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Disjoint Cycles in Directed Graphs on the Torus and the Klein Bottle

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Abstract. We give necessary and sufficient conditions for a directed graph embedded on the torus or the Klein bottle to contain pairwise disjoint circuits, each of a given orientation and homotopy, and in a given order. For the Klein bottle, the theorem is new. For the torus, the theorem was proved before by P.D. Seymour. This paper gives a shorter proof of that result.

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1. Introduction.

Let S be the torus or the Klein bottle. We call a function $\phi : S \longrightarrow S$ a *shift* if there exists a continuous function $\Phi : S \times [0,1] \longrightarrow S$ such that

(1) (i) $\Phi(x,0) = x$, $\Phi(x,1) = \phi(x)$, for all $x \in S$,

(ii) $\Phi(\cdot,t)$ is a homeomorphism on S, for each $t \in [0,1]$.

Let G be a directed graph embedded on S (without crossings). Let $C_1,\ldots,C_k$ be pairwise disjoint simple closed curves on S. We characterize when there exists a shift of S bringing each C_i to a directed cycle in G (with the same orientation as C_i), under the assumption that $S \setminus C_1$ is a cylinder. (This is automatically the case if S is the torus.)

For the torus, this characterization was given in [2]. In this paper, we give a shorter proof, while for the Klein bottle the result is new. For general compact surfaces a characterization is given in [1], except for the cases considered in the present paper.

In studying this problem, we assume without loss of generality that $C_1,\ldots,C_k$ occur in this order around S. That is, we assume that there exists a closed curve D_0 crossing each of $C_1,\ldots,C_k$ exactly once, and in this order. If S is the torus, each curve D gives a natural interpretation of 'left' and 'right' with respect to D. If S is the Klein bottle, we choose for each curve D some interpretation of 'left' and 'right', arbitrarily but fixed when going along D from its beginning point to its end point. Define a sequence

(2) $(\alpha_1,\ldots,\alpha_k)$

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by: $\alpha_i = +1$ if C_i crosses D_0 from left to right, and $\alpha_i = -1$ if C_i crosses D_0 from right to left.

Let D be any curve on S, with end points in faces of G. We assume here and below that any such curve has only a finite number of intersections with G. Moreover, we assume that each intersection with G is in a vertex. (We can add a vertex at each intersection.)

We say that a crossing of D with any C_i is *positive* if it is a crossing in the same direction as D_0, and *negative* otherwise. If D has π positive crossings with C_1 and ν negative crossings with C_1, then the *winding number* $w(D)$ of D is equal to $\pi - \nu$.

Let D traverse vertices $v_1, \ldots, v_m$ of G, in this order (repetition allowed). We associate with D a sequence

$$(3) \qquad i_G(D) = (X_1, \ldots, X_m),$$

where each X_j is a subset of $\{+1, -1\}$. Set X_j is defined as follows. Consider the segment of D when traversing v_j, going from face F to face F', say, of G. Let $e_1, \ldots, e_d$ be the edges incident with v_j, choosing indices in such a way that $F, e_1, \ldots, e_t, F', e_{t+1}, \ldots, e_d$ occur in this order clockwise at v_j, for some t. Then $+1 \in X_j$ if and only if at least one of $e_1, \ldots, e_t$ is directed towards v_j and at least one of $e_{t+1}, \ldots, e_d$ is directed away from v_j. Similarly, $-1 \in X_j$ if and only if at least one of $e_1, \ldots, e_t$ is directed away from v_j and at least one of $e_{t+1}, \ldots, e_d$ is directed towards v_j.

For any finite sequence x and any integer $w > 0$ we define x^w as the concatenation of w copies of x. If $x = (\xi_1, \ldots, \xi_s)$ and $y = (\eta_1, \ldots, \eta_t)$, then we let $x \prec y$ if there exist indices $1 \leq j_1 < j_2 < \ldots < j_s \leq t$ such that $\xi_i \in \eta_{j_i}$ for $i = 1, \ldots, s$. Moreover, $x \ll y$ if $x' \prec y$ for some cyclic permutation x' of x.

2. The torus.

We now first consider the torus.

Theorem 1. *Let S be the torus. Then there exists a shift of S bringing $C_1, \ldots, C_k$ to directed cycles in G, if and only if for each closed curve D of positive winding number one has:*

$$(4) \qquad (\alpha_1, \ldots, \alpha_k)^{w(D)} \ll i_G(D).$$

Proof. Necessity of the condition is trivial. Suppose now that the condition is satisfied. We may assume that each face of G is an open disk. (In any face F not being an open disk, we can put a new vertex v, with arcs from v to each vertex incident with F.)

We consider the torus as being the quotient space of $\mathbb{C} \setminus \{0\}$ by identifying any $y, z \in \mathbb{C}$ if $z = 2^u y$ for some integer u. Let $\pi : \mathbb{C} \setminus \{0\} \longrightarrow S$ be the quotient map. We make this construction in such a way that each lifting of each C_i to $\mathbb{C} \setminus \{0\}$ is a closed curve enclosing 0. More precisely, there exist closed curves Γ_i ($i \in \mathbb{Z}$) so that $\pi \circ \Gamma_i = C_i$ for each $i \in \mathbb{Z}$, taking indices of C_i mod k. We can take the indices in such a way that Γ_{i+1} encloses Γ_i, and such that $\Gamma_{i+k} = 2\Gamma_i$ for each integer i. Moreover, we assume that Γ_i has clockwise orientation if $\alpha_i = +1$ and anti-clockwise orientation if $\alpha_i = -1$ (taking indices of α_i mod k).

The inverse image $H := \pi^{-1}[G]$ of G is an infinite graph embedded in $\mathbb{C} \setminus \{0\}$. For any curve P on $\mathbb{C} \setminus \{0\}$ we denote $i_H(P) := i_G(\pi \circ P)$. (So $i_H(P)$ can be defined similarly as we defined $i_G(P)$ above.)

Now for each integer i, let $\mathcal{R}_i$ be the set of faces F of H so that there exists an integer $t \leq i$ and a curve P starting in a face enclosed by Γ_t and ending in F, such that

$$(5) \qquad (\alpha_t, \alpha_{t+1}, \ldots, \alpha_i) \not\prec i_H(P).$$

We show

Claim. *$\bigcup \mathcal{R}_i$ is bounded, for each integer i.*

Proof. We may assume that in the definition of $\mathcal{R}_i$ we can restrict P to curves traversing at most kf faces of H, where f denotes the number of faces of G.

Let P be a curve starting in a face enclosed by Γ_t and ending in F, satisfying (5). Suppose P traverses more than kf faces. We show that there exists a $t' \leq i$ and a curve P' starting in a face enclosed by $\Gamma_{t'}$ and ending in F such that

$$(6) \qquad (\alpha_{t'}, \alpha_{t'+1}, \ldots, \alpha_i) \not\prec i_H(P').$$

and such that P' traverses fewer faces of H than P does.

Since P traverses more than kf faces of H, there exists a face F' of G so that $\pi \circ P$ traverses F' more than k times. So P can be decomposed as $P = P_0 \cdot P_1 \cdot P_2 \cdot \ldots \cdot P_k \cdot P_{k+1}$, where for each $j = 1, \ldots, k$, $\pi \circ P_j$ is a curve with end points in F', intersecting G at least once. Without loss of generality, each such $\pi \circ P_j$ is a closed curve.

For $j = 0, \ldots, k$, let h_j be the smallest integer h for which

$$(7) \qquad (\alpha_t, \alpha_{t+1}, \ldots, \alpha_h) \not\prec i_H(P_0 \cdot P_1 \cdot \ldots \cdot P_j).$$

Then there exist j', j'' so that $0 \leq j' < j'' \leq k$ and so that $h_{j'} \equiv h_{j''} \pmod{k}$. Let $h' := h_{j'}$ and $h'' := h_{j''}$.

Since $\pi \circ (P_{j'+1} \cdot \ldots \cdot P_{j''})$ is a closed curve on S, there exists a $z \in \mathbb{C} \setminus \{0\}$ and a $u \in \mathbb{Z}$ so that $P_{j'+1} \cdot \ldots \cdot P_{j''}$ goes from z to $2^u z$.

Suppose $ku > h'' - h'$. Since the closed curve $\pi \circ (P_{j'+1} \cdot \ldots \cdot P_{j''})$ has winding number u, we know

$$(8) \qquad (\alpha_1, \ldots, \alpha_{ku}) \ll i_H(P_{j'+1} \cdot \ldots \cdot P_{j''}).$$

Hence

$$(9) \qquad (\alpha_{h'}, \alpha_{h'+1}, \ldots, \alpha_{h''}) \prec i_H(P_{j'+1} \cdot \ldots \cdot P_{j''}),$$

since $h' \equiv h'' \pmod{k}$. Since $(\alpha_t, \ldots, \alpha_{h'-1}) \prec i_H(P_0 \cdot \ldots \cdot P_{j'})$ (by definition of $h' = h_{j'}$), (9) implies $(\alpha_t, \ldots, \alpha_{h''}) \prec i_H(P_0 \cdot \ldots \cdot P_{j''})$. This contradicts the definition of $h'' = h_{j''}$.

So $ku \leq h'' - h'$. Consider the curve

$$(10) \qquad P' := (2^u(P_0 \cdot \ldots \cdot P_{j'})) \cdot P_{j''+1} \cdot \ldots \cdot P_{k+1}.$$

Let $t' := t + ku$. Then $t' = t + ku \leq t + h'' - h' \leq i$ (since $t \leq h'$ and $h'' \leq i$). Now

$$(11) \qquad (\alpha_{t'}, \alpha_{t'+1}, \ldots, \alpha_{h''}) \not\prec i_H(2^u(P_0 \cdot \ldots \cdot P_{j'}))$$

(as $(\alpha_{t'}, \alpha_{t'+1}, \ldots, \alpha_{h'+ku}) = (\alpha_t, \alpha_{t+1}, \ldots, \alpha_{h'}) \not\prec i_H(P_0 \cdot \ldots \cdot P_{j'}) = i_H(2^u(P_0 \cdot \ldots \cdot P_{j'}))$, by definition of $h' = h_{j'}$, and as $h' + ku \leq h''$). Moreover,

$$(12) \qquad (\alpha_{h''}, \alpha_{h''+1}, \ldots, \alpha_i) \not\prec i_H(P_{j''+1} \cdot \ldots \cdot P_{k+1})$$

(since otherwise $(\alpha_t, \ldots, \alpha_i) \prec i_H(P)$, as $(\alpha_t, \ldots, \alpha_{h''-1}) \prec i_H(P_0 \cdot \ldots \cdot P_{j''})$, by definition of $h'' = h_{j''}$).

(11) and (12) directly imply (6).

End of proof of the Claim.

Clearly, each face F enclosed by Γ_i belongs to $\mathcal{R}_i$ (since we can take $t = i$ and for P any curve remaining in F). Moreover, $\mathcal{R}_{i+k}$ can be obtained from $\mathcal{R}_i$ by multiplying the faces in $\mathcal{R}_i$ by 2.

The faces in $\mathcal{R}_i$ induce a connected subgraph of the dual graph of H, as one easily checks. Hence the arcs on the boundary of the unbounded connected component of $\mathbb{C} \setminus \overline{\bigcup \mathcal{R}_i}$ form a simple closed curve; call it Δ_i.

Then Δ_i is oriented clockwise if $\alpha_i = +1$, and anti-clockwise if $\alpha_i = -1$. This follows from the fact that any arc a of H on the boundary of $\overline{\bigcup \mathcal{R}_i}$ is oriented clockwise if $\alpha_i = +1$, and anti-clockwise if $\alpha_i = -1$ (clockwise and anti-clockwise with respect to $\bigcup \overline{\mathcal{R}_i}$). To see this, let a be incident with faces $F \in \mathcal{R}_i$ and $F' \notin \mathcal{R}_i$. By definition of $\mathcal{R}_i$, there exists a $t \leq i$ and a curve P starting in a face enclosed by Γ_t and ending in F, satisfying (5). We can extend P to a curve P' ending in F', by crossing a. Since $F' \notin \mathcal{R}_i$, $(\alpha_t, \ldots, \alpha_i) \prec i_H(P')$. Hence α_i must belong to the last set occurring in $i_H(P')$, giving the required statement.

Moreover, for each integer i, Δ_i is enclosed by Δ_{i+1}, without intersections. This follows from the fact that if F belongs to $\mathcal{R}_i$, then each face F' having a vertex in common with F belongs to $\mathcal{R}_{i+1}$. Indeed, by definition of $\mathcal{R}_i$, there exists a $t \leq i$ and a curve P starting in a face enclosed by Γ_t and ending in F, satisfying (5). We can extend P to a curve P' ending in F', by traversing a vertex incident with both F and F'. From (5) one derives $(\alpha_t, \ldots, \alpha_{i+1}) \not\prec i_H(P')$. Hence $F' \in \mathcal{R}_{i+1}$.

Since also $\Delta_{i+k} = 2\Delta_i$ for each i, it follows that $\pi \circ \Delta_1, \ldots, \pi \circ \Delta_k$ give disjoint closed curves on the torus S, of the same orientations as $C_1, \ldots, C_k$, respectively, and in the same order as $C_1, \ldots, C_k$. Shifting $C_1, \ldots, C_k$ to $\pi \circ \Delta_1, \ldots, \pi \circ \Delta_k$ gives the required shift. ∎

3. The Klein bottle.

We next consider the Klein bottle. Define $\alpha_i := -\alpha_{i-k}$ for $i = k+1, \ldots, 2k$.

Theorem 2. *Let S be the Klein bottle, such that $S \setminus C_1$ is a cylinder. Then there exists a shift of S bringing $C_1, \ldots, C_k$ to directed cycles in G, if and only if for each orientation-preserving closed curve D of positive winding number one has:*

$$(13) \qquad (\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{2k})^{w(D)/2} \ll i_G(D).$$

Proof. The proof is similar to that of Theorem 1. We now consider the Klein bottle as being the quotient space of $\mathbb{C} \setminus \{0\}$ by identifying any $y, z \in \mathbb{C}$ if $z = 2^u y$ for some even integer u or $z = 2^u \overline{y}$ for some odd integer u. Again, let $\pi : \mathbb{C} \setminus \{0\} \longrightarrow S$ be the quotient map, in such a way that there exist closed curves Γ_i ($i \in \mathbb{Z}$) so that $\pi \circ \Gamma_i = C_i$ for each $i \in \mathbb{Z}$, taking indices of C_i mod k. We can take the indices in such a way that Γ_{i+1} encloses Γ_i, and such that $\Gamma_{i+2k} = 2\Gamma_i$ for each integer i. Moreover, we assume that Γ_i has clockwise orientation if $\alpha_i = +1$ and anti-clockwise orientation if $\alpha = -1$, now taking indices of α_i mod $2k$.

Also the remainder of the proof is similar to that of Theorem 1. ∎

References.

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