Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

G. Ding, A. Schrijver, P.D. Seymour

Disjoint paths in a planar graph - a general theorem

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).
Disjoint Paths in a Planar Graph - a General Theorem

Guoli Ding  
Rutgers Center for Operations Research,  
Rutgers University,  
New Brunswick, N.J. 08903,  
U.S.A.

A. Schrijver  
Centrum voor Wiskunde en Informatica  
Kruislaan 413,  
1098 SJ Amsterdam,  
The Netherlands

P.D. Seymour  
Bellcore,  
445 South Street,  
Morristown, N.J. 07960,  
U.S.A.

Abstract. Let \( D = (V, A) \) be a directed planar graph, let \( (r_1, s_1), \ldots, (r_k, s_k) \) be pairs of vertices on the boundary of the unbounded face, let \( A_1, \ldots, A_k \) be subsets of \( A \) and let \( H \) be a collection of unordered pairs from \( \{1, \ldots, k\} \). We give necessary and sufficient conditions for the existence of a directed \( r_i - s_i \) path \( P_i \) in \( (V, A_i) \) (for \( i = 1, \ldots, k \)), such that \( P_i \) and \( P_j \) are vertex-disjoint whenever \( \{i, j\} \in H \).

1980 Mathematics Subject Classification: 05C35, 05C38, 05C70.  
Key Words and Phrases: disjoint, path, planar graph.

1. Introduction

Let \( D = (V, A) \) be a directed graph, let \( (r_1, s_1), \ldots, (r_k, s_k) \) be pairs of vertices of \( D \), let \( A_1, \ldots, A_k \) be subsets of \( A \) and let \( H \) be a collection of unordered pairs from \( \{1, \ldots, k\} \). We are interested under which conditions there exist directed paths \( P_1, \ldots, P_k \) so that:

\[(1)\]

(i) \( P_i \) is a directed \( r_i - s_i \) path in \( (V, A_i) \) (\( i = 1, \ldots, k \));

(ii) \( P_i \) and \( P_j \) are vertex-disjoint for each \( \{i, j\} \in H \).

In Section 3 we will discuss some special cases of this problem.

Since the problem is NP-complete, we may not expect a nice set of necessary and sufficient conditions characterizing the existence of paths satisfying (1). The problem is NP-complete even if we restrict the problem to instances with \( D \) planar, \( k = 2, A_1 = A_2 = A \) and \( H = \{1, 2\} \). Moreover, it is NP-complete when restricted to \( A_1 = \ldots = A_k = A, H \) is the collection of all pairs from \( \{1, \ldots, k\} \), and \( D \) arises from an undirected planar graph by replacing each edge by two opposite arcs.

In this paper we give necessary and sufficient conditions for the problem when

\[(2)\]

\( D \) is planar and the vertices \( r_1, s_1, \ldots, r_k, s_k \) all belong to the boundary of one fixed face \( I \).

The characterization extends one given by Robertson and Seymour [1]. In fact if (2) holds, there is an easy, greedy-type algorithm for finding the path \( P_i \), as we shall discuss below.

Let \( D \) be embedded in the plane \( \mathbb{R}^2 \). We identify \( D \) with its image in the plane. Without loss of generality we may assume \( I \) to be the unbounded face. (Each face is considered as an open region.) Moreover, we may assume that the boundary \( \text{bd}(I) \) of \( I \) is a simple closed
curve. This is no restriction, since we can extend \( D \) by new arcs, as long as we do not include them in any \( A_i \) and as long as we keep \( r_1, s_1, \ldots, r_k, s_k \) on \( \text{bd}(I) \).

We say that two pairs \((r, s)\) and \((r', s')\) of vertices on \( \text{bd}(I) \) cross if each \( r - s \) curve in \( \mathbb{R}^2 \setminus I \) intersects each \( r' - s' \) curve in \( \mathbb{R}^2 \setminus I \). Clearly, the following is a necessary condition for the existence of paths satisfying (1):

\[
(3) \quad \text{cross-freeness condition: if } \{i, j\} \in H \text{ then } (r_i, s_i) \text{ and } (r_j, s_j) \text{ do not cross.}
\]

Now the following algorithm finds paths as in (1) if (2) holds. First check if the cross-freeness condition is satisfied. If not, our problem has no solution. If the cross-freeness condition is satisfied, choose a pair \((r_i, s_i)\) so that the shortest of the two \( r_i - s_i \) paths along \( \text{bd}(I) \) is as short as possible (over all \( i = 1, \ldots, k \)). Without loss of generality, \( i = k \). Let \( Q \) be this shortest \( r_k - s_k \) path along \( \text{bd}(I) \). If \((V, A_k)\) does not contain any \( r_k - s_k \) path, then there are no paths satisfying (1). If \((V, A_k)\) does contain an \( r_k - s_k \) path, let \( P_k \) be the (unique) directed \( r_k - s_k \) path in \((V, A_k)\) which is nearest to \( Q \). Next repeat the algorithm for \( D, (r_1, s_1), \ldots, (r_{k-1}, s_{k-1}) \), removing from any \( A_i \) with \( \{i, k\} \in H \) all those arcs incident with some vertex in \( P_k \). After at most \( k \) iterations we either find paths as required, or we find that no such paths exist.

The correctness of the algorithm follows from the following observation. Suppose there exist paths \( Q_1, \ldots, Q_k \) as required. Then, if \( k \) is as above, we may assume without loss of generality that \( Q_k \) is equal to \( P_k \). Indeed, also \( Q_1, \ldots, Q_{k-1}, P_k \) form a solution, since if \( P_k \) intersects some \( Q_i \), then also \( Q_k \) intersects \( Q_i \).

We describe a second necessary condition. Let \( C \) be some curve in \( \mathbb{R}^2 \), starting in \( I \) and ending in some face \( F \). Let \( f(C) \) and \( l(C) \) denote the first and last point of intersection of \( C \) with \( D \). Let \( i_1, \ldots, i_n \) be indices from \( \{1, \ldots, k\} \) such that:

\[
(4) \quad \begin{align*}
(\text{i}) & \quad f(C), r_{i_1}, s_{i_1}, \ldots, r_{i_n}, s_{i_n} \text{ are all distinct;} \\
(\text{ii}) & \quad \text{the } r_{i_j} - s_{i_j} \text{ part of } \text{bd}(I) \text{ containing } f(C) \text{ is contained in the } r_{i_{j+1}} - s_{i_{j+1}} \text{ part of } \text{bd}(I) \text{ containing } f(C), \text{ for } j = 1, \ldots, n - 1; \\
(\text{iii}) & \quad \{i_j, i_{j+1}\} \in H \text{ for } j = 1, \ldots, n - 1.
\end{align*}
\]

For each \( j = 1, \ldots, n \) we define a set \( W_j \) as follows. If \( f(C), r_{i_j}, s_{i_j} \) occur clockwise around \( \text{bd}(I) \), \( W_j \) is the set of points \( p \) on \( D \) traversed by \( C \) such that some arc in \( A_{i_j} \) is entering \( C \) at \( p \) from the left and some arc in \( A_{i_j} \) is leaving \( C \) at \( p \) from the right. Similarly, if \( f(C), r_{i_j}, s_{i_j} \) occur anti-clockwise around \( \text{bd}(I) \), \( W_j \) is the set of points \( p \) of \( D \) traversed by \( C \) such that some arc in \( A_{i_j} \) is entering \( C \) at \( p \) from the right and some arc in \( A_{i_j} \) is leaving \( C \) at \( p \) from the left.

We say that \( C \) fits \( i_1, \ldots, i_n \) if there exist distinct points \( p_1, \ldots, p_n \) so that \( p_j \in W_j \) for \( j = 1, \ldots, n \) and so that \( C \) traverses \( p_1, \ldots, p_n \) in this order. Now we have the following condition:

\[
(5) \quad \text{cut condition: each curve } C \text{ starting and ending in } I \text{ fits each choice of } i_1, \ldots, i_n \text{ satisfying } (4), \text{ whenever } (f(C), l(C)) \text{ crosses each } (r_{i_j}, s_{i_j}) \text{ for } j = 1, \ldots, n.
\]
2. The theorem

We now prove:

**Theorem.** Let $D = (V, A)$ be a directed planar graph, embedded in the plane $\mathbb{R}^2$, let $(r_1, s_1), \ldots, (r_k, s_k)$ be pairs of vertices of $D$ on $\text{bd}(I)$, with $r_i \neq s_i$ for $i = 1, \ldots, n$, let $A_1, \ldots, A_k$ be subsets of $A$, and let $H$ be a set of unordered pairs from $\{1, \ldots, k\}$.

Then there exist paths $P_1, \ldots, P_k$ satisfying (1), if and only if the cross-freeness condition (3) and the cut condition (5) hold.

**Proof.** Necessity of the conditions is trivial. To see sufficiency, we assume without loss of generality that the arcs on $\text{bd}(I)$ do not belong to any $A_i$. (We can add new arcs to $D$ (but not to any $A_i$), without violating the cross-freeness and cut conditions.)

Choose an arbitrary point $p_0$ on $\text{bd}(I)$, not being a vertex of $D$. For each $i = 1, \ldots, k$, let $Q_i$ be that of the two $r_i - s_i$ parts of $\text{bd}(I)$ that does not contain $p_0$. For each $i = 1, \ldots, k$, let $F_i$ be the set of faces $F \neq I$ of $D$ for which there exists a curve $C$ starting in $I$ and ending in $F$, such that $f(C) \in Q_i$, and such that $C$ does not fit some choice of $i_1, \ldots, i_n$ satisfying (4) with $i_n = i$.

Note that, since no arc on $\text{bd}(I)$ belongs to $A_i$, each arc in $Q_i$ is on the boundary of $\bigcup F_i$. Let $B_i$ be the set of arcs on the boundary of $\bigcup F_i$ but not in $Q_i$. We show:

(6) $B_i$ is contained in $A_i$ and contains a directed $r_i - s_i$ path.

Assume without loss of generality that $r_1, p_0, s_1$ occur in this order clockwise around $\text{bd}(I)$. Let $a$ be an arc on the boundary of $\bigcup F_i$ and not in $Q_i$. We show that $a$ belongs to $A_i$ and that $a$ is oriented clockwise with respect to $\bigcup F_i$.

Let $a$ separate faces $F \in F_i$ and $F' \notin F_i$. By definition of $F_i$, there exists a curve $C$ starting in $I$ and ending in $F$, such that $f(C) \in Q_i$, and such that $C$ does not fit some choice $i_1, \ldots, i_n$ satisfying (4) with $i_n = i$. Now extend $C$ to $F'$ by crossing $a$, obtaining a curve $C'$.

If $C'$ does not fit $i_1, \ldots, i_n$, then $F' = I$ (as $F' \notin F_i$). Then, however, $C'$ violates the cut condition.

So $C'$ does fit $i_1, \ldots, i_n$. Since $C$ itself does not fit $i_1, \ldots, i_n$, this implies that $a$ belongs to $A_i$ and that $a$ is oriented clockwise with respect to $\bigcup F_i$. This proves (6).

Choose for each $i = 1, \ldots, k$ a directed $r_i - s_i$ path $P_i$ in $B_i$. We finally show that if $(i, j) \in H$ then $P_i$ and $P_j$ are vertex-disjoint. Assume without loss of generality that $i = 1, j = 2$, and let $(1, 2) \in H$. Suppose some vertex $v$ is traversed both by $P_1$ and by $P_2$. Hence $v$ is incident with some face $F_1$ in $F_1$ and with some face $F_2$ in $F_2$. It follows that there exists a curve $C$ from $I$ to $F_1$ such that $f(C) \in Q_1$ and such that $C$ does not fit indices $i_1, \ldots, i_n$ satisfying (4) with $i_n = 1$.

By the cross-freeness condition, we know that parts $Q_1$ and $Q_2$ of $\text{bd}(I)$ are either contained in each other or are disjoint.

First assume that they are contained in each other, say $Q_1 \subseteq Q_2$. Then each face $F' \neq I$ incident with $v$ is contained in $F_2$. To see this, we can extend curve $C$ via $v$ to $F'$, yielding curve $C'$. As $C$ does not fit $i_1, \ldots, i_n = 1$, it follows that $C'$ does not fit $i_1, \ldots, i_n = 1, i_{n+1} = 2$. So $F' \notin F_2$. As this holds for each face $F' \neq I$ incident with $v$, no arc incident with $v$ belongs to $B_2$, and hence $P_2$ does not traverse $v$. 

3
Next assume that $Q_1$ and $Q_2$ are disjoint. (So $p_0$ is inbetween of $Q_1$ and $Q_2$.) Since $F_2$ belongs to $\mathcal{F}_2$, there exists a curve $C'$ from $I$ to $F_2$ not fitting indices $i'_1, \ldots, i'_n$ satisfying (4) (adapted to $C', i'_1, \ldots, i'_n$), such that $f(C') \in Q_2$ and such that $i_{n'} = 2$.

Connect the curves $C$ and $C'$ by a $F_1 - F_2$ curve via $v$, yielding a curve $C''$ from $I$ to $I$. Then $C''$ does not fit $i_1, \ldots, i_n, i'_{n'}, \ldots, i'_1$, as one easily checks. This violates the cut condition.

The theorem can be seen to give a 'good characterization'.

3. Special cases

In this section we describe some special cases of the problem and the theorem.

First, let $G = (V, E)$ be an undirected planar graph, embedded in $\mathbb{R}^2$. Let $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ be pairs of vertices of $G$, each on the boundary of the unbounded face $I$ of $G$. Robertson and Seymour [1] proved that there exist pairwise vertex-disjoint paths $P_1, \ldots, P_k$ in $G$ where $P_i$ connects $r_i$ and $s_i$ for $i = 1, \ldots, k$, if and only if no two of the pairs $\{r_i, s_i\}$ cross and each vertex cut of $G$ contains at least as many vertices as it separates pairs from $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$.

This follows trivially from our theorem by replacing each arc by two opposite arcs, and taking for $H$ the collection of all pairs from $\{1, \ldots, k\}$.

The second special case generalizes the first. Let $G = (V, E)$ be an undirected planar graph, embedded in $\mathbb{R}^2$. Let $R_1, \ldots, R_t$ be pairwise disjoint sets of vertices of $G$, all on the boundary of the unbounded face $I$ of $G$.

We say that two sets $R$ and $R'$ of vertices on the boundary of $I$ cross if some pair of vertices in $R$ crosses some pair of vertices in $R'$. We say that a cut separates a set $R$ of vertices, if the cut separates $\{r, s\}$ for some $r, s$ in $R$.

Robertson and Seymour [1] proved more generally that there exist pairwise vertex-disjoint trees $T_1, \ldots, T_t$ in $G$ such that $T_i$ covers $R_i$ ($i = 1, \ldots, t$), if and only if no two of the $R_i$ cross and if each vertex cut of $G$ contains at least as many vertices as it separates sets from $R_1, \ldots, R_t$.

This follows from the theorem by replacing each edge of $G$ by two opposite edges, by taking as pairs $\{(r_1, s_1), \ldots, (r_k, s_k)\}$ all pairs $\{r, s\}$ for which there exists an $i \in \{1, \ldots, t\}$ such that $r, s \in R_i$, and by taking for $H$ all pairs $\{j, j'\}$ from $\{1, \ldots, k\}$ for which $r_j, s_{j''} r_{j''}$ and $s_{j''}$ not all belong to the same set among $R_1, \ldots, R_t$. (We take each $A_j$ to be equal to the full arc set.)

As a third special case, consider a planar directed graph $D = (V, A)$ and a collection of ordered pairs $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ on the boundary of the unbounded face $I$ (with $r_i \neq s_i$ for $i = 1, \ldots, k$). Then the theorem implies that there exist a directed $r_i - s_i$ path $P_i$, for $i = 1, \ldots, k$ so that $P_1, \ldots, P_k$ are pairwise vertex-disjoint, if and only if no two of the $(r_i, s_i)$ cross and for each cut $C$ not intersecting any of $r_1, s_1, \ldots, r_k, s_k$, the following cut condition holds:
if \( C \) separates \((r_{i_1}, s_{i_1}), \ldots, (r_{i_n}, s_{i_n})\), in this order, then \( C \) contains vertices \( p_1, \ldots, p_n \), in this order so that for each \( j = 1, \ldots, n \):

- if \( r_{i_j} \) is at the left hand side of \( C \) then at least one arc of \( D \) is entering \( C \) at \( p_j \) from the left and at least one arc of \( D \) is leaving \( C \) at \( p_j \) from the right;

- if \( r_{i_j} \) is at the right hand side of \( C \) then at least one arc of \( D \) is entering \( C \) at \( p_j \) from the right and at least one arc of \( D \) is leaving \( C \) at \( p_j \) from the left.

This follows by taking for \( H \) the set of all pairs from \( \{1, \ldots, k\} \) and taking each \( A_i \) equal to \( A \).

More generally, let \( D = (V, A) \) be a planar directed graph, let \( R_1, \ldots, R_k \) be sets of vertices on the boundary of the unbounded face \( I \) of \( D \), and let, for each \( i = 1, \ldots, k \), \( r_i \) be some vertex from \( R_i \). The theorem gives necessary and sufficient conditions for the existence of pairwise vertex-disjoint rooted trees \( T_1, \ldots, T_k \) in \( D \), where \( T_i \) has root \( r_i \) and covers \( R_i \) (\( i = 1, \ldots, k \)). Again this follows straightforwardly with reductions like the above.

Finally, let \( D = (V, A) \) be a planar directed graph and let \( R_1, \ldots, R_k \) be sets of vertices on the boundary of the unbounded face \( I \) of \( G \). Again it is straightforward to derive necessary and sufficient conditions for the existence of pairwise vertex-disjoint strongly connected subgraphs \( D_1, \ldots, D_k \) such that \( D_i \) covers \( R_i \) for \( i = 1, \ldots, k \).

Reference


April 11, 1990