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A Simpler Proof and a Generalization of the
Zero-Trees Theorem

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ABSTRACT

Z. Füredi and D. J. Kleitman proved that if an integer weight is assigned to each edge of a complete graph on
$p + 1$ vertices, then some spanning tree has total weight divisible by $p$. We obtain a simpler proof by generalizing
the result to hypergraphs.

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1. INTRODUCTION

The following theorem is due to Z. Füredi and D. J. Kleitman [2]. (It was conjectured by A. Bialostocki and P. Dierker [1], who proved the case when \( p \) is prime.)

(1.1) Let \( \Gamma \) be a finite abelian group of order \( p \), and let \( \omega : E(K_{p+1}) \rightarrow \Gamma \) be some function. Then there is a spanning tree \( T \) of \( K_{p+1} \) with \( \omega(T) = 0 \).

\( K_n \) denotes the complete graph with \( n \) vertices; \( E(G) \) denotes the set of edges of a graph \( G \); \( \omega(T) \) means \( \Sigma(\omega(e) : e \in E(T)) \), where the summation is in \( \Gamma \).

We shall give a simpler proof of (1.1). For inductive purposes, it is advantageous to prove a version of (1.1) for complete uniform hypergraphs, because it is then easy to reduce the general problem to the case when \( p \) is prime.

Thus, let \( V \) be a finite set. A hypergraph in \( V \) is a collection of subsets of \( V \); and it is \( r \)-uniform if each of these subsets has cardinality \( r \). (In this paper, all our hypergraphs will be \( r \)-uniform for some \( r \).) If \( H \) is a hypergraph, we denote \( \bigcup(e : e \in H) \) by \( V(H) \). A hypergraph \( T \) is connected if \( T \neq \emptyset \) and for every partition \( (A, B) \) of \( V(T) \) such that \( A \) and \( B \) are both nonempty there is a member \( e \in T \) with \( e \cap A, e \cap B \) both non-empty. It is easy to see that if \( T \) is connected and \( r \)-uniform then \( |V(T)| \leq (r-1)|T|+1 \); and if equality holds we say that \( T \) is a tree. (If \( r = 2 \), this coincides with the usual definition of a tree for graphs, except for trees with \( \leq 1 \) vertex.) If \( H \) is \( r \)-uniform, and \( T \subseteq H \) is a tree, we call it a tree of \( H \); and if \( V(T) = V(H) \) we call it a spanning tree of \( H \). If \( V \) is a finite set with \( |V| \geq r \), we denote by \( \binom{V}{r} \) the collection of all \( r \)-element subsets of \( V \). We shall prove the following generalization of (1.1).

(1.2) Let \( \Gamma \) be a finite abelian group of order \( p \), let \( r \geq 2 \) be an integer, let \( V \) be a set of cardinality \( p(r-1)+1 \), and let \( \omega : \binom{V}{r} \rightarrow \Gamma \) be some function. Then there is a spanning tree \( T \) of \( \binom{V}{r} \) with \( \omega(T) = 0 \).

\( \omega(T) \) means \( \Sigma(\omega(e) : e \in T) \).

2. THE PROOF OF (1.2)

We require several lemmas. First, we shall need the following, which is a special case of the Cauchy-Davenport theorem (see [3]). (It can also be proved directly in a couple of lines, as the reader may verify.)
(2.1) Let $p$ be prime, let $A \subseteq \mathbb{Z}_p$, and let $b, c \in \mathbb{Z}_p$ be distinct. If $1 \leq |A| \leq p - 1$ then

$$|\{a + b : a \in A\} \cup \{a + c : a \in A\}| > |A|.$$ 

If $T$ is an $r$-uniform tree, we say that $f \in T$ is a leaf of $T$ if there exists $u \in f$ such that $e \cap f \subseteq \{v\}$ for every $e \in T - \{f\}$. We call such an element $u$ a root of the leaf $e$. If $T, T'$ are trees in $\binom{V}{r}$ with leaves $e, e'$ respectively, and $T - \{e\} = T' - \{e'\}$, we say that $T'$ is obtained from $T$ by shifting a leaf. If $T, T' \subseteq \binom{V}{r}$ are trees, we say that $T$ is shiftable to $T'$ if there is a sequence $T = T_1, T_2, ..., T_k = T'$ of trees in $\binom{V}{r}$ such that $T_{i+1}$ is obtained from $T_i$ by shifting a leaf for $1 \leq i \leq k - 1$. This is evidently an equivalence relation, and in fact all trees in $\binom{V}{r}$ of the same cardinality are shiftable to one another, but we only need a weaker result, the following.

(2.2) Let $r \geq 2, k \geq 1$ be integers, let $|V| \geq k(r - 1) + 2$, and let $v_0 \in V$. Let $T_0$ be a tree in $\binom{V}{r}$ with $|T_0| = k$. Then $T_0$ is shiftable to a tree $T$ with $v_0 \notin V(T)$.

Proof. We may assume that $k \geq 2$, for the result is clear if $k = 1$. If $T$ is a tree in $\binom{V}{r}$ with $v_0 \in V(T)$ and $f$ is a leaf of $T$, we define $d(T, f)$ to be the unique $d \geq 1$ such that there is a sequence $v_0 = v_1, e_1, v_2, e_2, ..., v_d, e_d = f$ satisfying

(i) $v_1, v_2, ..., v_d \in V(T)$ are all distinct, and so are $e_1, e_2, ..., e_d \in T$

(ii) $v_i \in e_{i-1}$ for $2 \leq i \leq d$, and $v_i \in e_i$ for $1 \leq i \leq d$.

Let us choose a tree $T$ in $\binom{V}{r}$ such that $T_0$ is shiftable to $T$ and $v_0 \in V(T)$, and a leaf $f$ of $T$, in such a way that $d(T, f)$ is maximum. Let $u$ be a root of $f$. Since $|T| \geq 2$ it follows that $T$ has at least two leaves; let $f'$ be another leaf, with root $u'$. Since $d(T, f') \leq d(T, f)$ it follows that $v_0 \notin f - \{u\}$. Choose $v \in f - \{u\}$, and let $e = (f' - \{u'\}) \cup \{v\}$. Now $T' = (T - \{f'\}) \cup \{e\}$ is shiftable from $T$ and hence from $T_0$, and $e$ is a leaf of it, and if $v_0 \notin f' - \{u'\}$ then $d(T', e) > d(T, f)$, a contradiction. Thus $v_0 \notin f' - \{u'\}$, and since $V(T) \neq V$, the result
follows. ■

Again, let \( r \geq 2, k \geq 1 \) and let \( |V| \geq k(r-1)+1 \). We say that \( S \subseteq \binom{V}{r} \) is a \((V, k)\)-blocker if \( |S \cap T| \neq \emptyset \) for every tree \( T \) in \( \binom{V}{r} \) with \( |T| = k \). Our third lemma is the following.

(2.3) Let \( r \geq 2, k \geq 1 \) be integers, and let \( |V| = k(r-1)+1 \). If \( S \subseteq \binom{V}{r} \) is a \((V, k)\)-blocker then \( S \) includes a spanning tree of \( \binom{V}{r} \).

Proof. The result holds if \( k = 1 \), and so we may assume that \( k \geq 2 \) and proceed by induction on \( k \). Since there is a spanning tree and we may assume that it is not included in \( S \), it follows that \( \emptyset \neq S \subseteq \binom{V}{r} \). Thus, we may choose \( e, f \in \binom{V}{r} \) with \( |e \cap f| = r-1 \) and \( e \in S, f \notin S \). Let \( V \setminus (e \cap f) = V' \). If \( T' \) is a spanning tree of \( \binom{V}{r} \) then \( T' \cup \{f\} \) is a spanning tree of \( \binom{V}{r} \), and so \( S \cap (T' \cup \{f\}) \neq \emptyset \), that is, \( S' \cap T' \neq \emptyset \), where \( S' = S \cap \binom{V}{r} \). Hence \( S' \) is a \((V', k-1)\)-blocker, and so \( S' \) includes a spanning tree \( T' \) of \( \binom{V}{r} \), from the inductive hypothesis. Then \( T' \cup \{e\} \subseteq S \) is a spanning tree of \( \binom{V}{r} \), as required. ■

We shall use (2.1)-(2.3) to prove the following, which is the main step in the proof of (1.2).

(2.4) Let \( p \) be prime, let \( k \geq 1, r \geq 2 \) be integers with \( k \leq p \), let \( V \) be a set of cardinality \( k(r-1)+1 \), and let \( w : \binom{V}{r} \to \mathbb{Z}_p \) be some function. Then either

(i) there are \( k \) spanning trees \( T_1, \ldots, T_k \) with \( w(T_1), \ldots, w(T_k) \) all distinct, or

(ii) \( k \geq 2 \) and there is a monochromatic \((V, k-1)\)-blocker.

(A subset \( S \subseteq \binom{V}{r} \) is monochromatic if the restriction of \( w \) to \( S \) is constant.)

Proof. The result holds if \( k = 1 \), and so we may assume that \( k \geq 2 \) and proceed by induction on \( k \). We say that \( X \subseteq V \) is joint if \( |X| = r-1 \) and \( X = f_1 \cap f_2 \) for some \( f_1, f_2 \in \binom{V}{r} \) with \( w(f_1) \neq w(f_2) \). We assume that (i) is false. We may assume that

1. Some set \( X \subseteq V \) is joint.
For \( \binom{V}{r} \) is a \((V, k - 1)\)-blocker since \( k \geq 2 \), and so we may assume that \( w \) is non-constant on \( \binom{V}{r} \), for otherwise (ii) holds. The claim follows.

(2) If \( X \) is joint then \( k \geq 3 \) and there exists a monochromatic \((V - X, k - 2)\)-blocker.

For let \( X \subseteq V \) be joint. Suppose that there are \( k - 1 \) spanning trees \( T_1, \ldots, T_{k-1} \) of \( \binom{V - X}{r} \) with \( w(T_1) \), ..., \( w(T_{k-1}) \) all distinct. Choose \( f_1, f_2 \in \binom{V}{r} \) with \( f_1 \cap f_2 = X \) and \( w(f_1) \neq w(f_2) \). Now \( T_1 \cup \{f_1\} \) and \( T_1 \cup \{f_2\} \) are spanning trees of \( \binom{V}{r} \) for \( 1 \leq i \leq k - 1 \), and

\[
|\{w(T_i) + w(f_1) : 1 \leq i \leq k - 1\} \cup \{w(T_i) + w(f_2) : 1 \leq i \leq k - 1\}| \geq k
\]

by (2.1). Hence (i) holds, a contradiction. Thus, there do not exist \( k - 1 \) such spanning trees. From our inductive hypothesis applied to \( V - X \) the claim follows.

In particular, from (1) and (2) we deduce that \( k \geq 3 \). For each joint set \( X \), let \( S(X) \) be a monochromatic \((V - X, k - 2)\)-blocker, and let \( w(e) = q(X) \) for all \( e \in S(X) \).

(3) There exists \( q \in Z_p \) such that \( q(X) = q \) for every joint set \( X \).

For let \( X_1, X_2 \) be joint; we shall show that \( q(X_1) = q(X_2) \). Let \( X_1 \cup X_2 \subseteq Z \subseteq V \), where \( |Z| = 2^r - 2 \). Now \( S(X_1) \) is a \((V - X_1, k - 2)\)-blocker, and so \( S(X_1) \cap \binom{V - Z}{r} \) is a \((V - Z, k - 2)\)-blocker. By (2.3), there is a spanning tree \( T \) of \( \binom{V - Z}{r} \) with \( T \subseteq S(X_1) \). Similarly, \( S(X_2) \cap \binom{V - Z}{r} \) is a \((V - Z, k - 2)\)-blocker, and so \( S(X_2) \cap T \neq \emptyset \). Hence \( S(X_1) \cap S(X_2) \neq \emptyset \), and the claim follows.

Let us say a tree \( T \subseteq \binom{V}{r} \) is bad if \( |T| = k - 1 \) and \( w(e) \neq q \) for all \( e \in T \).

(4) If \( f_1 \) is a leaf of a bad tree \( T \), and \( f_2 \in \binom{V}{r} \) with \( |f_2 \cap V(T - \{f_1\})| \leq 1 \), then \( w(f_2) = w(f_1) \).

For let \( V' = V(T - \{f_1\}) \). If \( X \subseteq V - V' \) is joint then \( S(X) \cap (T - \{f_1\}) \neq \emptyset \), which is impossible by (3) since \( T \) is bad. Thus no subset of \( V - V' \) is joint, and the claim follows.

In particular,

(5) If \( T \) is a bad tree and \( T \) is shiftable to \( T' \) then \( T' \) is bad.
Now by (1), there is a joint set $X$. If there is a bad tree, then by $(r - 1)$ applications of (2.2), it is shiftable to a tree $T$ with $X \cap V(T) = \emptyset$; and by (5), $T$ is bad. But then $T \cap S(X) \neq \emptyset$, a contradiction as before. We deduce that there is no bad tree, and so $\{ e \in (V) : w(e) = q \}$ is a $(V, k - 1)$-blocker. Thus (ii) holds, as required.

Finally, we use (2.4) to prove (1.2).

Proof of (1.2).

We proceed by induction on $p$. If $p$ is prime, then $\Gamma \cong \mathbb{Z}_p$ and by (2.4) with $k = p$, either

(i) there are $p$ spanning trees $T_1, \ldots, T_p$ with $w(T_1), \ldots, w(T_p)$ all distinct; but then one of them is zero, as required, or

(ii) for some $q \in \Gamma$ there is a $(V, p - 1)$-blocker $S$ such that $w(e) = q$ for all $e \in S$; but then $S$ is a $(V, p)$-blocker and hence includes a spanning tree $T$, and $w(T) = \Sigma(q : e \in T) = 0$ as required.

We may assume then that $p$ is not prime, and so $\Gamma$ has a proper subgroup $\Gamma'$, of order $p'$ say. Let $\Gamma''$ be the quotient group $\Gamma/\Gamma'$, of order $p''$ say where $p = p'p''$, and let $\phi : \Gamma \to \Gamma''$ be the homomorphism with kernel $\Gamma'$. For each $e \in (V)$, we define $w''(e) = \phi(w(e)) \in \Gamma''$. Let $r' = p''(r - 1) + 1$. For each $f \subseteq V$ with $|f| = r'$, we define $w'(f)$ as follows. From our inductive hypothesis applied to $(V)$, $\Gamma''$ and $w''$, there is a spanning tree $T(f)$ of $(V)$ such that $w''(T(f)) = 0$, that is, $w(T(f)) \in \Gamma'$. We define $w'(f) = w(T(f))$. From our inductive hypothesis applied to $(V)$, $\Gamma'$ and $w'$, there is a spanning tree $T'$ of $(V)$ with $w'(T') = 0$. Let $T = \bigcup(T(f) : f \in T')$; then $T$ is a spanning tree of $(V)$ and

$$w(T) = \sum_{f \in T'} \sum_{e \in T(f)} w(e) = \sum_{f \in T'} w'(f) = 0$$

as required. $\blacksquare$
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