant under strong equivalence. This implies that representations with the same transfer function that are strongly irreducible (and therefore minimal with respect to the rank of E) are strongly equivalent. Analogous results are given by Grimm in [5]: representations with the same input-output relation that are minimal with respect to the size of E are shown to be related by operations of strong equivalence. One may consider these results as versions of the state space isomorphism theorem for D representations.

In this paper we will consider D representations without making any assumptions on $sE - A$ being square or invertible. For that reason our definition of minimality is formulated in terms of three indices: the rank of E, the column defect of E (dim ker E) and the row defect of E (codim im E). We define a D representation to be minimal if each of these three indices is minimal within the set of D representations that correspond to the same behaviour. In analogy with the standard state space case, one would expect that some kind of observability is required for a D representation to be minimal under external equivalence. Indeed, observability turns out to be a necessary condition. However, more conditions are needed to characterize minimality under external equivalence. In this paper we derive four conditions that are necessary and sufficient. In proving this, a prominent role is played by the so-called pencil representation (P representation):

$$\begin{aligned} \sigma G z &= F z \\ y &= H_y z \\ u &= H_u z. \end{aligned} \tag{1.5}$$

Here F and G are linear mappings from Z to X, where Z is the space of internal variables and X is the equation space. The pencil form is suited as well for representing systems for which a transfer function doesn't exist or is non-proper. The representation is called minimal if both dim Z and dim X are minimal. It is shown in [6, 14] that a minimal P representation can be realized directly from the behaviour of the system in a natural way.

In the existing literature, minimality under external equivalence has been characterized in terms of the matrices F, G, H_y, and H_u and the transformation group for minimal P representations has been derived. In these results the pencil representation is defined in a slightly different way by taking the input and output variables together as a vector w of so-called 'external variables'. The vector w takes its values in the space of external variables, denoted by $W(=Y \oplus U)$. A P representation is then given by

$$\begin{aligned} \sigma G z &= F z \\ w &= H z \end{aligned} \tag{1.6}$$

Before summarizing the existing results, we remark that there are other possibilities of representing a system in general first order form. In [15, 16] Willems introduced representations of the form:

$$\sigma E x + F x + G w = 0. \tag{1.7}$$

A first order representation of this form has the property that the variable x is a state variable. It should be noted that this is not true for the variable z in our P representation: the variable z consists of state variables as well as so-called 'driving variables' (see [14]).

We now summarize the existing results on P representations in the following propositions, which can also be found in [6, 13].

PROPOSITION 1.4 *A P representation given by (F,G,H) is minimal under external equivalence if and only if the following conditions hold:*

(i) *G is surjective*

(ii) *$[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ is injective*

(iii) *$[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank for all $s \in \mathbb{C}$.*

PROPOSITION 1.5 *Two minimal P representations (F, G, H) and $(\tilde{F}, \tilde{G}, \tilde{H})$ are externally equivalent if and only if there exist invertible matrices S and T such that*

$$\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} = \begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} T.$$

In [6] it is shown that a close connection exists between a P representation and a D representation: an algorithm is given for rewriting a P representation in descriptor form in such a way that minimality is preserved. In the next section we present an algorithm with a similar property for rewriting a D representation in pencil form. Both algorithms are used for deriving minimality results for D representations in section 3. The connection between a P representation and a D representation is further exploited in section 4 to derive the transformation group with respect to external equivalence for minimal D representations.

2. ALGORITHMS

In this section we present algorithms for obtaining a D representation from a P representation and vice versa. These algorithms will be used in the next section where we derive results on the minimality of D representations by using the known results for P representations that were mentioned in the introduction. For that reason it is important that both algorithms preserve minimality. There is a trivial way to rewrite a P representation in descriptor form. Starting from the P representation (F, G, H_y, H_u), we obtain an equivalent D representation that is given by

$$\begin{aligned} \sigma \begin{bmatrix} G \\ 0 \end{bmatrix} \xi &= \begin{bmatrix} F \\ H_u \end{bmatrix} \xi + \begin{bmatrix} 0 \\ -I \end{bmatrix} u \\ y &= H_y \xi. \end{aligned} \tag{2.1}$$

However when (F, G, H_y, H_u) is minimal, the representation (2.1) is not necessarily minimal. For example, as a starting point one could take a minimal standard state space representation in pencil form, i. e.

$G=[I \quad 0]$, $F=[A \quad B]$, $H_y=[C \quad D]$ and $H_u=[0 \quad I]$, which yields the D representation

$$\sigma\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -I \end{bmatrix}u$$

$$y = [C \quad D]\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{2.2}$$

In this case, (2.2) is clearly not a minimal representation. The following algorithm, which already appeared in [6], does a better job at preserving minimality properties.

ALGORITHM 1 Let a P representation be given by (F, G, H_y, H_u). Decompose the internal variable space Z as $Z_0 \oplus Z_1 \oplus Z_2$ where $Z_1 = \ker G \cap \ker H_u$, and $Z_1 \oplus Z_2 = \ker G$. Accordingly, write

$$G = [G_0 \quad 0 \quad 0], \qquad F = [F_0 \quad F_1 \quad F_2],$$
$$H_y = [H_{00} \quad H_{01} \quad H_{02}], \qquad H_u = [H_{u0} \quad 0 \quad H_{u2}]. \tag{2.3}$$

The matrix H_{u2} has full column rank, and by renumbering the u-variables if necessary, we can write

$$H_{u0} = \begin{bmatrix} H_{10} \\ H_{20} \end{bmatrix}, \qquad H_{u2} = \begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} \tag{2.4}$$

where H_{22} is invertible (or empty, if $\ker G \subset \ker H_u$). Define descriptor matrices by

$$E = \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} \bar{F}_0 & F_1 \\ \bar{H}_{10} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \bar{F}_2 \\ -I & \bar{H}_{12} \end{bmatrix},$$
$$C = \begin{bmatrix} \bar{H}_{00} & H_{01} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & \bar{H}_{02} \end{bmatrix} \tag{2.5}$$

with

$$\begin{aligned} \bar{F}_0 &= F_0 - F_2 H_{22}^{-1} H_{20} \\ \bar{H}_{00} &= H_{00} - H_{02} H_{22}^{-1} H_{20} \\ \bar{H}_{10} &= H_{10} - H_{12} H_{22}^{-1} H_{20} \\ \bar{F}_2 &= F_2 H_{22}^{-1} \\ \bar{H}_{02} &= H_{02} H_{22}^{-1} \\ \bar{H}_{12} &= H_{12} H_{22}^{-1}. \end{aligned} \tag{2.6}$$

The essence of the above construction is that as many z-variables as possible are replaced by u-variables, while at the same time a minimum number of y-variables is introduced as 'descriptor state variables'. In the following lemma we make precise how certain properties are transformed under the algorithm. Later it will become clear that these properties are actually minimality properties.

LEMMA 2.1 *Let (E, A, B, C, D) be a* D *representation that results from applying Algorithm 1 to a* P *representation, given by (F, G, H_y, H_u). Then the two representations are externally equivalent, and furthermore the following holds:*

(i) $\operatorname{rank} E = \operatorname{rank} G$

(ii) $\dim\ker E = \dim(Y \cap H[\ker G]) + \dim\ker[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$.

(iii) $\operatorname{codim}\operatorname{im} E = \operatorname{codim}(Y + H[\ker G]) + \operatorname{codim}\operatorname{im} G$

(iv) $\dim\ker[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}} = \dim\ker[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$

(v) $\operatorname{codim}\operatorname{im}[E \quad B] \leqslant \operatorname{codim}\operatorname{im} G$

Moreover we have the following implications:
(vi) *G is surjective $\Rightarrow [E \quad B]$ is surjective*
(vii) *$[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ is injective $\Rightarrow [E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ is injective*
(viii) *$[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank for all $s \in \mathbb{C} \Rightarrow [sE^{\mathrm{T}} - A^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank for all $s \in \mathbb{C}$.*

PROOF The only operations that are involved in Algorithm 1 are:
- choosing another basis for the internal variables
- reordering u-components
- multiplying an equation from the left by a constant invertible matrix.

It is therefore immediate (see [11]) that the resulting D representation is externally equivalent with (F, G, H_y, H_u). Further, equality (i) is trivial while (ii) follows from

$$\begin{aligned}\dim\ker E &= \dim(\ker G \cap \ker H_u)\\ &= \dim(\ker G \cap \ker H) + \dim(Y \cap H[\ker G])\\ &= \dim\ker[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}} + \dim(Y \cap H[\ker G]).\end{aligned} \tag{2.7}$$

Denoting the number of rows of H_{12} by m_1 we have

$$\begin{aligned}\operatorname{codim}\operatorname{im} E &= \operatorname{codim}\operatorname{im} G_0 + m_1\\ &= \operatorname{codim}\operatorname{im} G + \operatorname{codim}(Y + H[\ker G]).\end{aligned} \tag{2.8}$$

This implies (iii). Equality (iv) is again trivial while (v) follows from

$$\begin{aligned}\operatorname{codim}\operatorname{im}[E \quad B] &= \operatorname{codim}\operatorname{im}\begin{bmatrix} G_0 & 0 & 0 & \bar{F}_2 \\ 0 & 0 & -I & \bar{H}_{12} \end{bmatrix}\\ &= \operatorname{codim}\operatorname{im}[G \quad F_2 H_{22}^{-1}] \leqslant \operatorname{codim}\operatorname{im} G.\end{aligned} \tag{2.9}$$

The implications (vi) and (vii) follow immediately from (v) and (iv) respectively. Implication (viii) can be easily verified by considering the matrix equality

$$\begin{bmatrix} sG_0 - \bar{F}_0 & -F_1 \\ -\bar{H}_{10} & 0 \\ \bar{H}_{00} & H_{01} \\ 0 & 0 \end{bmatrix} = \\ = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} sG - F_0 & -F_1 & -F_2 \\ H_{00} & H_{01} & H_{02} \\ H_{10} & 0 & H_{12} \\ H_{20} & 0 & H_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ -H_{22}^{-1} H_{20} & 0 \end{bmatrix}. \tag{2.10}$$

Next we present an algorithm for obtaining a P representation from a D representation:

ALGORITHM 2 Let a D representation be given by (E, A, B, C, D). Decompose the descriptor space X_d as $X_{d1} \oplus X_{d2}$ where $X_{d2} = \ker E$. Decompose the equation space X_e as $X_{e1} \oplus X_{e2} \oplus X_{e3}$ where $X_{e1} = \operatorname{im} E$ and $X_{e1} \oplus X_{e2} = \operatorname{im}[E \quad B]$. Accordingly write

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \quad C = [C_1 \quad C_2]. \tag{2.11}$$

The matrix B_2 is now surjective. By renumbering the u-variables if necessary, we can write

$$\begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ 0 & 0 \end{bmatrix}, \quad D = [D_1 \quad D_2] \tag{2.12}$$

where B_{22} is invertible. Define pencil matrices as:

$$G = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_{11} \\ A_{31} & A_{32} & 0 \end{bmatrix}, \quad H_y = [\bar{C}_1 \quad \bar{C}_2 \quad \bar{D}_1],$$

$$H_u = \begin{bmatrix} 0 & 0 & I \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_{21} \end{bmatrix} \tag{2.13}$$

with

$$\begin{aligned} \bar{A}_{11} &= A_{11} - B_{12}B_{22}^{-1}A_{21} \\ \bar{A}_{12} &= A_{12} - B_{12}B_{22}^{-1}A_{22} \\ \bar{B}_{11} &= B_{11} - B_{12}B_{22}^{-1}B_{21} \\ \bar{C}_1 &= C_1 - D_2B_{22}^{-1}A_{21} \\ \bar{C}_2 &= C_2 - D_2B_{22}^{-1}A_{22} \\ \bar{D}_1 &= D_1 - D_2B_{22}^{-1}B_{21} \\ \bar{A}_{21} &= -B_{22}^{-1}A_{21} \\ \bar{A}_{22} &= -B_{22}^{-1}A_{22} \\ \bar{B}_{21} &= -B_{22}^{-1}B_{21}. \end{aligned} \tag{2.14}$$

LEMMA 2.2 *Let* (F, G, H_y, H_u) *be a* P *representation that results from applying Algorithm 2 to a* D *representation, given by* (E, A, B, C, D). *Then the following equalities hold:*

(i) $\operatorname{rank} G = \operatorname{rank} E$

(ii) $\operatorname{codim}\operatorname{im} G = \operatorname{codim}\operatorname{im}[E \quad B]$

(iii) $\dim\ker G = \dim\ker E + \dim B^{-1}[\operatorname{im} E]$

(iv) $\dim\ker[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}} = \dim(\ker E \cap \ker C \cap A^{-1}[\operatorname{im} E])$

Moreover we have the following implications:

(v) $[E \quad B]$ *is surjective* $\Rightarrow$ G *is surjective*

(vi) $\ker E \cap \ker C \cap A^{-1}[\operatorname{im} E] = \{0\} \Rightarrow [G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *is injective*

(vii) $[sE^{\mathrm{T}} - A^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ *has full column rank for all* $s \in \mathbb{C} \Rightarrow [sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *has full column rank for all* $s \in \mathbb{C}$.

PROOF The external equivalence of the two representations follows from the same argument as in the proof of the previous lemma. Next, the equalities (i) and (ii) are immediate. Denoting the number of columns of B_{21} by m_1, equality (iii) follows from:

$$\begin{aligned} \dim\ker G &= \dim\ker E + m_1 \\ &= \dim\ker E + \dim U - \dim \operatorname{im}[B_{21} \quad B_{22}] \\ &= \dim\ker E + \dim\ker[B_{21} \quad B_{22}] \\ &= \dim\ker E + \dim B^{-1}[\operatorname{im} E]. \end{aligned} \tag{2.15}$$

Equality (iv) follows from:

$$\begin{aligned} \dim\ker \begin{bmatrix} G \\ H \end{bmatrix} &= \dim\ker \begin{bmatrix} \bar{C}_2 \\ \bar{A}_{22} \end{bmatrix} \\ &= \dim\ker \begin{bmatrix} I & -D_2 B_{22}^{-1} \\ 0 & -B_{22}^{-1} \end{bmatrix} \begin{bmatrix} C_2 \\ A_{22} \end{bmatrix} \\ &= \dim\ker \begin{bmatrix} C_2 \\ A_{22} \end{bmatrix} \\ &= \dim(\ker E \cap \ker C \cap A^{-1}[\operatorname{im} E]). \end{aligned} \tag{2.16}$$

The implications (v) and (vi) follow immediately from (iii) and (iv) respectively. Implication (vii) can be easily verified by considering the matrix equality

$$\begin{bmatrix} sI-\bar{A}_{11} & -\bar{A}_{12} & -\bar{B}_{11} \\ -A_{31} & -A_{32} & 0 \\ \bar{C}_1 & \bar{C}_2 & \bar{D}_1 \\ 0 & 0 & I \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_{21} \end{bmatrix} =$$

$$= \begin{bmatrix} I & -B_{12}B_{22}^{-1} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & D_2 B_{22}^{-1} & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & B_{22}^{-1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} sI-A_{11} & -A_{12} & -B_{11} \\ -A_{21} & -A_{22} & -B_{21} \\ -A_{31} & -A_{32} & 0 \\ C_1 & C_2 & D_1 \\ 0 & 0 & I \end{bmatrix}. \tag{2.17}$$

3. Minimality

In this section we derive necessary and sufficient conditions for the minimality under external equivalence of a D representation (E, A, B, C, D) that are stated completely in terms of the matrices E, A, B, C and D. In [6] we considered minimality in a polynomial setting. There we assumed that the system is represented by a set of autoregressive equations:

$$R_1(\sigma)y + R_2(\sigma)u = 0$$

where $R_1(s)$ and $R_2(s)$ are polynomial matrices. Denoting the sum of the minimal row indices of the matrix $[R_1(s) \quad R_2(s)]$ by n, we gave the following characterization of minimality which involves an invariant subspace $W^0 \subset W$.

Theorem 3.1 ([6]) *Let a* D *representation be given by* (E, A, B, C, D). *The representation is minimal under external equivalence if and only if*

(i) rank $E = n$

(ii) $\dim \ker E = \dim (Y \cap W^0)$

(iii) $\operatorname{codim} \operatorname{im} E = \operatorname{codim} (Y + W^0)$.

Intuitively speaking, the subspace W^0 is spanned by the minimum number of 'driving variables' of the system; when W^0 coincides with the input space we are dealing with a system with a strictly causal input-output structure. A formal definition of W^0 in terms of the behaviour of the system is given in [6, 14]. As already mentioned in the introduction, a P representation (F, G, H) can be obtained directly from the behaviour. It is therefore not surprising that W^0 can be easily expressed in terms of the matrices F, G and H. In [6] we proved the following proposition:

PROPOSITION 3.2 *Assume that a* P *representation, given by* (F, G, H), *satisfies the following conditions:*

(i) *G is surjective*

(ii) *$[G^T \quad H^T]^T$ is injective.*

Then we have

$$W^0 = H[\ker G]. \tag{3.1}$$

Using the above proposition together with the properties of Algorithm 2 as expressed in Lemma 2.2 we are now able to express W^0 in terms of the matrices of a D representation.

LEMMA 3.3 *Assume that a* D *representation, given by* (E, A, B, C, D), *satisfies the following conditions:*

(i) *$[E \quad B]$ is surjective*

(ii) *$[E^T \quad C^T]^T$ is injective*

(iii) *$A[\ker E] \subset \operatorname{im} E$.*

Then we have

$$Y \cap W^0 = C[\ker E] \tag{3.2}$$

$$\pi_U W^0 = B^{-1}[\operatorname{im} E]. \tag{3.3}$$

PROOF Application of Algorithm 2 to our D representation yields a P representation that, according to Lemma 2.2, satisfies the conditions of Prop. 3.2. We may therefore conclude that

$$W^0 = \operatorname{im} \begin{bmatrix} C_2 & D_1 - D_2 B_{22}^{-1} B_{21} \\ 0 & I \\ 0 & -B_{22}^{-1} B_{21} \end{bmatrix}, \tag{3.4}$$

where the matrices are partitioned as in Algorithm 2. It now follows immediately that $Y \cap W^0 = C[\ker E]$ and $\pi_U W^0 = B^{-1}[\operatorname{im} E]$.

In [6] we showed that conditions (i) and (ii) of the above lemma are necessary conditions for the minimality of a D representation. We now prove that this also holds true for condition (iii).

LEMMA 3.4 *If the* D *representation, given by* (E, A, B, C, D), *is minimal under external equivalence, then the following holds:*

$$A[\ker E] \subset \operatorname{im} E. \tag{3.5}$$

PROOF Suppose that the condition is not fulfilled, while (E, A, B, C, D) is minimal. Then, by a suitable choice of coordinates, we can represent the system as:

$$\sigma\xi_1 = A_{11}\xi_1 + A_{12}\xi_2 + B_1 u \tag{3.6}$$

$$0 = A_{21}\xi_1 + A_{22}\xi_2 + B_2 u \tag{3.7}$$

$$y = C_1\xi_1 + C_2\xi_2 + Du. \tag{3.8}$$

We now have $A_{22} \neq 0$ and without restrictions we can assume that A_{22} is of the form $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

Equation (3.7) then splits into two equations:

$$0 = A_{211}\xi_1 + \tilde{\xi}_2 + B_{21}u \tag{3.9}$$

$$0 = A_{212}\xi_1 + B_{22}u. \tag{3.10}$$

Expressing $\tilde{\xi}_2$ in ξ_1 and u and substituting this expression into equations (3.6) and (3.8) leads to an equivalent D representation $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ for which

$$\dim \ker \tilde{E} = \dim \ker E - \operatorname{rank} A_{22} \tag{3.11}$$

$$\operatorname{codim} \operatorname{im} \tilde{E} = \operatorname{codim} \operatorname{im} E - \operatorname{rank} A_{22}. \tag{3.12}$$

Since we supposed that $\operatorname{rank} A_{22} > 0$ this contradicts our assumption on the minimality of (E, A, B, C, D).

REMARK 3.5 In [12] a D representation (E, A, B, C, D) is defined to contain non-dynamic modes if there exist invertible constant matrices M and N such that

$$M[sE - A]N = \begin{bmatrix} s\tilde{E} - \tilde{A} & 0 \\ 0 & I \end{bmatrix} \tag{3.13}$$

where I is an identity matrix of appropriate size. It is not difficult to show that a D representation contains non-dynamic modes if and only if it does not satisfy the condition of the above lemma. Grimm also made a remark on this in [5].

REMARK 3.6 In the literature (see [7] and references therein) various definitions of observability and controllability/reachability at infinity for D representations were given. It is easy to see that these definitions coincide when there are no non-dynamic modes.

In order to derive the main theorem of this section we will use the characterization of minimality for P representations that was mentioned in the introduction. As stated before, the algorithms of section 2 will be used for that purpose and it is therefore important that they preserve minimality. In [6] we already showed that this is the case for Algorithm 1. We now prove that the same property holds for Algorithm 2.

LEMMA 3.7 *Let (F, G, H) be a P representation that results from applying Algorithm 2 to a D representation that is minimal under external equivalence. Then (F, G, H) is also minimal under external equivalence.*

PROOF From Lemma 2.2 it follows that G is surjective. Furthermore the minimality of rank G follows immediately from the minimality of rank E since in both Algorithm 1 and Algorithm 2 we have

$$\operatorname{rank} G = \operatorname{rank} E.$$

In order to conclude that the P representation is minimal we still have to prove that $\dim \ker G$ is minimal. Using Lemma 3.3 together with Lemma 2.2, we have

$$\begin{aligned} \dim \ker G &= \dim \ker E + \dim B^{-1}[\operatorname{im} E] \\ &= \dim C[\ker E] + \dim B^{-1}[\operatorname{im} E] \\ &= \dim (Y \cap W^0) + \dim (\pi_U W^0) \\ &= \dim W^0. \end{aligned} \tag{3.14}$$

From [6] we may now conclude that $\dim \ker G$ is minimal and moreover that the P representation (F, G, H) is minimal.

We now present the first main result of this paper:

THEOREM 3.8 *Let a* D *representation be given by* (E, A, B, C, D). *The representation is minimal under external equivalence if and only if the following conditions hold:*

(i) $[E \quad B]$ *is surjective*

(ii) $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ *is injective*

(iii) $A[\ker E] \subset \operatorname{im} E$

(iv) $[sE^{\mathrm{T}} - A^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ *has full column rank for all* $s \in \mathbb{C}$.

PROOF From Lemma 3.4 and the remark preceding it, it follows immediately that for a minimal D representation the conditions (i), (ii) and (iii) should hold. In order to prove (iv) we apply Algorithm 2 to the representation. According to Lemma 3.7 the P representation (F, G, H) that is obtained in this way is minimal. This implies that $[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ should have full column rank for all $s \in \mathbb{C}$ (Prop. 1.4). Condition (iv) now easily follows from matrix equality (2.14). Conversely, when Algorithm 2 is applied to a D representation for which conditions (i)-(iv) hold, Lemma 2.2 yields that the resulting P representation satisfies the conditions of Prop. 1.4 and is therefore minimal. From this it follows that rank E is minimal. Furthermore since conditions (i) and (ii) are assumed to be satisfied we can use Lemma 3.3 to derive

$$\begin{aligned}\dim \ker E &= \dim C[\ker E] \\ &= \dim Y \cap W^0 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\operatorname{codim} \operatorname{im} E &= \operatorname{codim} B^{-1}[\operatorname{im} E] \\ &= \operatorname{codim} \pi_U W^0. \end{aligned} \tag{3.16}$$

By Theorem 3.1 this proves that the D representation is minimal.

4. TRANSFORMATIONS

In [5] Grimm proves that minimal D representations that have the same input/output relation are related by operations of strong equivalence. In this section we present a similar result in our context of external equivalence. We first prove that strong equivalence is a stronger concept than external equivalence.

PROPOSITION 4.1 *Let* (E, A, B, C, D) *and* $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *be* D *representations that are strongly equivalent. Then the two representations are externally equivalent.*

PROOF It is clear that 'trivial augmentations' and 'trivial deflations' do not affect the behaviour of the system. Furthermore multiplying the system matrix from the left by

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$

can be considered as a 'reformulation of constraints' in the terminology of [11] and this proves that the resulting representation is externally equivalent to the original one. In the same way, right multiplication by

$$\begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}$$

where $EY = 0$ can be considered as a 'change of internal variables' (see again [11]). Finally, left multiplication by

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$$

where $XE = 0$ is a trivial operation: multiplying the equation

$$\sigma E\xi_1 = A\xi_1 + B\xi_2 \tag{4.1}$$

from the left by X gives:

$$0 = XA\xi_1 + XB\xi_2. \tag{4.2}$$

This can of course, without affecting the system, be added to the equation

$$y = C\xi_1 + D\xi_2 \tag{4.3}$$

yielding

$$y = (C + XA)\xi_1 + (D + XB)\xi_2. \tag{4.4}$$

From the above we can conclude that (E, A, B, C, D) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are externally equivalent.

Before presenting the main theorem of this section we need the following lemma:

LEMMA 4.2 *Let a minimal* D *representation be given by* (E, A, B, C, D). *Decompose the equation space* X *as* $X_1 \oplus X_2$ *where* $X_1 = \operatorname{im} E$. *Decompose the input space* U *as* $\pi_U W^0 \oplus U_2$. *Accordingly write*

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Then B_{22} *is invertible.*

PROOF Since (E, A, B, C, D) is minimal, Theorem 3.1 yields that

$$\operatorname{codim} \operatorname{im} E = \operatorname{codim} \pi_U W^0. \tag{4.5}$$

From this it follows immediately that B_{22} is square. Next we prove that B_{22} is injective. Let $u \in U_2$ be such that $B_{22}u = 0$. Then $Bu \in \operatorname{im} E$, so $u \in B^{-1}[\operatorname{im} E]$. Using the equality

$$B^{-1}[\operatorname{im} E] = \pi_U W^0 \tag{4.6}$$

(Lemma 3.3) it follows that $u = 0$ from which it can be concluded that B_{22} is injective. This proves that B_{22} is invertible.

THEOREM 4.3 *Let* (E, A, B, C, D) *and* $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *be* D *representations that are minimal under external equivalence. Then the two representations are externally equivalent if and only if they are related by operations of strong equivalence.*

PROOF The 'if' part follows immediately from the above proposition. In order to prove the 'only if' part, it should be noted that we can arrive at D representations of the form (2.11) by using operations of strong equivalence. For that reason we can assume that our D representations (E, A, B, C, D) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are already of that form. Next apply Algorithm 2 while using the decomposition of U from the above lemma. This yields externally equivalent P representations (F, G, H) and $(\tilde{F}, \tilde{G}, \tilde{H})$ that are minimal by Lemma 3.7. We can now use existing knowledge on the transformation group for externally equivalent minimal P representations. According to Prop. 1.5, there exist invertible matrices S and T such that

$$\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} = \begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} T. \tag{4.7}$$

Writing this in further detail gives:

$$\begin{bmatrix} S & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & D_2B_{22}^{-1} & I & 0 \\ 0 & 0 & 0 & I \\ 0 & B_{22}^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} & -B_{11} \\ -A_{21} & 0 & -B_{21} \\ C_1 & C_2 & D_1 \\ 0 & 0 & I \end{bmatrix} =$$

$$= \begin{bmatrix} I & -\tilde{B}_{12}\tilde{B}_{22}^{-1} & 0 & 0 \\ 0 & \tilde{D}_2\tilde{B}_{22}^{-1} & I & 0 \\ 0 & 0 & 0 & I \\ 0 & \tilde{B}_{22}^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} sI-\tilde{A}_{11} & -\tilde{A}_{12} & -\tilde{B}_{11} \\ -\tilde{A}_{21} & 0 & -\tilde{B}_{21} \\ \tilde{C}_1 & \tilde{C}_2 & \tilde{D}_1 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{bmatrix}. \tag{4.8}$$

It now follows immediately that

$$\begin{aligned} &T_1 = S, \\ &T_2 = T_3 = T_7 = T_8 = 0, \\ &T_9 = I, \end{aligned} \tag{4.9}$$

and T_5 is invertible. It is easily checked that this implies that also

$$\begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -B_{12}B_{22}^{-1} & 0 \\ 0 & D_2B_{22}^{-1} & I \\ 0 & B_{22}^{-1} & 0 \end{bmatrix} \begin{bmatrix} sI-A_{11} & -A_{12} & -B_{11} & -B_{12} \\ -A_{21} & 0 & -B_{21} & -B_{22} \\ C_1 & C_2 & D_1 & D_2 \end{bmatrix} = \tag{4.10}$$

$$= \begin{bmatrix} I & -\tilde{B}_{12}\tilde{B}_{22}^{-1} & 0 \\ 0 & \tilde{D}_2\tilde{B}_{22}^{-1} & I \\ 0 & \tilde{B}_{22}^{-1} & 0 \end{bmatrix} \begin{bmatrix} sI-\tilde{A}_{11} & -\tilde{A}_{12} & -\tilde{B}_{11} & -\tilde{B}_{12} \\ -\tilde{A}_{21} & 0 & -\tilde{B}_{21} & -\tilde{B}_{22} \\ \tilde{C}_1 & \tilde{C}_2 & \tilde{D}_1 & \tilde{D}_2 \end{bmatrix} \begin{bmatrix} S & 0 & 0 & 0 \\ T_4 & T_5 & T_6 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Multiplying from the left by

$$\begin{bmatrix} I & -\tilde{B}_{12}\tilde{B}_{22}^{-1} & 0 \\ 0 & \tilde{D}_2\tilde{B}_{22}^{-1} & I \\ 0 & \tilde{B}_{22}^{-1} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 & \tilde{B}_{12} \\ 0 & 0 & \tilde{B}_{22} \\ 0 & I & -\tilde{D}_2 \end{bmatrix} \tag{4.11}$$

yields:

$$\begin{bmatrix} S & -SB_{12}B_{22}^{-1} + \tilde{B}_{12}B_{22}^{-1} & 0 \\ 0 & \tilde{B}_{22}B_{22}^{-1} & 0 \\ 0 & D_2B_{22}^{-1} - \tilde{D}_2B_{22}^{-1} & I \end{bmatrix} \begin{bmatrix} sI-A_{11} & -A_{12} & -B_{11} & -B_{12} \\ -A_{21} & 0 & -B_{21} & -B_{22} \\ C_1 & C_2 & D_1 & D_2 \end{bmatrix} =$$

$$= \begin{bmatrix} sI-\tilde{A}_{11} & -\tilde{A}_{12} & -\tilde{B}_{11} & -\tilde{B}_{12} \\ -\tilde{A}_{21} & 0 & -\tilde{B}_{21} & -\tilde{B}_{22} \\ \tilde{C}_1 & \tilde{C}_2 & \tilde{D}_1 & \tilde{D}_2 \end{bmatrix} \begin{bmatrix} S & 0 & 0 & 0 \\ T_4 & T_5 & T_6 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{4.12}$$

Defining

$$M = \begin{bmatrix} S & -SB_{12}B_{22}^{-1} + \tilde{B}_{12}B_{22}^{-1} \\ 0 & \tilde{B}_{22}B_{22}^{-1} \end{bmatrix}, \quad N = \begin{bmatrix} S & 0 \\ T_4 & T_5 \end{bmatrix},$$

$$X = [0 \quad D_2B_{22}^{-1} - \tilde{D}_2B_{22}^{-1}] \text{ and } Y = \begin{bmatrix} 0 & 0 \\ T_6 & 0 \end{bmatrix} \tag{4.13}$$

we see that M and N are invertible and

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sE-A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s\tilde{E}-\tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}. \tag{4.14}$$

This proves that the representations are related by operations of strong equivalence.

5. Conclusions

In this paper we have characterized the minimality of a descriptor representation in terms of the matrices E, A, B, C and D. The conditions that are necessary and sufficient for minimality under external equivalence can be summarized as:

- absence of non-dynamic modes
- controllability and observability at infinity
- observability at finite modes.

These conditions coincide with the conditions for minimality under input-output equivalence that were derived by Grimm in [5] except for the fact that controllability at the finite modes is not required in our case. This is to be expected since the main difference between external equivalence and input-output equivalence is the way in which the uncontrollable modes are treated; the uncontrolled behaviour, i. e. the set of time trajectories of the output variables that are not influenced by the input variables, remains invariant under external equivalence whereas it can be removed under input-output equivalence.

Next we have obtained the transformation group for minimal descriptor representations. We found that the transformations under which minimal descriptor representations are externally equivalent coincide with the operations of strong equivalence as introduced by Verghese *et al.* Combining this result with the results of our previous paper [6] we can conclude that the realization procedure of [6] leads to a minimal descriptor representation that is unique up to operations of strong equivalence.

The basic tool that we used in this paper was the pencil representation. Results on minimality and transformations for pencil representations were exploited. It should be noted that the transformation group for minimal pencil representations consists of isomorphisms, while more complex operations, namely those of strong equivalence, constitute the transformation group for minimal descriptor representations. The pencil form, that also appeared as a natural result of our realization procedure of [6], is therefore perhaps a more convenient first order form than the descriptor form to describe linear time invariant systems.

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