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Edge-Disjoint Circuits in Graphs on the Torus

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Abstract. We give necessary and sufficient conditions for a given graph embedded on the torus, to contain edge-disjoint cycles of prescribed homotopies (under the assumption of a 'parity' condition).

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1. Introduction

We prove a theorem on edge-disjoint cycles of prescribed homotopies in an undirected graph embedded on the torus. It forms a sharpening (integer version) of a theorem proved in [1] for general compact orientable surfaces.

Let $G = (V, E)$ be an undirected graph embedded on the torus T, and let $C_1, \ldots, C_k$ be closed curves on T. We are interested in conditions under which:

(1) there exist pairwise edge-disjoint cycles $\tilde{C}_1, \ldots, \tilde{C}_k$ in G so that $\tilde{C}_i$ is freely homotopic to C_i, for $i = 1, \ldots, k$.

Here by embedding we mean: without intersecting edges. We will identify an embedded graph with its image in T. A *closed curve* on T is a continuous function $C : S_1 \longrightarrow T$, where S_1 is the unit circle in the complex plane.

A *cycle* in G is a sequence $(v_0, e_1, v_1, \ldots, e_d, v_d)$ so that e_i is an edge connecting v_{i-1} and v_i $(i = 1, \ldots, d)$, with $v_0 = v_d$. In a natural way we can identify such a cycle in G with a closed curve on T. We call a collection of cycles *pairwise edge-disjoint* if no two cycles have an edge in common, and moreover, no cycle traverses the same edge more than once.

Two closed curves C and $\tilde{C}$ on T are called *freely homotopic*, in notation: $C \sim \tilde{C}$, if there exists a continuous function $\Phi : S_1 \times [0,1] \longrightarrow T$ so that $\Phi(z, 0) = C(z)$ and $\Phi(z, 1) = \tilde{C}(z)$ for each $z \in S_1$. (So there is no point fixed.)

A necessary for (1) is the following *cut condition*: for each closed curve D on T, intersecting G only a finite number of times and not intersecting V, one has:

$$\text{(2)} \qquad \operatorname{cr}(G, D) \geq \sum_{i=1}^{k} \operatorname{mincr}(C_i, D).$$

Here we use the notation (for closed curves C and D):

$$\text{(3)} \qquad \begin{aligned} \operatorname{cr}(G, D) &:= |\{z \in S_1 \mid D(z) \in G\}|, \\ \operatorname{cr}(C, D) &:= |\{(y, z) \in S_1 \times S_1 \mid C(y) = D(z)\}|, \\ \operatorname{mincr}(C, D) &:= \min\{\operatorname{cr}(\tilde{C}, \tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D\}. \end{aligned}$$

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Condition (2) is not sufficient for (1), as is shown by the following example, where the wriggled lines indicate closed curves C_1 and C_2 and where the torus arises by identifying the two segments α and identifying the two segments β:

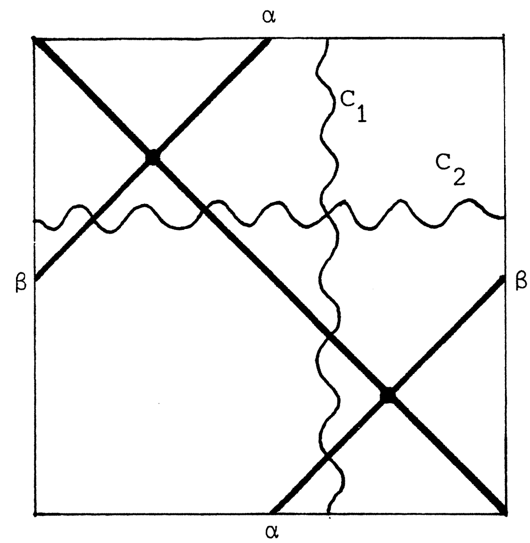

Figure 1

A second example arises by taking for G a graph consisting of two vertices, each attached with a loop (pairwise disjoint and nonullhomotopic), and for C_1 a closed curve going twice around one of the loops:

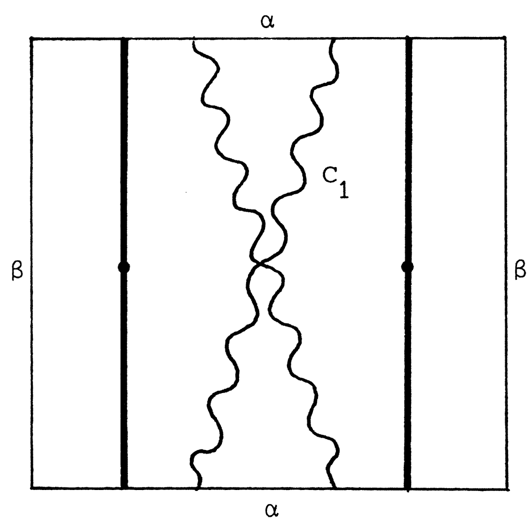

Figure 2

We show that (2) is sufficient for (1) if each C_i is *simple* (i.e., is a one-to-one function), and the following *parity condition* holds:

(4) (parity condition) for each closed curve D on T, not intersecting vertices of G, the number of crossings of D with edges of G, plus the number of crossings with $C_1, \ldots, C_k$, is an even number.

One easily checks that the parity condition implies that each vertex of G has even degree.

Theorem. *Let $G = (V, E)$ be a graph embedded on the torus T, and let $C_1, \ldots, C_k$ be simple closed curves on T, such that the parity condition holds. Then there exist pairwise edge-disjoint closed cycles $\tilde{C}_1, \ldots, \tilde{C}_k$ in G so that $\tilde{C}_i \sim C_i$ ($i = 1, \ldots, k$), if and only if the cut condition holds.*

(We do not require the $\tilde{C}_i$ to be simple – they may have self-intersections at vertices of G.)

Figures 1 and 2 show that we cannot delete the parity or the simple-ness condition. For general compact compact orientable surfaces the cut condition only implies the existence of a 'fractional' solution to (1).

2. Closed curves on the torus and their crossings

Before proving the theorem (in Section 3), we show an inequality for the function $\text{mincr}(C, D)$ defined in (3). This inequality is essential in our proof, and does not hold for compact orientable surfaces other than the sphere and the torus.

Let $D_1, D_2 : S_1 \longrightarrow T$ be closed curves on T with $D_1(1) = D_2(1)$. Let $D_1 \cdot D_2$ denote the concatenation of D_1 and D_2. That is, $D_1 \cdot D_2 : S_1 \longrightarrow T$ is defined by: $(D_1 \cdot D_2)(z) := D_1(z^2)$ if $\text{Im} z \geq 0$ and $(D_1 \cdot D_2)(z) := D_2(z^2)$ if $\text{Im} z < 0$. Then:

Proposition. $\text{mincr}(C, D_1 \cdot D_2) \leq \text{mincr}(C, D_1) + \text{mincr}(C, D_2)$.

Proof. Identify the torus T with the product $S_1 \times S_1$ of two copies of the unit circle S_1 in the complex plane $\mathbb{C}$. For $m, n \in \mathbb{Z}$ we define the closed curve $C_{m,n} : S_1 \longrightarrow S_1 \times S_1$ by:

$$C_{m,n}(z) := (z^m, z^n) \qquad \text{for } z \in S_1. \tag{5}$$

The closed curves $C_{m,n}$ form a system of representatives for the free homotopy classes of closed curves on T. Now for $m, n, m', n' \in \mathbb{Z}$:

$$\text{mincr}(C_{m,n}, C_{m',n'}) = |\det \begin{pmatrix} m & n \\ m' & n' \end{pmatrix}| = |mn' - m'n|. \tag{6}$$

To see the Proposition, we may assume that $D_1 = C_{m',n'}$ and $D_2 = C_{m'',n''}$ for some $m', n', m'', n'' \in \mathbb{Z}$. Then $D_1 \cdot D_2 \sim C_{m'+m'',n'+n''}$. Hence choosing m, n so that $C \sim C_{m,n}$:

$$\begin{aligned} \text{mincr}(C, D_1 \cdot D_2) &= |m(n' + n'') - (m' + m'')n| \\ &\leq |mn' - m'n| + |mn'' - m''n| = \text{mincr}(C, D_1) + \text{mincr}(C, D_2). \end{aligned} \tag{7}$$

∎

3. Proof of the theorem

The cut condition clearly is necessary. To see sufficiency, suppose the cut condition is satisfied, but cycles as required do not exist. We assume that we have a counterexample $G = (V, E)$ with

$$(8) \qquad \sum_{v \in V} 2^{\deg(v)}$$

as small as possible. Here $\deg(v)$ denotes the degree of vertex v.

We first show:

(9) each vertex of G has degree at most 4.

Suppose to the contrary that vertex v has degree $2d \geq 6$:

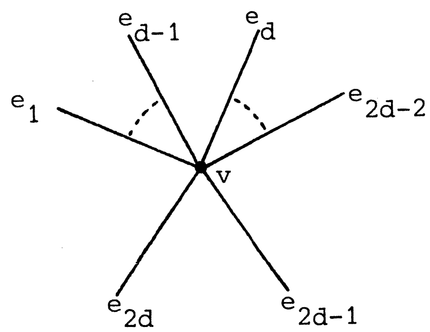

Figure 3

Replace it by:

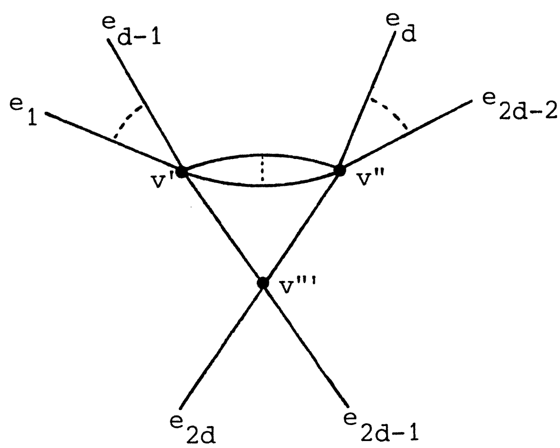

Figure 4

where there are $d - 2$ parallel edges connecting v' and v''. For the new graph G' again the cut condition holds (as we may assume that the cut D does not intersect the 'new' edges in Figure 4, since we can make a detour through the original edges without increasing

$\mathrm{cr}(G', D))$. However, for G' the sum (8) has decreased (since $2^{2d-2} + 2^{2d-2} + 2^4 < 2^{2d}$). So in G' cycles as required exist. This directly gives cycles as required in the original graph G, contradicting our assumption. This shows (9).

We next show that in each vertex v of G of degree 4 the following holds. Consider a neighbourhood $N \simeq \mathbb{C}$ of v not containing any other vertex than v:

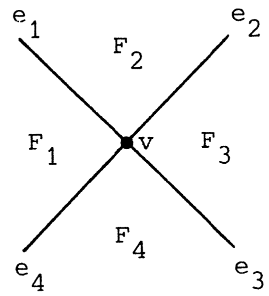

Figure 5

Here $F_1, \ldots, F_4$ stand for the intersections of faces with N. Then there exist a closed curve $D : S_1 \longrightarrow T \setminus V$ such that:

(10)
(i) D contains a subcurve contained in N connecting F_1 and F_3;
(ii) $\mathrm{cr}(G, D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D)$.

Suppose such a curve does not exist. Replace N as in Figure 5 by:

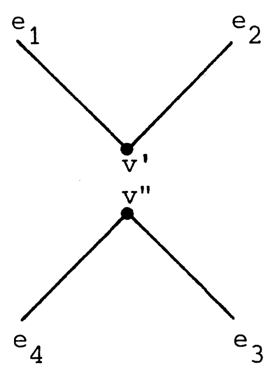

Figure 6

Since any packing of cycles as required in the new graph G' would yield a required packing in the original graph G, and since for G' the sum (8) has decreased, the cut condition does not hold for G'. That is,

$$\text{(11)} \qquad \mathrm{cr}(G', D) < \sum_{i=1}^{k} \mathrm{mincr}(C_i, D)$$

for some closed curve D not intersecting any vertex of G'. We may assume that D does not traverse v. Let p be the number of (pairwise disjoint) subcurves of D contained in N and connecting F_1 and F_3 (in one direction or the other). As $\mathrm{cr}(G', D) < \mathrm{cr}(G, D)$ we know $p \geq 1$. Choose D so that p is as small as possible. We show $p = 1$. Assume $p \geq 2$.

Let P be any curve in N from F_1 to F_3 not intersecting v, v' or v'', and only crossing e_1 and e_2. Then we may assume that:

(12)

(i) $D = P \cdot D_1 \cdot P \cdot D_2$ where D_1 and D_2 are paths from F_3 to F_1,

or (ii) $D = P \cdot D_1 \cdot P^{-1} \cdot D_2$ where D_1 is a path from F_3 to F_3, and D_2 is a path from F_1 to F_1

(P^{-1} denotes the path reverse to P). If (12)(i) holds, then (using the Proposition):

$$\text{(13)} \qquad \begin{aligned} &\mathrm{cr}(G', D) = \mathrm{cr}(G', P \cdot D_1) + \mathrm{cr}(G', P \cdot D_2) \\ &\geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, P \cdot D_1) + \sum_{i=1}^{k} \mathrm{mincr}(C_i, P \cdot D_2) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D), \end{aligned}$$

since $P \cdot D_1$ and $P \cdot D_2$ are closed curves containing fewer than p subcurves in N connecting F_1 and F_3.

If (12)(ii) holds (again using the Proposition):

$$\text{(14)} \qquad \begin{aligned} &\mathrm{cr}(G', D) \geq \mathrm{cr}(G', D_1) + \mathrm{cr}(G', D_2) \\ &\geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, P \cdot D_1 \cdot P^{-1}) + \sum_{i=1}^{k} \mathrm{mincr}(C_i, D_2) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D), \end{aligned}$$

since D_1 and D_2 are closed curves containing fewer than p subcurves in N connecting F_1 and F_3.

Both (13) and (14) contradict (11). So $p = 1$. Hence $\mathrm{cr}(G', D) = \mathrm{cr}(G, D) - 2$. Therefore, by (11), $\mathrm{cr}(G, D) < 2 + \sum_{i=1}^{k} \mathrm{mincr}(C_i, D)$. It follows by the parity condition that D satisfies (10).

Now by the 'homotopic circulation theorem' in [1], the cut condition implies the existence of a 'fractional' packing of cycles. That is, there exist cycles

$$\text{(15)} \qquad C_{1,1}, \ldots, C_{1,t_1}, C_{2,1}, \ldots, C_{2,t_2}, \ldots, C_{k,1}, \ldots, C_{k,t_k}$$

in G and rational numbers

$$\text{(16)} \qquad \lambda_{1,1}, \ldots, \lambda_{1,t_1}, \lambda_{2,1}, \ldots, \lambda_{2,t_2}, \ldots, \lambda_{k,1}, \ldots, \lambda_{k,t_k} > 0$$

satisfying:

$$\text{(17)} \qquad \begin{aligned} &\text{(i)} \quad C_{i,j} \sim C_i && (i = 1, \ldots, k; j = 1, \ldots, t_i), \\ &\text{(ii)} \quad \sum_{j=1}^{t_i} \lambda_{i,j} = 1 && (i = 1, \ldots, k), \\ &\text{(iii)} \quad \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{i,j} \chi^{C_{i,j}}(e) \leq 1 && (e \in E). \end{aligned}$$

Here $\chi^C(e)$ denotes the number of times cycle C traverses edge e.

We may assume that no $C_{i,j}$ after arriving in a vertex v via an edge e, it immediately returns over the same edge e backwards.

We show:

(18) for each i, j, if $C_{i,j}$ arrives in a vertex v via edge e, say, then next leaves v via the edge opposite to e.

(If e_1, e_2, e_3, e_4 are the edges incident to v in cyclic order, then e_1 and e_3 are called *opposite*; similarly for e_2 and e_4.) To see this, suppose that cycle $C_{1,1}$ say, contains $\ldots, e_1, v, e_2, \ldots$ (where v, e_1, e_2, e_3, e_4 are as in Figure 5). Let $D : S_1 \longrightarrow T \setminus V$ be a closed curve satisfying (10). We may assume that D crosses e_1 and e_2 successively.

However, since $C_{1,1}$ contains $\ldots, e_1, v, e_2, \ldots$, we know

$$\mathrm{cr}(C_{1,1}, D) > \mathrm{mincr}(C_{1,1}, D). \tag{19}$$

This gives the contradiction

$$\begin{aligned} \mathrm{cr}(G, D) &\geq \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{i,j} \mathrm{cr}(C_{i,j}, D) \\ &> \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{i,j} \mathrm{mincr}(C_{i,j}, D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \end{aligned} \tag{20}$$

This proves (18). It follows from (18) that any two of the $C_{1,1}, \ldots, C_{k,t_k}$ are pairwise edge-disjoint or are the same (up to cyclic permutation and reversion). (No $C_{i,j}$ makes more than one orbit of a cycle, as it is homotopic to a simple closed curve C_i.) This implies that we can select from $C_{1,1}, \ldots, C_{k,t_k}$ pairwise edge-disjoint cycles as required. ∎

Reference

[1] A. Schrijver, Decomposition of graphs on surfaces and a homotopic circulation theorem, *Journal of Combinatorial Theory, Series B*, to appear.

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