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# Disjoint Paths in a Planar Graph - a General Theorem

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**Abstract.** Let $D = (V, A)$ be a directed planar graph, let $(r_1, s_1), \ldots, (r_k, s_k)$ be pairs of vertices on the boundary of the unbounded face, let $A_1, \ldots, A_k$ be subsets of $A$, and let $H$ be a collection of unordered pairs from $\{1, \ldots, k\}$. We give necessary and sufficient conditions for the existence of a directed $r_i - s_i$ path $P_i$ in $(V, A_i)$ (for $i = 1, \ldots, k$), such that $P_i$ and $P_j$ are vertex-disjoint whenever $\{i, j\} \in H$.

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## 1. Introduction

Let $D = (V, A)$ be a directed graph, let $(r_1, s_1), \ldots, (r_k, s_k)$ be pairs of vertices of $D$, let $A_1, \ldots, A_k$ be subsets of $A$, and let $H$ be a collection of unordered pairs from $\{1, \ldots, k\}$. We are interested under which conditions there exist directed paths $P_1, \ldots, P_k$ so that:

(1) (i) $P_i$ is a directed $r_i - s_i$ path in $(V, A_i)$ $(i = 1, \ldots, k)$;

(ii) $P_i$ and $P_j$ are vertex-disjoint for each $\{i, j\} \in H$.

In Section 3 we will discuss some special cases of this problem.

Since the problem is NP-complete, we may not expect a nice set of necessary and sufficient conditions characterizing the existence of paths satisfying (1). The problem is NP-complete even if we restrict the problem to instances with $D$ planar, $k = 2, A_1 = A_2 = A$ and $H = \{\{1, 2\}\}$. Moreover, it is NP-complete when restricted to $A_1 = \ldots = A_k = A$, $H$ is the collection of all pairs from $\{1, \ldots, k\}$, and $D$ arises from an undirected planar graph by replacing each edge by two opposite arcs.

In this paper we give necessary and sufficient conditions for the problem when

(2) $D$ is planar and the vertices $r_1, s_1, \ldots, r_k, s_k$ all belong to the boundary of one fixed face $I$.

The characterization extends one given by Robertson and Seymour [1]. In fact if (2) holds, there is an easy, greedy-type algorithm for finding the path $P_i$, as we shall discuss below.

Let $D$ be embedded in the plane $\mathbb{R}^2$. We identify $D$ with its image in the plane. Without loss of generality we may assume $I$ to be the unbounded face. (Each face is considered as an *open* region.) Moreover, we may assume that the boundary bd($I$) of $I$ is a simple closed curve. This is no restriction, since we can extend $D$ by new arcs, as long as we do not include them in any $A_i$ and as long as we keep $r_1, s_1, \ldots, r_k, s_k$ on bd$(I)$.

We say that two pairs $(r, s)$ and $(r', s')$ of vertices on bd$(I)$ *cross* if each $r - s$ curve in $\mathbb{R}^2 \setminus I$ intersects each $r' - s'$ curve in $\mathbb{R}^2 \setminus I$. Clearly, the following is a necessary condition for the existence of paths satisfying (1):

(3) *cross-freeness condition*: if $\{i, j\} \in H$ then $(r_i, s_i)$ and $(r_j, s_j)$ do not cross.

Now the following algorithm finds paths as in (1) if (2) holds. First check if the cross-freeness condition is satisfied. If not, our problem has no solution. If the cross-freeness condition is satisfied, choose a pair $(r_i, s_i)$ so that the shortest of the two $r_i - s_i$ paths along bd$(I)$ is as short as possible (over all $i = 1, \ldots, k$). Without loss of generality, $i = k$. Let $Q$ be this shortest $r_k - s_k$ path along bd$(I)$. If $(V, A_k)$ does not contain any $r_k - s_k$ path, then there are no paths satisfying (1). If $(V, A_k)$ does contain an $r_k - s_k$ path, let $P_k$ be the (unique) directed $r_k - s_k$ path in $(V, A_k)$ which is nearest to $Q$. Next repeat the algorithm for $D, (r_1, s_1), \ldots, (r_{k-1}, s_{k-1})$, removing from any $A_i$ with $\{i, k\} \in H$ all those arcs incident with some vertex in $P_k$. After at most $k$ iterations we either find paths as required, or we find that no such paths exist.

The correctness of the algorithm follows from the following observation. Suppose there exist paths $Q_1, \ldots, Q_k$ as required. Then, if $k$ is as above, we may assume without loss of generality that $Q_k$ is equal to $P_k$. Indeed, also $Q_1, \ldots, Q_{k-1}, P_k$ form a solution, since if $P_k$ intersects some $Q_i$, then also $Q_k$ intersects $Q_i$.

We describe a second necessary condition. Let $C$ be some curve in $\mathbb{R}^2$, starting in $I$ and ending in some face $F$. let $f(C)$ and $l(C)$ denote the first and last point of intersection of $C$ with $D$. Let $i_1, \ldots, i_n$ be indices from $\{1, \ldots, k\}$ such that:

(4) (i) $f(C), r_{i_1}, s_{i_1}, \ldots, r_{i_n}, s_{i_n}$ are all distinct;

(ii) the $r_{i_j} - s_{i_j}$ part of bd$(I)$ containing $f(C)$ is contained in the $r_{i_{j+1}} - s_{i_{j+1}}$ part of bd$(I)$ containing $f(C)$, for $j = 1, \ldots, n-1$;

(iii) $\{i_j, i_{j+1}\} \in H$ for $j = 1, \ldots, n-1$.

For each $j = 1, \ldots, n$ we define a set $W_j$ as follows. If $f(C), r_{i_j}, s_{i_j}$ occur clockwise around bd$(I)$, $W_j$ is the set of points $p$ on $D$ traversed by $C$ such that some arc in $A_{i_j}$ is entering $C$ at $p$ from the left and some arc in $A_{i_j}$ is leaving $C$ at $p$ from the right. Similarly, if $f(C), r_{i_j}, s_{i_j}$ occur anti-clockwise around bd$(I)$, $W_j$ is the set on points $p$ of $D$ traversed by $C$ such that some arc in $A_{i_j}$ is entering $C$ at $p$ from the right and some arc in $A_{i_j}$ is leaving $C$ at $p$ from the left.

We say that $C$ *fits* $i_1, \ldots, i_n$ if there exist distinct points $p_1, \ldots, p_n$ so that $p_j \in W_j$ for $j = 1, \ldots, n$ and so that $C$ traverses $p_1, \ldots, p_n$ in this order. Now we have the following condition:

(5) *cut condition*: each curve $C$ starting and ending in $I$ fits each choice of $i_1, \ldots, i_n$ satisfying (4), whenever $(f(C), l(C))$ crosses each $(r_{i_j}, s_{i_j})$ $(j = 1, \ldots, n)$.

## 2. The theorem

We now prove:

**Theorem.** *Let $D = (V, A)$ be a directed planar graph, embedded in the plane $\mathbb{R}^2$, let $(r_1, s_1), \ldots, (r_k, s_k)$ be pairs of vertices of $D$ on bd($I$), with $r_i \neq s_i$ for $i = 1, \ldots, n$, let $A_1, \ldots, A_k$ be subsets of $A$, and let $H$ be a set of unordered pairs from $\{1, \ldots, k\}$.*

*Then there exist paths $P_1, \ldots, P_k$ satisfying* (1), *if and only if the cross-freeness condition* (3) *and the cut condition* (5) *hold.*

**Proof.** Necessity of the conditions is trivial. To see sufficiency, we assume without loss of generality that the arcs on bd($I$) do not belong to any $A_i$. (We can add new arcs to $D$ (but not to any $A_i$), without violating the cross-freeness and cut conditions.)

Choose an arbitrary point $p_0$ on bd($I$), not being a vertex of $D$. For each $i = 1, \ldots, k$, let $Q_i$ be that of the two $r_i - s_i$ parts of bd($I$) that does not contain $p_0$. For each $i = 1, \ldots, k$, let $\mathcal{F}_i$ be the set of faces $F \neq I$ of $D$ for which there exists a curve $C$ starting in $I$ and ending in $F$, such that $f(C) \in Q_i$ and such that $C$ does not fit some choice of $i_1, \ldots, i_n$ satisfying (4) with $i_n = i$.

Note that, since no arc on bd($I$) belongs to $A_i$, each arc in $Q_i$ is on the boundary of $\bigcup \mathcal{F}_i$. Let $B_i$ be the set of arcs on the boundary of $\bigcup \mathcal{F}_i$ but not in $Q_i$. We show:

(6) $\qquad$ $B_i$ is contained in $A_i$ and contains a directed $r_i - s_i$ path.

Assume without loss of generality that $r_i, p_0, s_i$ occur in this order clockwise around bd($I$). Let $a$ be an arc on the boundary of $\bigcup \mathcal{F}_i$ and not in $Q_i$. We show that $a$ belongs to $A_i$ and that $a$ is oriented clockwise with respect to $\bigcup \mathcal{F}_i$.

Let $a$ separate faces $F \in \mathcal{F}_i$ and $F' \notin \mathcal{F}_i$. By definition of $\mathcal{F}_i$, there exists a curve $C$ starting in $I$ and ending in $F$, such that $f(C) \in Q_i$ and such that $C$ does not fit some choice $i_1, \ldots, i_n$ satisfying (4) with $i_n = i$. Now extend $C$ to $F'$ by crossing $a$, obtaining a curve $C'$.

If $C'$ does not fit $i_1, \ldots, i_n$, then $F' = I$ (as $F' \notin \mathcal{F}_i$). Then, however, $C'$ violates the cut condition.

So $C'$ does fit $i_1, \ldots, i_n$. Since $C$ itself does not fit $i_1, \ldots, i_n$, this implies that $a$ belongs to $A_i$ and that $a$ is oriented clockwise with respect to $\bigcup \mathcal{F}_i$. This proves (6).

Choose for each $i = 1, \ldots, k$ a directed $r_i - s_i$ path $P_i$ in $B_i$. We finally show that if $\{i, j\} \in H$ then $P_i$ and $P_j$ are vertex-disjoint. Assume without loss of generality that $i = 1, j = 2$, and let $\{1, 2\} \in H$. Suppose some vertex $v$ is traversed both by $P_1$ and by $P_2$. Hence $v$ is incident with some face $F_1$ in $\mathcal{F}_1$ and with some face $F_2$ in $\mathcal{F}_2$. It follows that there exist a curve $C$ from $I$ to $F_1$ such that $f(C) \in Q_i$ and such that $C$ does not fit indices $i_1, \ldots, i_n$ satisfying (4) with $i_n = 1$.

By the cross-freeness condition, we know that parts $Q_1$ and $Q_2$ of bd($I$) are either contained in each other or are disjoint.

First assume that they are contained in each other, say $Q_1 \subseteq Q_2$. Then each face $F' \neq I$ incident with $v$ is contained in $\mathcal{F}_2$. To see this, we can extend curve $C$ via $v$ to $F'$, yielding curve $C'$. As $C$ does not fit $i_1, \ldots, i_n = 1$, it follows that $C'$ does not fit $i_1, \ldots, i_n = 1, i_{n+1} = 2$. So $F' \in \mathcal{F}_2$. As this holds for each face $F' \neq I$ incident with $v$, no arc incident with $v$ belongs to $B_2$, and hence $P_2$ does not traverse $v$.

Next assume that $Q_1$ and $Q_2$ are disjoint. (So $p_0$ is inbetween of $Q_1$ and $Q_2$.) Since $F_2$ belongs to $\mathcal{F}_2$, there exists a curve $C'$ from $I$ to $F_2$ not fitting indices $i'_1, \ldots, i'_{n'}$ satisfying (4) (adapted to $C', i'_1, \ldots, i'_{n'}$), such that $f(C') \in Q_2$ and such that $i'_{n'} = 2$.

Connect the curves $C$ and $C'$ by a $F_1 - F_2$ curve via $v$, yielding a curve $C''$ from $I$ to $I$. Then $C''$ does not fit $i_1, \ldots, i_n, i'_{n'}, \ldots, i'_1$, as one easily checks. This violates the cut condition. ∎

The theorem can be seen to give a 'good characterization'.

## 3. Special cases

In this section we describe some special cases of the problem and the theorem.

First, let $G = (V, E)$ be an undirected planar graph, embedded in $\mathbb{R}^2$. Let $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ be pairs of vertices of $G$, each on the boundary of the unbounded face $I$ of $G$. Robertson and Seymour [1] proved that there exist pairwise vertex-disjoint paths $P_1, \ldots, P_k$ in $G$ where $P_i$ connects $r_i$ and $s_i$ for $i = 1, \ldots, k$, if and only if no two of the pairs $\{r_i, s_i\}$ cross and each vertex cut of $G$ contains at least as many vertices as it separates pairs from $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$.

This follows trivially from our theorem by replacing each arc by two opposite arcs, and taking for $H$ the collection of all pairs from $\{1, \ldots, k\}$.

The second special case generalizes the first. Let $G = (V, E)$ be an undirected planar graph, embedded in $\mathbb{R}^2$. Let $R_1, \ldots, R_t$ be pairwise disjoint sets of vertices of $G$, all on the boundary of the unbounded face $I$ of $G$.

We say that two sets $R$ and $R'$ of vertices on the boundary of $I$ *cross* if some pair of vertices in $R$ crosses some pair of vertices in $R'$. We say that a cut *separates* a set $R$ of vertices, if the cut separates $\{r, s\}$ for some $r, s$ in $R$.

Robertson and Seymour [1] proved more generally that there exist pairwise vertex-disjoint trees $T_1, \ldots, T_t$ in $G$ such that $T_i$ covers $R_i$ ($i = 1, \ldots, t$), if and only if no two of the $R_i$ cross and if each vertex cut of $G$ contains at least as many vertices as it separates sets from $R_1, \ldots, R_t$.

This follows from the theorem by replacing each edge of $G$ by two opposite edges, by taking as pairs $(r_1, s_1), \ldots, (r_k, s_k)$ all pairs $(r, s)$ for which there exists an $i \in \{1, \ldots, t\}$ such that $r, s \in R_i$, and by taking for $H$ all pairs $\{j, j'\}$ from $\{1, \ldots, k\}$ for which $r_j, s_j, r_{j'}$ and $s_{j'}$ not all belong to the same set among $R_1, \ldots, R_t$. (We take each $A_j$ to be equal to the full arc set.)

As a third special case, consider a planar directed graph $D = (V, A)$ and a collection of ordered pairs $(r_1, s_1), \ldots, (r_k, s_k)$ on the boundary of the unbounded face $I$ (with $r_i \neq s_i$ for $i = 1, \ldots, k$). Then the theorem implies that there exist a directed $r_i - s_i$ path $P_i$, for $i = 1, \ldots, k$ so that $P_1, \ldots, P_k$ are pairwise vertex-disjoint, if and only if no two of the $(r_i, s_i)$ cross and for each cut $C$ not intersecting any of $r_1, s_1, \ldots, r_k, s_k$, the following *cut condition* holds:

(7) if $C$ separates $(r_{i_1}, s_{i_1}), \ldots, (r_{i_n}, s_{i_n})$, in this order, then $C$ contains vertices $p_1, \ldots, p_n$, in this order so that for each $j = 1, \ldots, n$:

- if $r_{i_j}$ is at the left hand side of $C$ then at least one arc of $D$ is entering $C$ at $p_j$ from the left and at least one arc of $D$ is leaving $C$ at $p_j$ from the right;
- if $r_{i_j}$ is at the right hand side of $C$ then at least one arc of $D$ is entering $C$ at $p_j$ from the right and at least one arc of $D$ is leaving $C$ at $p_j$ from the left.

This follows by taking for $H$ the set of all pairs from $\{1, \ldots, k\}$ and taking each $A_i$ equal to $A$.

More generally, let $D = (V, A)$ be a planar directed graph, let $R_1, \ldots, R_t$ be sets of vertices on the boundary of the unbounded face $I$ of $D$, and let, for each $i = 1, \ldots, k$, $r_i$ be some vertex from $R_i$. The theorem gives necessary and sufficient conditions for the existence of pairwise vertex-disjoint rooted trees $T_1, \ldots, T_k$ in $D$, where $T_i$ has root $r_i$ and covers $R_i$ $(i = 1, \ldots, k)$. Again this follows straightforwardly with reductions like the above.

Finally, let $D = (V, A)$ be a planar directed graph and let $R_1, \ldots, R_k$ be sets of vertices on the boundary of the unbounded face $I$ of $G$. Again it is straightforward to derive necessary and sufficient conditions for the existence of pairwise vertex-disjoint strongly connected subgraphs $D_1, \ldots, D_k$ such that $D_i$ covers $R_i$ for $(i = 1, \ldots, k)$.

## Reference

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