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Unique Normal Forms for Disjoint Unions of Conditional Term Rewriting Systems

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ABSTRACT

In [14] we have shown that every term rewriting system with the unique normal form property can be conservatively extended to a confluent term rewriting system with the same set of normal forms. This paper gives a simplified construction, which moreover yields a positive answer to a conjecture in [14] stating that the normal form property is a modular property of left-linear term rewriting systems. We further show that the main result of [14]—the modularity of unique normal forms—can be generalized to semi-equational conditional term rewriting systems; however, for join and normal conditional term rewriting systems the method of [14] fails.

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Introduction

Starting with Toyama [19], several authors studied disjoint unions of term rewriting systems. The central issue is what properties of term rewriting systems are preserved under disjoint unions. Such a property is called 'modular'. Toyama [19] showed the modularity of confluence. In [20] Toyama refuted the modularity of strong normalization. His counterexample inspired Rusinowitch [18] to the formulation of sufficient conditions for the strong normalization of two strongly normalizing term rewriting systems. Rusinowitch's results were extended by the present author [15]. Barendregt and Klop gave an example showing that completeness (i.e. the combination of confluence and strong normalization) is not a modular property, see Toyama [20]. The restriction to left-linear term rewriting systems is sufficient for obtaining the modularity of completeness, as was shown by Toyama, Klop and Barendregt [21]. An interesting alternative approach to modularity is explored in Kurihara and Kaji [12]. Kurihara and Ohuchi [13] recently showed that 'simple termination' is a modular property. A term rewriting systems is said to be simply terminating if there exists a simplification ordering showing its strong normalization. In [14] we proved that the property of unique normal forms is a modular property by showing that every term rewriting systems with unique normal forms can be conservatively extended to a confluent term rewriting systems with the same set of normal forms (*). We also showed that the normal form property is not modular.

In this paper we will give a much simpler proof of (*). The resulting construction enables us to establish the modularity of the normal form property for left-linear term rewriting systems. It also facilitates the extension of the modularity of unique normal forms to the so-called semi-equational conditional term rewriting systems, a particular form of conditional term rewriting system. Conditional term rewriting systems are an important extension of term rewriting systems. They arise in the algebraic specification of abstract data types (Bergstra and Klop [1], Kaplan [10], Zhang and Rémy [22]). Furthermore, they provide a natural computational mechanism for integrating functional and logic programming (Dershowitz and Plaisted [5,6], Fribourg [7], Goguen and Meseguer [8]). In [16] we extended Toyama's confluence result for term rewriting systems to conditional term rewriting systems. We continued this line of research in [17] by extending the results of Rusinowitch [18], Middeldorp [15] and Kurihara and Kaji [12] to conditional term rewriting systems. Both papers clearly showed that conditional term rewriting can be very tricky. In this paper we will also encounter several statements that are obviously true for unconditional term rewriting systems, but nevertheless fail for conditional term rewriting systems. In fact, we will see that (*) is not true for join and normal systems, two other well-known types of conditional term rewriting systems. We finally show that the modularity of unique normal forms for semi-equational conditional term rewriting systems can be obtained by means of (*).

A concise introduction to term rewriting is given in the next section. Extensive surveys are Klop [11] and Dershowitz and Jouannaud [2]. Section 2 contains the simplified proof of (*). In Section 3 we show how this proof can be used to obtain the modularity of the normal form property for left-linear term rewriting systems. Section 4 studies the modularity of unique normal forms with respect to conditional term rewriting.

1. Preliminaries

Let $\mathcal{V}$ be a countably infinite set of *variables*. A *term rewriting system* (TRS for short) is a pair $(\mathcal{F}, \mathcal{R})$. The set $\mathcal{F}$ consists of *function symbols*; associated to every $f \in \mathcal{F}$ is its arity $n \geq 0$. Function symbols of arity 0 are called *constants*. The set of terms built from $\mathcal{F}$ and $\mathcal{V}$, notation $\mathcal{T}(\mathcal{F}, \mathcal{V})$, is the smallest set such that:

- $\mathcal{V} \subset \mathcal{T}(\mathcal{F}, \mathcal{V})$,
- if $f \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

Terms not containing variables are called *ground* or *closed* terms. The set of variables occurring in a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is denoted by $V(t)$. Identity (syntactic equality) of terms is denoted by $\equiv$. The set $\mathcal{R}$ consists of pairs (l, r) with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ subject to two constraints:
(1) the left-hand side l is not a variable,
(2) the variables which occur in the right-hand side r also occur in l.
Pairs (l, r) are called *rewrite rules* or *reduction rules* and will henceforth be written as $l \to r$. We usually present a TRS as a set of rewrite rules, without making explicit the set of function symbols. A rewrite rule $l \to r$ is *left-linear* if l does not contain multiple occurrences of the same variable. A *left-linear* TRS only contains left-linear rewrite rules. The rule $l \to r$ is *collapsing* if r is a single variable and it is *duplicating* if r contains more occurrences of some variable than l does.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that the set $\{x \in \mathcal{V} \mid \sigma(x) \not\equiv x\}$ is finite. This set is called the *domain* of σ and will be denoted by $\mathcal{D}(\sigma)$. Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, i.e. $\sigma(f(t_1, \ldots, t_n)) \equiv f(\sigma(t_1), \ldots, \sigma(t_n))$ for every n-ary function symbol f and terms $t_1, \ldots, t_n$. We call $\sigma(t)$ an *instance* of t. An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression). If $s, t_1, \ldots, t_n$ are terms and $x_1, \ldots, x_n$ mutually distinct variables then $s[x_i \leftarrow t_i \mid 1 \le i \le n]$ denotes the result of simultaneously replacing every occurrence of x_i in s by t_i $(i = 1, \ldots, n)$.

A *context* $C[\,, \ldots,\,]$ is a 'term' which contains at least one occurrence of a special symbol $\Box$. If $C[\,, \ldots,\,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the result of replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[\,]$. A term s is a *subterm* of a term t if there exists a context $C[\,]$ such that $t \equiv C[s]$.

The *rewrite relation* $\to_{\mathcal{R}}$ is defined as follows: $s \to_{\mathcal{R}} t$ if there exists a rewrite rule $l \to r$ in $\mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$ and $t \equiv C[\sigma(r)]$. The transitive-reflexive closure of $\to_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$; if $s \twoheadrightarrow_{\mathcal{R}} t$ we say that s *reduces* to t. We write $s \leftarrow_{\mathcal{R}} t$ if $t \to_{\mathcal{R}} s$; likewise for $s \twoheadleftarrow_{\mathcal{R}} t$. The transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to_{\mathcal{R}}^{+}$ and $\leftrightarrow_{\mathcal{R}}$ denotes the symmetric closure of $\to_{\mathcal{R}}$ (so $\leftrightarrow_{\mathcal{R}} = \to_{\mathcal{R}} \cup \leftarrow_{\mathcal{R}}$). The transitive-reflexive closure of $\leftrightarrow_{\mathcal{R}}$ is called *conversion* and denoted by $=_{\mathcal{R}}$. If $s =_{\mathcal{R}} t$ then s and t are *convertible*. Two terms t_1, t_2 are *joinable*, notation $t_1 \downarrow_{\mathcal{R}} t_2$, if there exists a term t_3 such that $t_1 \twoheadrightarrow_{\mathcal{R}} t_3 \twoheadleftarrow_{\mathcal{R}} t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2. The relation $\downarrow_{\mathcal{R}}$ is called *joinability*. We often omit the subscript $\mathcal{R}$.

A term s is a *normal form* if there are no terms t with $s \to t$. The set of normal forms of a TRS $(\mathcal{F}, \mathcal{R})$ is denoted by $\mathrm{NF}(\mathcal{F}, \mathcal{R})$. When no confusion can arise, we simply write $\mathrm{NF}(\mathcal{R})$. A TRS $\mathcal{R}$ is *strongly normalizing* (SN) if there are no infinite reduction sequences $t_1 \to t_2 \to t_3 \to \ldots$. In other words, every reduction sequence eventually ends in a normal form. A TRS $\mathcal{R}$ is *weakly normalizing* (WN) if every term reduces to a normal form. A TRS $\mathcal{R}$ is *confluent* or has the *Church-Rosser* property (CR) if for all terms s, t_1, t_2 with $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$ we have $t_1 \downarrow t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable ($t_1 = t_2 \;\Rightarrow\; t_1 \downarrow t_2$). A TRS $\mathcal{R}$ has *unique normal forms* (UN) if no distinct normal forms are convertible ($s = t$ and $s, t \in \mathrm{NF}(\mathcal{R})$ $\Rightarrow\; s \equiv t$). A TRS $\mathcal{R}$ has the *normal form property* (NF) if every term convertible with a normal form, reduces to that normal form ($s = t$ and $t \in \mathrm{NF}(\mathcal{R}) \;\Rightarrow\; s \twoheadrightarrow t$).

The next proposition relates the last three properties. The proof is very simple, see e.g. [14].

PROPOSITION 1.1. *Every confluent* TRS *has the normal form property and every* TRS *with the normal form property has unique normal forms. The reverse implications are not true in general.* $\Box$

2. Simple Construction

In this section we prove that every TRS with unique normal forms can be conservatively extended to a confluent TRS with the same set of normal forms. The construction in this paper is a considerable simplification of the one in [14]. For instance, we will see that it is sufficient to add at most one new constant whereas in [14] we employed infinitely many new function symbols.

Let $(\mathcal{F}, \mathcal{R})$ be a TRS with unique normal forms. First we consider the case that $\mathcal{F}$ contains at least one constant symbol. We will show that every equivalence class C of convertible terms contains a term t which can be used as a 'common reduct' in order to obtain confluence with respect to C.

DEFINITION 2.1.

(1) The set of equivalence classes of convertible terms is denoted by $\mathcal{C}$:

$$\mathcal{C} = \{\varnothing \neq C \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid C \text{ is closed under } =_{\mathcal{R}}\}.$$

(2) The subset of $\mathcal{C}$ consisting of all equivalence classes without a normal form is denoted by $\mathcal{C}^{\perp}$.

(3) If $C \in \mathcal{C}$ then $V_{fix}(C)$ denotes the set of variables occurring in every term $t \in C$:

$$V_{fix}(C) = \bigcap_{t \in C} V(t).$$

The next two propositions originate from [14]. For the sake of completeness, the proofs are repeated here.

PROPOSITION 2.2. *If $t \in C \in \mathcal{C}$ and $V(t) - V_{fix}(C) = \{x_1, \ldots, x_n\}$ then $t[x_i \leftarrow s_i \mid 1 \le i \le n] \in C$ for all terms $s_1, \ldots, s_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.*

PROOF. We first prove the statement for all terms $s_1, \ldots, s_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ with $V(s_i) \cap \{x_1, \ldots, x_n\} = \varnothing$ $(i = 1, \ldots, n)$. Define a sequence of terms $t_0, \ldots, t_n$ as follows:

$$t_0 \equiv t,$$

$$t_i \equiv t_{i-1}[x_i \leftarrow s_i] \quad \text{if } 0 < i \le n.$$

We will show that $t_i =_{\mathcal{R}} t$ by induction on i. The case $i = 0$ is trivial. Suppose the statement is true for all $i < k$ $(k > 0)$. Because $x_k \notin V_{fix}(C)$ there exists a term $u \in C$ such that $x_k \notin V(u)$. The induction hypothesis tells us that $t_{k-1} =_{\mathcal{R}} t$. This implies that

$$t_k \equiv t_{k-1}[x_k \leftarrow s_k] =_{\mathcal{R}} u[x_k \leftarrow s_k] \equiv u =_{\mathcal{R}} t.$$

Therefore $t_n \equiv t[x_1 \leftarrow s_1] \ldots [x_n \leftarrow s_n] \equiv t[x_i \leftarrow s_i \mid 1 \le i \le n] \in C$. Now let $s_1, \ldots, s_n$ be arbitrary terms of $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Choose distinct fresh variables $y_1, \ldots, y_n$ By the above argument we have $t[x_i \leftarrow y_i \mid 1 \le i \le n] \in C$ and because

$$V(t[x_i \leftarrow y_i \mid 1 \le i \le n]) - V_{fix}(C) = \{y_1, \ldots, y_n\}$$

we obtain

$$t[x_i \leftarrow y_i \mid 1 \le i \le n][y_i \leftarrow s_i \mid 1 \le i \le n] \equiv t[x_i \leftarrow s_i \mid 1 \le i \le n] \in C.$$

□

PROPOSITION 2.3. *If $C \in \mathcal{C}$ contains a normal form t then $V_{fix}(C) = V(t)$.*

PROOF. Let $s \in C$. We will show that $V(t) \subseteq V(s)$ by induction on the length of the conversion $s =_{\mathcal{R}} t$. The case of zero length is trivial. Let $s \leftrightarrow_{\mathcal{R}} s_1 =_{\mathcal{R}} t$. From the induction hypothesis we obtain $V(t) \subseteq V(s_1)$. If $s \rightarrow_{\mathcal{R}} s_1$ then $V(s_1) \subseteq V(s)$ and we are done. Assume $s \leftarrow_{\mathcal{R}} s_1$. We have to show that every variable of t occurs in s. Suppose to the contrary that there is a variable $x \in V(t)$ which does not occur in s. Choose a fresh variable y. Replacing every occurrence of x in the conversion $s_1 =_{\mathcal{R}} t$ yields a conversion $s_1' =_{\mathcal{R}} t'$. Notice that t' is a normal form of $\mathcal{R}$ different from t. Because $x \notin V(s)$ we obtain $s_1' \rightarrow_{\mathcal{R}} s$. But now we have the following conversion between t and t':

$$t =_{\mathcal{R}} s_1 \rightarrow_{\mathcal{R}} s \leftarrow_{\mathcal{R}} s_1' =_{\mathcal{R}} t',$$

which is impossible due to the unique normal forms of $\mathcal{R}$. We conclude that $V_{fix}(C) = V(t)$. □

The following proposition is not true if $\mathcal{F}$ does not contain constant symbols.

PROPOSITION 2.4. *If $C \in \mathcal{C}^{\perp}$ then there exists a term $t \in C$ such that $V_{fix}(C) = V(t)$.*

PROOF. Take an arbitrary term $s \in C$ and suppose that $V(s) - V_{fix}(C) = \{x_1, \ldots, x_n\}$. Let $t \equiv s[x_i \leftarrow c \mid 1 \le i \le n]$ where c is any closed term. Proposition 2.2 yields $t \in C$ and we have $V_{fix}(C) = V(t)$ by construction. □

According to the previous results we can define a mapping $\pi : \mathcal{C} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{V})$ with the following properties:

(1) $\pi(C) \in C$,

(2) if $C \in \mathcal{C}$ contains the normal form t then $\pi(C) \equiv t$,

(3) $V_{fix}(C) = V(\pi(C))$.

The term $\pi(C)$ will serve as a common reduct for C.

DEFINITION 2.5. The TRS $(\mathcal{F}, \mathcal{R}')$ is defined by

$$\mathcal{R}' = \mathcal{R} \cup \{t \rightarrow \pi(C) \mid t \in C \in \mathcal{C} \text{ and } t \not\equiv \pi(C)\}.$$

Due to the third property of π and the observation that every variable is a normal form, $\mathcal{R}'$ only contains legal rewrite rules.

PROPOSITION 2.6.

(1) *For all terms s, $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ we have $s =_{\mathcal{R}} t$ if and only if $s =_{\mathcal{R}'} t$.*

(2) $\mathrm{NF}(\mathcal{R}) = \mathrm{NF}(\mathcal{R}')$.

(3) *$\mathcal{R}'$ is confluent.*

PROOF. The first two properties are an immediate consequence of our construction. Suppose $s =_{\mathcal{R}'} t$. According to (1), s and t belong to the same class C of convertible terms. By definition, both terms rewrite in zero or one step to their common reduct $\pi(C)$. □

LEMMA 2.7. *Every* TRS *$(\mathcal{F}, \mathcal{R})$ with unique normal forms can be extended to a confluent* TRS *$(\mathcal{F}', \mathcal{R}')$ such that:*

(1) *for all terms s, $t \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ we have $s =_{\mathcal{R}} t$ if and only if $s =_{\mathcal{R}'} t$,*

(2) $\mathrm{NF}(\mathcal{F}, \mathcal{R}) = \mathrm{NF}(\mathcal{F}', \mathcal{R}')$.

PROOF. If $\mathcal{F}$ contains a constant symbol then the preceding definitions and propositions yield the desired result. So assume that $\mathcal{F}$ only contains function symbols with arity ≥ 1. Let $\perp$ be a fresh (i.e. $\perp \notin \mathcal{F}$) constant symbol and define $\mathcal{F}_1 = \mathcal{F} \cup \{\perp\}$ and $\mathcal{R}_1 = \mathcal{R} \cup \{\perp \rightarrow \perp\}$. The normal forms of $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{F}_1, \mathcal{R}_1)$ clearly coincide. The equivalence of $=_{\mathcal{R}}$ and $=_{\mathcal{R}_1}$ with respect to $\mathcal{T}(\mathcal{F}_1, \mathcal{V})$ is also easily proved. Hence $(\mathcal{F}_1, \mathcal{R}_1)$ has unique normal forms. Because $\mathcal{F}_1$ contains a constant symbol, we know already the existence of a confluent TRS $(\mathcal{F}_1, \mathcal{R}_1')$ such that the relations $=_{\mathcal{R}_1}$ and $=_{\mathcal{R}_1'}$ coincide and $\mathrm{NF}(\mathcal{R}_1) = \mathrm{NF}(\mathcal{R}_1')$. Therefore, $s =_{\mathcal{R}} t$ if and only if $s =_{\mathcal{R}_1'} t$ for all terms s, $t \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ and

$\mathrm{NF}(\mathcal{F}, \mathcal{R}) = \mathrm{NF}(\mathcal{F}_1, \mathcal{R}_1')$. □

3. NF is a Modular Property of Left-Linear Term Rewriting Systems

Before proving the main result of this paper, we introduce several notations and definitions for handling disjoint unions of TRS's. Most of them originate from Toyama [19].

DEFINITION 3.1. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be TRS's with disjoint alphabets (i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \varnothing$). The *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ of $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ is the TRS $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$.

DEFINITION 3.2. A property $\mathcal{P}$ of TRS's is called *modular* if for all disjoint TRS's $\mathcal{R}_1, \mathcal{R}_2$ the following equivalence holds:

$$\mathcal{R}_1 \oplus \mathcal{R}_2 \text{ has the property } \mathcal{P} \iff \text{both } \mathcal{R}_1 \text{ and } \mathcal{R}_2 \text{ have the property } \mathcal{P}.$$

Confluence was the first property for which the modularity has been established.

THEOREM 3.3 (Toyama [19]). *Confluence is a modular property of* TRS'*s*. □

In [14] we gave the following example, showing that NF is not a modular property.

EXAMPLE 3.4. Let $\mathcal{R}_1 = \{F(x, x) \to C\}$ and $\mathcal{R}_2 = \{a \to b,\ a \to c,\ b \to b,\ c \to c\}$. Both TRS's have the property NF. The following conversion shows that $F(b, c)$ is $\mathcal{R}_1 \oplus \mathcal{R}_2$-convertible to the normal form C:

$$F(b, c) \leftarrow F(a, c) \leftarrow F(a, a) \to C.$$

However, it is clear that $F(b, c)$ does not reduce to C. So $\mathcal{R}_1 \oplus \mathcal{R}_2$ is not NF.

Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be disjoint TRS's. Every term $t \in \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ can be viewed as an alternation of $\mathcal{F}_1$-parts and $\mathcal{F}_2$-parts. This structure is formalized in Definition 3.5 and illustrated in Figure 1.

NOTATION. We write $\mathcal{T}$ instead of $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ and we abbreviate $\mathcal{T}(\mathcal{F}_i, \mathcal{V})$ to $\mathcal{T}_i$ $(i = 1, 2)$.

DEFINITION 3.5.

(1) The *root symbol* of a term $t \in \mathcal{T}$, notation $root(t)$, is defined by

$$root(t) = \begin{cases} F & \text{if } t \equiv F(t_1, \ldots, t_n), \\ t & \text{otherwise.} \end{cases}$$

(2) Let $t \equiv C[t_1, \ldots, t_n]$ with $C[\,, \ldots,] \not\equiv \square$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if $C[\,, \ldots,]$ is a $\mathcal{F}_a$-context and $root(t_i) \in \mathcal{F}_b$ with $a \neq b$ for $i = 1, \ldots, n$ $(a, b \in \{1, 2\})$. The t_i's are the *principal* subterms of t.

(3) The *rank* of a term $t \in \mathcal{T}$ is defined by

$$rank(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T}_1 \cup \mathcal{T}_2, \\ 1 + \max\{rank(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!]. \end{cases}$$

(4) The set $S(t)$ of *special* subterms of a term $t \in \mathcal{T}$ is inductively defined by

$$S(t) = \begin{cases} \{t\} & \text{if } rank(t) = 1, \\ \{t\} \cup S(t_1) \cup \ldots \cup S(t_n) & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!]. \end{cases}$$

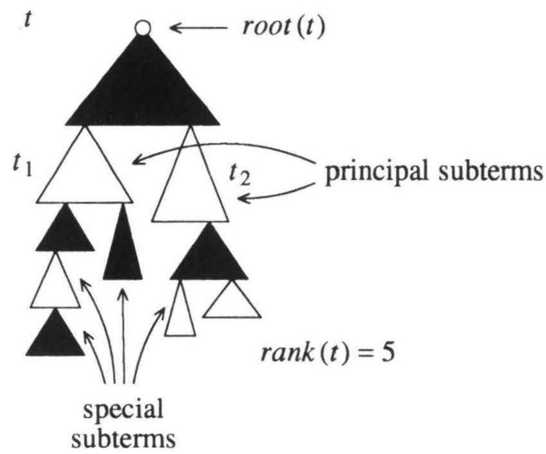

FIGURE 1.

To achieve better readability we will call the function symbols of $\mathcal{F}_1$ *black* and those of $\mathcal{F}_2$ *white*. Variables have no colour. A black (white) term does not contain white (black) function symbols, but may contain variables. In examples, black symbols will be printed as capitals and white symbols in lower case.

DEFINITION 3.6. Let $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}$. We write $<s_1, \ldots, s_n> \propto <t_1, \ldots, t_n>$ if $t_i \equiv t_j$ whenever $s_i \equiv s_j$, for all $1 \le i < j \le n$. The combination of $<s_1, \ldots, s_n> \propto <t_1, \ldots, t_n>$ and $<t_1, \ldots, t_n> \propto <s_1, \ldots, s_n>$ is abbreviated to $<s_1, \ldots, s_n> \infty <t_1, \ldots, t_n>$. This notation is used to code principal subterms by variables.

PROPOSITION 3.7. *If* $s \twoheadrightarrow t$ *then* $rank(s) \ge rank(t)$.

PROOF. Straightforward. □

DEFINITION 3.8. Let $s \to t$ by application of a rewrite rule r. We write $s \to^i t$ if $s \equiv C[\![s_1, \ldots, s_n]\!]$ and r is being applied in one of the s_j's and we write $s \to^o t$ otherwise. The relation $\to^i$ is called *inner* reduction and $\to^o$ is called *outer* reduction.

DEFINITION 3.9. Let $t \in \mathcal{T}$. The *topmost homogeneous part* of t, notation $top(t)$, is the result of replacing all principal subterms of t by □, i.e.

$$top(t) = \begin{cases} t & \text{if } rank(t) = 1, \\ C[\,, \ldots, \,] & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!]. \end{cases}$$

DEFINITION 3.10. We say that a rewrite step $s \to t$ is *destructive at level* 1 if the root symbols of s and t have different colours. The rewrite step $s \to t$ is *destructive at level* $n+1$ if $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to^i C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s_j \to t_j$ destructive at level n.

Notice that $s \to t$ is destructive at level 1 if and only if $s \to^o t$ and either $t \in V(top(s))$ or t is a principal subterm of s. The next definition introduces special notations for 'degenerate' cases of "$t \equiv C[\![t_1, \ldots, t_n]\!]$". Although it might give the impression of making mountains of molehills, it actually is very useful for cutting down the number of cases to consider in some of the following proofs.

DEFINITION 3.11. First we extend the notion of context as defined in Section 1. We write $C\langle\, , \ldots ,\, \rangle$ for a 'term' containing zero or more occurrences of $\square$ and $C\{\, , \ldots ,\, \}$ denotes a 'term' different from $\square$ itself, containing zero or more occurrences of $\square$. If $t \in \mathcal{T}$ and $t_1, \ldots, t_n$ are the (possibly zero) principal subterms of t (from left to right), then we write $t \equiv C\{\{t_1, \ldots, t_n\}\}$ provided $t \equiv C\{t_1, \ldots, t_n\}$. We write $t \equiv C\langle\!\langle t_1, \ldots, t_n \rangle\!\rangle$ if $t \equiv C\langle t_1, \ldots, t_n \rangle$ and either $C\langle\, , \ldots ,\, \rangle \not\equiv \square$ and $t_1, \ldots, t_n$ are the principal subterms of t or $C\langle\, , \ldots ,\, \rangle \equiv \square$ and $t \in \{t_1, \ldots, t_n\}$.

The next two propositions are very intuitive. Their straightforward proofs are left to the reader.

PROPOSITION 3.12. *If* $(\mathcal{F}_1, \mathcal{R}_1)$ *and* $(\mathcal{F}_2, \mathcal{R}_2)$ *are disjoint* TRS'*s then* $\mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2) = \mathrm{NF}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1) \cap \mathrm{NF}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_2)$. $\square$

PROPOSITION 3.13. *Let* $(\mathcal{F}_1, \mathcal{R}_1)$ *and* $(\mathcal{F}_2, \mathcal{R}_2)$ *be* TRS'*s such that* $\mathrm{NF}(\mathcal{F}_1, \mathcal{R}_1) = \mathrm{NF}(\mathcal{F}_2, \mathcal{R}_2)$. *If* $\mathcal{F}'$ *is a set of fresh function symbols, i.e.* $\mathcal{F}' \cap (\mathcal{F}_1 \cup \mathcal{F}_2) = \varnothing$, *then* $\mathrm{NF}(\mathcal{F}_1 \cup \mathcal{F}', \mathcal{R}_1) = \mathrm{NF}(\mathcal{F}_2 \cup \mathcal{F}', \mathcal{R}_2)$. $\square$

PROPOSITION 3.14. *Let* $(\mathcal{F}_1, \mathcal{R}_1)$ *and* $(\mathcal{F}_2, \mathcal{R}_2)$ *be disjoint* TRS'*s. If* $(\mathcal{F}_i', \mathcal{R}_i')$ *is an extension of* $(\mathcal{F}_i, \mathcal{R}_i)$ *with* $\mathrm{NF}(\mathcal{F}_i, \mathcal{R}_i) = \mathrm{NF}(\mathcal{F}_i', \mathcal{R}_i')$ $(i = 1, 2)$ *such that* $\mathcal{F}_1' \cap \mathcal{F}_2' = \varnothing$, *then* $\mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2) = \mathrm{NF}(\mathcal{R}_1' \oplus \mathcal{R}_2')$.

PROOF. Because $\mathcal{R}_1 \cup \mathcal{R}_2 \subseteq \mathcal{R}_1' \cup \mathcal{R}_2'$, we clearly have $\mathrm{NF}(\mathcal{R}_1' \oplus \mathcal{R}_2') \subseteq \mathrm{NF}(\mathcal{F}_1' \cup \mathcal{F}_2', \mathcal{R}_1 \cup \mathcal{R}_2)$. It is not difficult to see that $\mathrm{NF}(\mathcal{F}_1' \cup \mathcal{F}_2', \mathcal{R}_1 \cup \mathcal{R}_2) = \mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2)$. For the other inclusion we assume that $t \in \mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2)$. In particular, $t \in \mathrm{NF}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1)$ and $t \in \mathrm{NF}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_2)$. From Proposition 3.13 we obtain $t \in \mathrm{NF}(\mathcal{F}_1' \cup \mathcal{F}_2, \mathcal{R}_1')$ and hence $t \in \mathrm{NF}(\mathcal{F}_1' \cup \mathcal{F}_2', \mathcal{R}_1')$. Likewise we obtain $t \in \mathrm{NF}(\mathcal{F}_1' \cup \mathcal{F}_2', \mathcal{R}_2')$. Proposition 3.12 yields $t \in \mathrm{NF}(\mathcal{R}_1' \oplus \mathcal{R}_2')$. $\square$

THEOREM 3.15 (Middeldorp [14]). UN *is a modular property of* TRS'*s.*

PROOF. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be disjoint TRS's. We have to show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ has the property UN if and only if both $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are UN.

$\Rightarrow$ Trivial.

$\Leftarrow$ According to Lemma 2.7 we can extend $(\mathcal{F}_i, \mathcal{R}_i)$ to a confluent TRS $(\mathcal{F}_i', \mathcal{R}_i')$ with the same set of normal forms $(i = 1, 2)$. Without loss of generality we assume that $\mathcal{F}_1' \cap \mathcal{F}_2' = \varnothing$. Let $s =_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ be a conversion between normal forms of $\mathcal{R}_1 \oplus \mathcal{R}_2$. Clearly $s =_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$. Because s and t are normal forms with respect to $\mathcal{R}_1' \oplus \mathcal{R}_2'$ (Proposition 3.14), we can use Theorem 3.3 in order to obtain the desired $s \equiv t$.

$\square$

We will now show that NF is a modular property of left-linear TRS's. To this end, we assume that $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are disjoint left-linear TRS's with the property NF. By Proposition 1.1, $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ also have the property UN. So, like in the proof of the modularity of UN, we may extend $(\mathcal{F}_i, \mathcal{R}_i)$ to a confluent TRS $(\mathcal{F}_i', \mathcal{R}_i')$ with the same set of normal forms $(i = 1, 2)$. According to Lemma 2.7 we may further assume that $s =_{\mathcal{R}_i} t$ if and only if $s =_{\mathcal{R}_i'} t$ for all terms $s, t \in \mathcal{T}(\mathcal{F}_i', \mathcal{V})$ $(i = 1, 2)$. Without loss of generality we finally assume that $\mathcal{F}_1' \cap \mathcal{F}_2' = \varnothing$.

NOTATION. We abbreviate $\mathcal{T}(\mathcal{F}_1' \cup \mathcal{F}_2', \mathcal{V})$ to $\mathcal{T}'$ and we use $\mathcal{T}_i'$ as a shorthand for $\mathcal{T}(\mathcal{F}_i', \mathcal{V})$ $(i = 1, 2)$.

Consider a conversion $s =_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ between terms $s, t \in \mathcal{T}$ with $t \in \mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2)$. Just as in the proof of Theorem 3.15 we obtain $s =_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$ and $t \in \mathrm{NF}(\mathcal{R}_1' \oplus \mathcal{R}_2')$. Theorem 3.3 yields $s \twoheadrightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$. The question now arises how to transform this reduction into a $\mathcal{R}_1 \oplus \mathcal{R}_2$-reduction from s to t. Our solution consists of restricting the rewrite relation $\twoheadrightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$ in such a way that the resulting relation $\rightarrowtail$ is weakly normalizing and has the nice property that t is the only $\rightarrowtail$-normal form of s. The reader familiar with the work of Kurihara and Kaji [12] will notice the resemblance of $\rightarrowtail$ with their 'modular reduction'.

In the following we assume that all terms belong to $\mathcal{T}'$, unless stated otherwise.

DEFINITION 3.16. We write $s \rightarrowtail t$ if there exists a context $C[\ ]$ and terms s_1, t_1 such that $s \equiv C[s_1]$, $t \equiv C[t_1]$, $s_1 \in S(s)$, $s_1 \rightarrow_{\mathcal{R}_i}^{o+} t_1$ and $t_1 \in \mathrm{NF}(\rightarrow_{\mathcal{R}_i}^{o})$ for some $i \in \{1, 2\}$. We write $s \rightarrowtail^{o} t$ if $s \rightarrowtail t$ with $C[\ ] \equiv \Box$.

PROPOSITION 3.17. *The relation $\rightarrowtail$ is weakly normalizing.*

PROOF. We will show by induction on $rank(t)$ that every term $t \in \mathcal{T}'$ has a normal form with respect to $\rightarrowtail$. If $rank(t) = 1$ then $t \in \mathcal{T}_1'$ or $t \in \mathcal{T}_2'$. We consider without loss of generality only the former. Clearly $t \in \mathrm{NF}(\rightarrow_{\mathcal{R}_2})$. If $t \in \mathrm{NF}(\rightarrow_{\mathcal{R}_1})$ or if t does not have a normal form with respect to $\rightarrow_{\mathcal{R}_1}$, then $t \in \mathrm{NF}(\rightarrowtail)$. Otherwise $t \rightarrowtail t'$ for some $t' \in \mathrm{NF}(\rightarrow_{\mathcal{R}_1})$ and because $t' \in \mathrm{NF}(\rightarrow_{\mathcal{R}_2})$ we obtain $t' \in \mathrm{NF}(\rightarrowtail)$. Let $t \equiv C[\![t_1, \ldots, t_n]\!]$. Applying the induction hypothesis to $t_1, \ldots, t_n$ yields $\rightarrowtail$-normal forms $t_1', \ldots, t_n'$ such that $t_i \rightarrowtail \ldots \rightarrowtail t_i'$ for $i = 1, \ldots, n$. We clearly can write $C[t_1', \ldots, t_n'] \equiv C'\{\!\{s_1, \ldots, s_m\}\!\}$ for some $\rightarrowtail$-normal forms $s_1, \ldots, s_m$ $(m \geq 0)$ and 'context' $C'\{\ , \ldots, \ \}$. Choose fresh variables $X_1, \ldots, X_m$ such that $<s_1, \ldots, s_m> \infty <X_1, \ldots, X_m>$. Because $rank(C'\{X_1, \ldots, X_m\}) = 1$ we obtain a $\rightarrowtail$-normal form $C^*\langle X_{i_1}, \ldots, X_{i_p}\rangle$ of $C'\{X_1, \ldots, X_m\}$ from the induction hypothesis. Now we have the following $\rightarrowtail$-reduction sequence:

$$t \equiv C[\![t_1, \ldots, t_n]\!] \rightarrowtail \ldots \rightarrowtail C[t_1', \ldots, t_n'] \equiv C'\{\!\{s_1, \ldots, s_m\}\!\}$$
$$\rightarrowtail \ldots \rightarrowtail C^*\langle\!\langle s_{i_1}, \ldots, s_{i_p} \rangle\!\rangle.$$

It is not difficult to see that $C^*\langle\!\langle s_{i_1}, \ldots, s_{i_p} \rangle\!\rangle$ is a normal form with respect to $\rightarrowtail$. □

The other property of $\rightarrowtail$ is a bit harder to prove. We start with some technical propositions.

PROPOSITION 3.18. *If $s =_{\mathcal{R}_i}^{o} t$ and $t \in \mathrm{NF}(\rightarrow_{\mathcal{R}_i}^{o})$ then $s \twoheadrightarrow_{\mathcal{R}_i}^{o} t$.*

PROOF. We use induction on the length of the conversion $s =_{\mathcal{R}_i}^{o} t$. The case of zero length is trivial. Let $s \leftrightarrow_{\mathcal{R}_i}^{o} s_1 =_{\mathcal{R}_i}^{o} t$. From the induction hypothesis we obtain $s_1 \twoheadrightarrow_{\mathcal{R}_i}^{o} t$. If $s \rightarrow_{\mathcal{R}_i}^{o} s_1$ then we are done. Suppose $s \leftarrow_{\mathcal{R}_i}^{o} s_1$. It is easy to see that we may write

$$s \equiv C_1\langle\!\langle u_{j_1}, \ldots, u_{j_m} \rangle\!\rangle \leftarrow_{\mathcal{R}_i}^{o} s_1 \equiv C\{\!\{u_1, \ldots, u_n\}\!\} \twoheadrightarrow_{\mathcal{R}_i}^{o} t \equiv C_2\langle\!\langle u_{k_1}, \ldots, u_{k_p} \rangle\!\rangle$$

for some terms $u_1, \ldots, u_n$ and 'contexts' $C\{\ , \ldots, \ \}$, $C_1\langle\ , \ldots, \ \rangle$ and $C_2\langle\ , \ldots, \ \rangle$. Choose fresh variables $X_1, \ldots, X_n$ such that $<u_1, \ldots, u_n> \infty <X_1, \ldots, X_n>$. We have

$$C_1\langle X_{j_1}, \ldots, X_{j_m}\rangle \leftarrow_{\mathcal{R}_i} C\{X_1, \ldots, X_n\} \twoheadrightarrow_{\mathcal{R}_i} C_2\langle X_{k_1}, \ldots, X_{k_p}\rangle$$

with $C_2\langle X_{k_1}, \ldots, X_{k_p}\rangle \in \mathrm{NF}(\mathcal{R}_i)$. We obtain $C_1\langle X_{j_1}, \ldots, X_{j_m}\rangle \twoheadrightarrow_{\mathcal{R}_i} C_2\langle X_{k_1}, \ldots, X_{k_p}\rangle$. from the assumption that $\mathcal{R}_i$ has the normal form property. Instantiating this reduction yields $s \equiv C_1\langle\!\langle u_{j_1}, \ldots, u_{j_m} \rangle\!\rangle \twoheadrightarrow_{\mathcal{R}_i}^{o} C_2\langle\!\langle u_{k_1}, \ldots, u_{k_p} \rangle\!\rangle \equiv t$. □

PROPOSITION 3.19. *If* $s \rightarrow^{o}_{\mathcal{R}_i'} t$ *then* $s =^{o}_{\mathcal{R}_i} t$.

PROOF. Just as in the previous proof we may write $s \equiv C_1\{\{u_1, \ldots, u_n\}\} \rightarrow^{o}_{\mathcal{R}_i'} C_2\langle\langle u_{j_1}, \ldots, u_{j_m}\rangle\rangle \equiv t$. Choosing fresh variables $X_1, \ldots, X_n$ with $<u_1, \ldots, u_n> \infty <X_1, \ldots, X_n>$ yields $C_1\{X_1, \ldots, X_n\} \rightarrow_{\mathcal{R}_i'} C_2\langle X_{j_1}, \ldots, X_{j_m}\rangle$. Because $C_1\{X_1, \ldots, X_n\}$ and $C_2\langle X_{j_1}, \ldots, X_{j_m}\rangle$ belong to $\mathcal{T}_i'$, we have $C_1\{X_1, \ldots, X_n\} =_{\mathcal{R}_i} C_2\langle X_{j_1}, \ldots, X_{j_m}\rangle$ from which we immediately obtain $s \equiv C_1\{\{u_1, \ldots, u_n\}\} =^{o}_{\mathcal{R}_i} C_2\langle\langle u_{j_1}, \ldots, u_{j_m}\rangle\rangle \equiv t$. □

NOTATION. We write $s \approx^{o} t$ if $top(s) \equiv top(t)$.

The left-linearity of $\mathcal{R}_1$ and $\mathcal{R}_2$ is only (explicitly) used in the proof of the next proposition.

PROPOSITION 3.20. *If* $s \rightarrow^{o}_{\mathcal{R}_i} t$ *and* $s \approx^{o} s'$ *then there exists a term* t' *such that* $s' \rightarrow^{o}_{\mathcal{R}_i} t'$. *Furthermore, if* $s \rightarrow^{o}_{\mathcal{R}_i} t$ *is not destructive then we also have* $t \approx^{o} t'$.

PROOF. We have $s \equiv C_1\{\{u_1, \ldots, u_n\}\} \rightarrow^{o}_{\mathcal{R}_i} C_2\langle\langle u_{j_1}, \ldots, u_{j_m}\rangle\rangle \equiv t$ for some terms $u_1, \ldots, u_n$ and 'contexts' $C_1\{\ , \ldots,\ \}$ and $C_2\langle\ , \ldots,\ \rangle$. If $s \approx^{o} s'$ then $s' \equiv C_1\{\{u_1', \ldots, u_n'\}\}$ for some terms $u_1', \ldots, u_n'$ and because $\mathcal{R}_i$ is left-linear we can apply the same rewrite rule as in $s \rightarrow^{o}_{\mathcal{R}_i} t$ to the term s'. This gives us $s' \rightarrow^{o}_{\mathcal{R}_i} C_2\langle\langle u_{j_1}', \ldots, u_{j_m}'\rangle\rangle$ and we define $t' \equiv C_2\langle\langle u_{j_1}', \ldots, u_{j_m}'\rangle\rangle$. If $s \rightarrow^{o}_{\mathcal{R}_i} t$ is not destructive then $C_2\langle\ , \ldots,\ \rangle \not\equiv \Box$ and hence $top(t) \equiv C_2\langle\ , \ldots,\ \rangle \equiv top(t')$. So $t \approx^{o} t'$ by definition. □

PROPOSITION 3.21. *If* $t \in \mathrm{NF}(\rightarrow^{o}_{\mathcal{R}_i})$ *and* $t \approx^{o} t'$ *then* $t' \in \mathrm{NF}(\rightarrow^{o}_{\mathcal{R}_i})$.

PROOF. Immediate consequence of the previous proposition. □

PROPOSITION 3.22. *If* $s \rightarrow^{o}_{\mathcal{R}_i'} t$ *is destructive then* $t \in \mathrm{NF}(\rightarrow^{o}_{\mathcal{R}_i})$ *and* $s \twoheadrightarrow^{o} t$.

PROOF. The root symbol of s belongs to $\mathcal{F}_i'$ and, by Definition 3.10, $root(t) \notin \mathcal{F}_i'$. Therefore, t is not reducible with respect to $\rightarrow^{o}_{\mathcal{R}_i}$. Combining Propositions 3.18 and 3.19 yields $s \twoheadrightarrow^{o}_{\mathcal{R}_i} t$ and since $s \not\equiv t$ we obtain $s \twoheadrightarrow^{o} t$. □

PROPOSITION 3.23. *If* $s \twoheadrightarrow^{o} t$ *and* $s \approx^{o} s'$ *then there exists a term* t' *such that* $s' \twoheadrightarrow^{o} t'$.

PROOF. We have $s \rightarrow^{o+}_{\mathcal{R}_i} t$ with $t \in \mathrm{NF}(\rightarrow^{o}_{\mathcal{R}_i})$ for some $i \in \{1, 2\}$. We will show by induction on the length of $s \rightarrow^{o+}_{\mathcal{R}_i} t$ the existence of a term $t' \in \mathrm{NF}(\rightarrow^{o}_{\mathcal{R}_i})$ such that $s' \rightarrow^{o+}_{\mathcal{R}_i} t'$. If the length of $s \rightarrow^{o+}_{\mathcal{R}_i} t$ equals one, we apply Proposition 3.20 in order to obtain a term t' with $s' \rightarrow^{o}_{\mathcal{R}_i} t'$. If $s' \rightarrow^{o}_{\mathcal{R}_i} t'$ is destructive then $t' \in \mathrm{NF}(\rightarrow^{o}_{\mathcal{R}_i})$ by Proposition 3.22. Otherwise $t \approx^{o} t'$ by Proposition 3.20 and hence $t' \in \mathrm{NF}(\rightarrow^{o}_{\mathcal{R}_i})$ by Proposition 3.21. Next we assume that $s \rightarrow^{o}_{\mathcal{R}_i} s_1 \rightarrow^{o+}_{\mathcal{R}_i} t$. Proposition 3.22 shows that $s \rightarrow^{o}_{\mathcal{R}_i} s_1$ is not destructive. Proposition 3.20 yields a term s_1' with $s' \rightarrow^{o}_{\mathcal{R}_i} s_1'$ and $s_1 \approx^{o} s_1'$. From the induction hypothesis we obtain a term $t' \in \mathrm{NF}(\rightarrow^{o}_{\mathcal{R}_i})$ with $s_1' \rightarrow^{o+}_{\mathcal{R}_i} t'$. We conclude that $s' \twoheadrightarrow^{o} t'$. □

PROPOSITION 3.24. *If* $s \rightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$ *is destructive then* $s \notin \mathrm{NF}(\twoheadrightarrow)$.

PROOF. Easy consequence of Proposition 3.22. □

PROPOSITION 3.25. *If* $s \rightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$ *and* $s \in \mathrm{NF}(\twoheadrightarrow)$ *then* $t \in \mathrm{NF}(\twoheadrightarrow)$.

PROOF. We have $s \equiv C[s_1]$, $t \equiv C[t_1]$ and $s_1 \rightarrow^{o}_{\mathcal{R}_i'} t_1$ for some context $C[\ ]$, terms $s_1 \in S(s)$, t_1 and index $i \in \{1, 2\}$. The previous proposition shows that $s_1 \rightarrow^{o}_{\mathcal{R}_i'} t_1$ is not destructive. Hence $root(t_1) \in \mathcal{F}_i'$ and $t_1 \in S(t)$. It is not difficult to see that for every special subterm $t' \not\equiv t_1$ of t we can find a special subterm s' of s with $s' \approx^o t'$. Suppose $t \notin \mathrm{NF}(\twoheadrightarrow)$. Then there exists a $t' \in S(t)$ such that $t' \twoheadrightarrow^o t''$ for some term t''. Because $s \in \mathrm{NF}(\twoheadrightarrow)$, the previous proposition and the above remark show that this is only possible in case $t' \equiv t_1$. Since $root(t_1) \in \mathcal{F}_i'$, $t_1 \twoheadrightarrow^o t''$ implies $t_1 \rightarrow_{\mathcal{R}_i}^{o+} t''$ with $t'' \in \mathrm{NF}(\rightarrow_{\mathcal{R}_i}^o)$. Therefore $s_1 \rightarrow_{\mathcal{R}_i'}^o t_1 \rightarrow_{\mathcal{R}_i}^{o+} t''$. Proposition 3.19 yields $s_1 =_{\mathcal{R}_i}^o t''$ and we obtain $s_1 \twoheadrightarrow_{\mathcal{R}_i}^o t''$ from Proposition 3.18. Clearly $s_1 \not\equiv t''$. Hence $s_1 \twoheadrightarrow^o t''$, contradicting the assumption of s being in $\twoheadrightarrow$-normal form. □

PROPOSITION 3.26. *If* $s \twoheadrightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$, $s \in \mathrm{NF}(\twoheadrightarrow)$ *and* $t \in \mathrm{NF}(\mathcal{R}_1' \oplus \mathcal{R}_2')$ *then* $s \equiv t$.

PROOF. We use induction on the length of $s \twoheadrightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$. The case of zero length is trivial. Let $s \rightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} s_1 \twoheadrightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$. From Proposition 3.25 we obtain $s_1 \in \mathrm{NF}(\twoheadrightarrow)$ and hence we can apply the induction hypothesis to the sequence $s_1 \twoheadrightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$. This yields $s_1 \equiv t$. We clearly have $s \equiv C[s']$, $t \equiv C[t']$ and $s' \rightarrow_{\mathcal{R}_i'}^o t'$ for some context $C[\,]$, terms $s' \in S(s)$, t' and index $i \in \{1, 2\}$. Proposition 3.19 yields $s' =_{\mathcal{R}_i}^o t'$ and because $t' \in \mathrm{NF}(\rightarrow_{\mathcal{R}_i}^o)$ we have $s' \twoheadrightarrow_{\mathcal{R}_i}^o t'$ by Proposition 3.18. If $s' \rightarrow_{\mathcal{R}_i}^{o+} t'$ then $s' \twoheadrightarrow^o t'$, contradicting the assumption $s \in \mathrm{NF}(\twoheadrightarrow)$. Therefore $s' \equiv t'$ and $s \equiv t$. □

THEOREM 3.27. NF *is a modular property of left-linear* TRS'*s*.

PROOF. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be disjoint left-linear TRS's. We have to show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ has the property NF if and only if both $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ have the property NF.

$\Rightarrow$ Trivial.

$\Leftarrow$ Let $(\mathcal{F}_1', \mathcal{R}_1')$ and $(\mathcal{F}_2', \mathcal{R}_2')$ be TRS's satisfying the requirements made after Theorem 3.3 and consider a conversion $s =_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ between terms $s, t \in \mathcal{T}$ with $t \in \mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2)$. We clearly have $s =_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$ and we obtain $t \in \mathrm{NF}(\mathcal{R}_1' \oplus \mathcal{R}_2')$ from Proposition 3.14. Theorem 3.3 yields $s \twoheadrightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$. By Proposition 3.17, s has a normal form with respect to $\twoheadrightarrow$, say t'. In particular we have $s \twoheadrightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t'$. The confluence of $\mathcal{R}_1' \oplus \mathcal{R}_2'$ (Theorem 3.3) implies $t' \twoheadrightarrow_{\mathcal{R}_1' \oplus \mathcal{R}_2'} t$ and hence $t' \equiv t$ by Proposition 3.26. So $s \twoheadrightarrow \ldots \twoheadrightarrow t$ and because $\twoheadrightarrow$ is a restriction of $\twoheadrightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$, we obtain $s \twoheadrightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$. We conclude that $\mathcal{R}_1 \oplus \mathcal{R}_2$ has the property NF.

□

4. Conditional Term Rewriting Systems

The rewrite rules of a *conditional term rewriting system* (CTRS) have the form

$$l \rightarrow r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$$

with $s_1, \ldots, s_n, t_1, \ldots, t_n, l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. The equations $s_1 = t_1, \ldots, s_n = t_n$ are the *conditions* of the rewrite rule. Depending on the interpretation of the =–sign in the conditions, different rewrite relations can be associated to a given CTRS. In this paper we restrict ourselves to the three most common interpretations.

(1) *Join systems.*
In a join CTRS the =–sign in the conditions is interpreted as *joinability*. Formally: $s \rightarrow t$ if there exists a conditional rewrite rule $l \rightarrow r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$, a substitution σ and a context $C[\,]$ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) \downarrow \sigma(t_i)$ for all $i \in \{1, \ldots, n\}$. Rewrite rules of a join CTRS will henceforth be written as

$$l \rightarrow r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n.$$

(2) *Semi-equational systems.*
Semi-equational CTRS's are obtained by interpreting the =–sign in the conditions as *conversion*.

(3) *Normal systems.*
In a normal CTRS the rewrite rules are subject to the constraint that every t_i is a ground normal form with respect to the rewrite relation obtained by interpreting the =-sign in the conditions as *reduction* ($\twoheadrightarrow$). Rewrite rules of a normal CTRS will be presented as

$$l \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_n \twoheadrightarrow t_n.$$

This classification originates essentially from Bergstra and Klop [1]. The nomenclature stems from Dershowitz, Okada and Sivakumar [4].

The restrictions we impose on CTRS's $\mathcal{R}$ in any of the three formulations are the same as for unconditional TRS's: if $l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n$ is a rewrite rule of $\mathcal{R}$ then l is not a single variable and variables occurring in r also occur in l.

Conditional term rewriting is inherently more complicated than ordinary term rewriting, see Bergstra and Klop [1] and Kaplan [10]. Several well-known results for TRS's have been shown not to hold for CTRS's. Sufficient conditions for confluence and strong normalization of CTRS's can be found in [1], [3], [4], [9] and [10]. In two recent papers ([16] and [17]) we studied CTRS's from the modularization point of view. In [16] we extended Toyama's confluence result for disjoint unions of TRS's to CTRS's.

THEOREM 4.1 (Middeldorp [16]). *Confluence is a modular property of join, semi-equational and normal* CTRS*'s.* □

Strong and weak normalization were the theme of [17]. In this section we are concerned with the modularity of unique normal forms. We first observe that the proof of Theorem 3.15 does not extend to join CTRS's because not every join CTRS with unique normal forms can be extended to a confluent join CTRS with the same set of normal forms.

EXAMPLE 4.2. Let

$$\mathcal{R} = \begin{cases} A \to B \\ A \to C \\ B \to B \\ D \to E \Leftarrow B \downarrow C \\ D \to F. \end{cases}$$

Clearly $\mathcal{R}$ has the property UN. However, there does not exist a confluent CTRS $\mathcal{R}'$ such that $\mathcal{R} \subseteq \mathcal{R}'$ and the normal forms of $\mathcal{R}$ and $\mathcal{R}'$ coincide: if such a $\mathcal{R}'$ were to exist then $B \downarrow_{\mathcal{R}'} C$ and therefore $D \to_{\mathcal{R}'} E$ which contradicts either the confluence of $\mathcal{R}'$ or the equality of NF$(\mathcal{R})$ and NF$(\mathcal{R}')$.

The same remark holds for normal CTRS's, as can be seen by replacing the fourth rule of $\mathcal{R}$ in the previous example by the rule $D \to E \Leftarrow B \twoheadrightarrow C$. In the remainder of this section we will show that the method for proving the modularity of UN for TRS's does extend to semi-equational CTRS's.

A careful inspection of the proofs in Section 2 reveals that Lemma 2.7 is also true for semi-equational CTRS's. Only part (1) and (2) of Proposition 2.6 need some further elaboration. As a matter of fact, this is precisely the place were join and normal CTRS's fail. The following definition is fundamental for establishing properties of (semi-equational) CTRS's.

DEFINITION 4.3. Let $\mathcal{R}$ be a semi-equational CTRS. We inductively define TRS's $\mathcal{R}_i$ for $i \geq 0$ as follows:

$$\mathcal{R}_0 = \{s \to t \mid s \equiv C[\sigma(l)] \text{ and } t \equiv C[\sigma(r)] \text{ for some context } C[\,],\text{ substitution } \sigma \text{ and unconditional rewrite rule } l \to r \in \mathcal{R}\},$$

$$\mathcal{R}_{i+1} = \{s \to t \mid s \equiv C[\sigma(l)] \text{ and } t \equiv C[\sigma(r)] \text{ for some context } C[\,],\text{ substitution } \sigma \text{ and rewrite rule } l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n \in \mathcal{R} \text{ such that } \sigma(s_j) =_{\mathcal{R}_i} \sigma(t_j) \text{ for } j = 1, \ldots, n\}.$$

We have $s \to_{\mathcal{R}} t$ if and only if $s \to_{\mathcal{R}_i} t$ for some $i \geq 0$. The *depth* of $s \to_{\mathcal{R}} t$ is defined as the minimum i such that $s \to_{\mathcal{R}_i} t$. Depths of conversions $s =_{\mathcal{R}} t$ are similarly defined.

Proposition 4.4 is the analogue of the first two parts of Proposition 2.6 for semi-equational CTRS's. The reader is invited to check that the proof fails for join and normal CTRS's.

PROPOSITION 4.4.
(1) *For all terms* $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ *we have* $s =_{\mathcal{R}} t$ *if and only if* $s =_{\mathcal{R}'} t$.
(2) $\mathrm{NF}(\mathcal{R}) = \mathrm{NF}(\mathcal{R}')$.

PROOF.
(1) If $s =_{\mathcal{R}} t$ then $s =_{\mathcal{R}'} t$ since $\mathcal{R}'$ is an extension of $\mathcal{R}$. For the other direction it is sufficient to prove that $s \to_{\mathcal{R}'} t$ implies $s =_{\mathcal{R}} t$. This will be done by induction on the depth of $s \to_{\mathcal{R}'} t$. If the depth equals zero then there exists a context $C[\,]$, an unconditional rewrite rule $l \to r \in \mathcal{R}'$ and a substitution σ such that $s \equiv C[\sigma(l)]$ and $t \equiv C[\sigma(r)]$. If $l \to r \in \mathcal{R}$ then we clearly have $s \to_{\mathcal{R}} t$. Otherwise $r \equiv \pi(C)$ with $l \in C \in \mathcal{C}$ and we obtain $l =_{\mathcal{R}} r$ and hence $s =_{\mathcal{R}} t$. If the depth of $s \to_{\mathcal{R}'} t$ equals $n+1$ $(n \geq 0)$, then there exists a context $C[\,]$, a conditional rewrite rule $l \to r \Leftarrow s_1 = t_1, \ldots, s_m = t_m \in \mathcal{R}$ and a substitution σ such that $s \equiv C[\sigma(l)]$, $t \equiv C[\sigma(r)]$ and $\sigma(s_i) =_{\mathcal{R}'} \sigma(t_i)$ for $i = 1, \ldots, m$ with depth less than or equal to n. Notice that $\mathcal{R}' - \mathcal{R}$ only contains unconditional rewrite rules. A straightforward induction on the length of the conversion $\sigma(s_i) =_{\mathcal{R}'} \sigma(t_i)$ yields $\sigma(s_i) =_{\mathcal{R}} \sigma(t_i)$ $(i = 1, \ldots, m)$. Therefore $\sigma(l) \to_{\mathcal{R}} \sigma(r)$ and $s \to_{\mathcal{R}} t$.
(2) The inclusion $\mathrm{NF}(\mathcal{R}') \subseteq \mathrm{NF}(\mathcal{R})$ is evident. Suppose there exists a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $t \in \mathrm{NF}(\mathcal{R})$ and $t \notin \mathrm{NF}(\mathcal{R}')$. One easily shows that t cannot be reducible with respect to a rewrite rule of $\mathcal{R}' - \mathcal{R}$. Hence there exists a context $C[\,]$, a rewrite rule $l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n \in \mathcal{R}$ $(n \geq 0)$ and a substitution σ such that $t \equiv C[\sigma(l)]$ and $\sigma(s_i) =_{\mathcal{R}'} \sigma(t_i)$ for $i = 1, \ldots, n$. Part (1) shows that $\sigma(s_i) =_{\mathcal{R}} \sigma(t_i)$ $(i = 1, \ldots, n)$ which implies $t \to_{\mathcal{R}} C[\sigma(r)]$, contradicting the assumption $t \in \mathrm{NF}(\mathcal{R})$. We conclude that $\mathrm{NF}(\mathcal{R}) = \mathrm{NF}(\mathcal{R}')$.

□

We obtain the following result.

LEMMA 4.5. *Every semi-equational* CTRS $(\mathcal{F}, \mathcal{R})$ *with unique normal forms can be extended to a confluent semi-equational* CTRS $(\mathcal{F}', \mathcal{R}')$ *such that:*
(1) *for all terms* $s, t \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ *we have* $s =_{\mathcal{R}} t$ *if and only if* $s =_{\mathcal{R}'} t$,
(2) $\mathrm{NF}(\mathcal{F}, \mathcal{R}) = \mathrm{NF}(\mathcal{F}', \mathcal{R}')$.

□

The other key result used in the proof of Theorem 3.15, that is to say Proposition 3.14, is not true in its full generality for semi-equational CTRS's. Fortunately, we will see that it is sufficient to prove this result only for confluent extensions. The complicated proof of the next proposition, which is evidently true for unconditional TRS's (even without the confluence requirement, see Proposition

3.12), is almost identical to the proof of Lemma 4.27 in [17], where the same result is shown to hold for join CTRS's. In order not to disrupt the discussion, we refrain from repeating the proof.

PROPOSITION 4.6. *If* $(\mathcal{F}_1, \mathcal{R}_1)$ *and* $(\mathcal{F}_2, \mathcal{R}_2)$ *are disjoint confluent semi-equational* CTRS*'s then* $\mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2) = \mathrm{NF}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1) \cap \mathrm{NF}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_2)$. □

Proposition 3.13 is not true for semi-equational CTRS's, as is shown in the next example.

EXAMPLE 4.7. Consider the semi-equational CTRS's

$$\mathcal{R}_1 = \begin{cases} A & \to B \\ A & \to C \\ F(x, y) & \to D \Leftarrow x = y \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} A & \to B \\ A & \to C \\ F(x, x) & \to D \\ F(A, x) & \to D \Leftarrow A = x \\ F(B, x) & \to D \Leftarrow B = x \\ F(C, x) & \to D \Leftarrow C = x \\ F(D, x) & \to D \Leftarrow D = x \\ F(F(x, y), z) & \to D \Leftarrow F(x, y) = z \end{cases}$$

with $\mathcal{F}_1 = \mathcal{F}_2 = \{A, B, C, D, F\}$. It is not difficult to show that $\mathrm{NF}(\mathcal{F}_1, \mathcal{R}_1) = \mathrm{NF}(\mathcal{F}_2, \mathcal{R}_2)$. Take $\mathcal{F}' = \{g\}$ with g a unary function symbol and let $t \equiv F(g(B), g(C))$. Clearly $t \in \mathrm{NF}(\mathcal{F}_2 \cup \mathcal{F}', \mathcal{R}_2)$. However, $t \to_{\mathcal{R}_1} D$ since $g(B) =_{\mathcal{R}_1} g(C)$. Notice that both systems are not confluent.

The following restricted version of Proposition 3.13 for semi-equational CTRS's can be obtained using similar techniques as in the proof of Lemma 4.27 from [17].

PROPOSITION 4.8. *Let* $(\mathcal{F}_1, \mathcal{R}_1)$ *and* $(\mathcal{F}_2, \mathcal{R}_2)$ *be semi-equational* CTRS*'s such that* $\mathrm{NF}(\mathcal{F}_1, \mathcal{R}_1) = \mathrm{NF}(\mathcal{F}_2, \mathcal{R}_2)$ *and* $(\mathcal{F}_2, \mathcal{R}_2)$ *is confluent. If* $\mathcal{F}'$ *is a set of fresh function symbols then* $\mathrm{NF}(\mathcal{F}_1 \cup \mathcal{F}', \mathcal{R}_1) \subseteq \mathrm{NF}(\mathcal{F}_2 \cup \mathcal{F}', \mathcal{R}_2)$. □

PROPOSITION 4.9. *Let* $(\mathcal{F}_1, \mathcal{R}_1)$ *and* $(\mathcal{F}_2, \mathcal{R}_2)$ *be disjoint semi-equational* CTRS*'s. If* $(\mathcal{F}_i', \mathcal{R}_i')$ *is a confluent extension of* $(\mathcal{F}_i, \mathcal{R}_i)$ *with* $\mathrm{NF}(\mathcal{F}_i, \mathcal{R}_i) = \mathrm{NF}(\mathcal{F}_i', \mathcal{R}_i')$ $(i = 1, 2)$ *such that* $\mathcal{F}_1' \cap \mathcal{F}_2' = \varnothing$*, then* $\mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2) = \mathrm{NF}(\mathcal{R}_1' \oplus \mathcal{R}_2')$.

PROOF. Similar to the proof of Proposition 3.14. The application of Proposition 4.7 and 4.8 (instead of Proposition 3.12 and 3.13) is justified by the confluence of $(\mathcal{F}_1', \mathcal{R}_1')$ and $(\mathcal{F}_2', \mathcal{R}_2')$. □

The next example shows why confluence is required in Proposition 4.9.

EXAMPLE 4.10. Let $\mathcal{F}_1 = \mathcal{F}_1' = \{F, C\}$, $\mathcal{F}_2 = \mathcal{F}_2' = \{a, b, c\}$, $\mathcal{R}_1 = \mathcal{R}_1' = \{F(x, y) \to C \Leftarrow x = y\}$, $\mathcal{R}_2 = \{a \to b\}$ and $\mathcal{R}_2' = \mathcal{R}_2 \cup \{a \to c\}$. The term $F(b, c)$ belongs to $\mathrm{NF}(\mathcal{R}_1 \oplus \mathcal{R}_2)$ because b and c are not convertible with respect to $\mathcal{R}_1 \oplus \mathcal{R}_2$. However, $F(b, c) \to_{\mathcal{R}_1' \oplus \mathcal{R}_2'} C$ since $b \leftarrow_{\mathcal{R}_2} a \to_{\mathcal{R}_2'} c$.

Therefore $NF(\mathcal{R}_1 \oplus \mathcal{R}_2) \neq NF(\mathcal{R}_1' \oplus \mathcal{R}_2')$ even though $NF(\mathcal{F}_2, \mathcal{R}_2) = NF(\mathcal{F}_2', \mathcal{R}_2')$.

Putting all pieces together, we obtain the modularity of UN for semi-equational CTRS's.

THEOREM 4.11 UN *is a modular property of semi-equational* CTRS'*s.*

PROOF. The proof is the same as the proof of Theorem 3.15, apart from using Lemma 4.5, Proposition 4.9 and Theorem 4.1 instead of Lemma 2.7, Proposition 3.14 and Theorem 3.3. □

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