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J.W. Cohen

On the attained waiting time

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On the Attained Waiting Time

J.W. Cohen
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Summary
By using properties of up- and downcrossings of the sample functions of the work load process and of the attained waiting time process for a G/G/1 queueing model it is shown that both processes have the same stationary distribution, if such distributions do exist.

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Sakasegawa and Wolff [1] show by using sample function arguments that for the FIFO G/G/1 queueing model the workload process $\mathbf{v}_t$ and the attained waiting time process $\boldsymbol{\eta}_t$ possess the same stationary distribution, if such distributions exist. However their proof is some what artificial (see their use of preemptive LIFO).

A direct proof proceeds as follows. Consider a busy cycle $\mathbf{c}$ with $\mathbf{n}$ the number of customers served; $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{\mathbf{n}}$ are the service times of these customers, $\mathbf{w}_1, \ldots, \mathbf{w}_{\mathbf{n}}$ their successive actual waiting times, $\mathbf{i}$ is the idle time, so

$$\mathbf{c} = \boldsymbol{\tau}_1 + \cdots + \boldsymbol{\tau}_{\mathbf{n}} + \mathbf{i}. \tag{1}$$

The attained service time $\boldsymbol{\eta}_t$ at epoch t is by definition the time between t and the arrival epoch of the customer being served at epoch t. In the figure below the sample function of the work load process $\mathbf{v}_t$ and the corresponding $\boldsymbol{\eta}_t$-process during the busy cycle $\mathbf{c}$ are shown, with $\mathbf{n}=4$.

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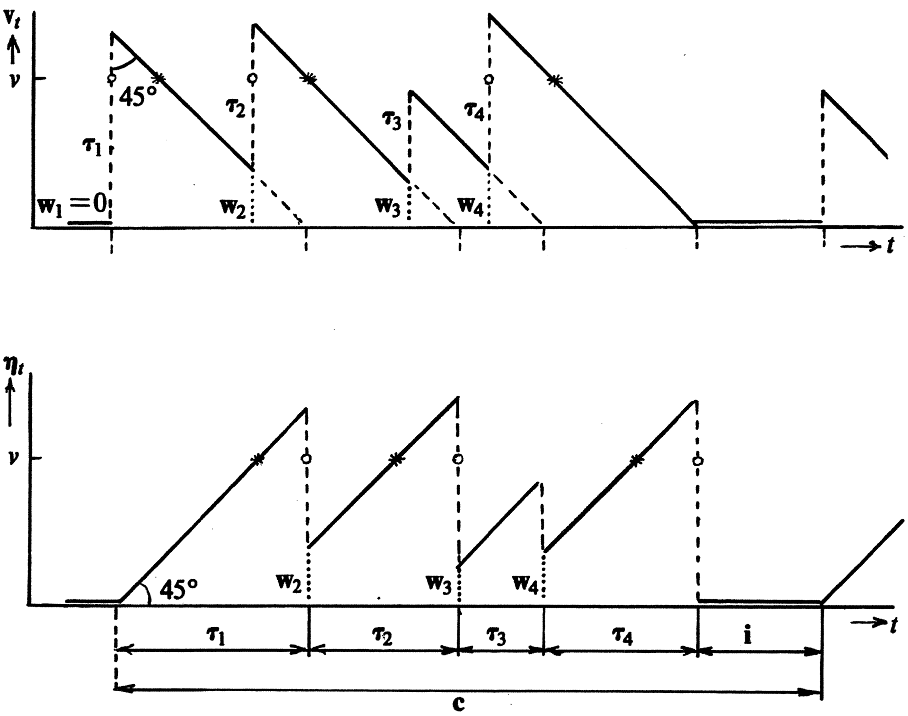

Define for $v \geqslant 0$,

$$\begin{aligned}
&\mathbf{d}(v) := \#\ \text{downcrossings of } \mathbf{v}_t,\ 0 \leqslant t \leqslant \mathbf{c} \text{ with level } v, (*)\\
&\mathbf{u}(v) := \#\ \text{upcrossings }\ ,,\ \mathbf{v}_t,\ 0 \leqslant t \leqslant \mathbf{c}\ ,,\ \ ,,\ \ v, (o)
\end{aligned} \tag{2}$$

$$\begin{aligned}
&\boldsymbol{\delta}(v) := \#\ \text{upcrossings of } \boldsymbol{\eta}_t,\ 0 \leqslant t \leqslant \mathbf{c}\ ,,\ \ ,,\ \ v, (*)\\
&\boldsymbol{\omega}(v) := \#\ \text{downcrossings }\ ,,\ \boldsymbol{\eta}_t,\ 0 \leqslant t \leqslant \mathbf{c}\ ,,\ \ ,,\ \ v, (o).
\end{aligned} \tag{3}$$

Note that in the figure $\mathbf{d}(v)=3$; the upcrossings are there indicated by o, the downcrossings by $*$. It is immediately evident from the geometry of the sample functions, cf. [2], [3], that with probability one, for $v \geqslant 0$,

$$\mathbf{d}(v) = \mathbf{u}(v), \qquad \boldsymbol{\delta}(v) = \boldsymbol{\omega}(v), \tag{4}$$

$$\mathbf{u}(v) = \boldsymbol{\delta}(v); \tag{5}$$

and

$$\mathbf{d}(v) = \frac{\mathrm{d}}{\mathrm{d}v} \int_0^{\mathbf{c}} (\mathbf{v}_t < v)\mathrm{d}t, \quad \boldsymbol{\delta}(v) = \frac{\mathrm{d}}{\mathrm{d}v} \int_0^{\mathbf{c}} (\boldsymbol{\eta}_t < v)\mathrm{d}t, \tag{6}$$

where we use the notation

$$(\mathbf{v}_t < v) \equiv 1_{\mathbf{v}_t < v} \text{ and } \int_0^{\mathbf{c}} (\mathbf{v}_t < v)\mathrm{d}t \equiv \int_0^{\infty} (\mathbf{v}_t < v,\ \mathbf{c} \geqslant t)\mathrm{d}t, \tag{7}$$

for the indicator function and the integral. Since

$$\mathbf{i} = \{\int_0^{\mathbf{c}} (\mathbf{v}_t < v)\mathrm{d}t\}_{v=0+} = \{\int_0^{\mathbf{c}} (\boldsymbol{\eta}_t < v)\mathrm{d}t\}_{v=0+}, \tag{8}$$

integration of (6), using the boundary conditions (8) yields, via (4) and (5), that with prob. 1,

$$\int_0^{\mathbf{c}} (\mathbf{v}_t < v)\mathrm{d}t = \int_0^{\mathbf{c}} (\boldsymbol{\eta}_t < v)\mathrm{d}t, \quad v \geqslant 0. \tag{9}$$

Because

$$(\mathbf{v}_t < v) = 1 - (\mathbf{v}_t \geqslant v),$$

we have from (9)

$$\int_0^{\mathbf{c}} (\mathbf{v}_t \geqslant v)\mathrm{d}t = \int_0^{\mathbf{c}} (\boldsymbol{\eta}_t \geqslant v)\mathrm{d}t,$$

which is theorem 1 of [1].

For the GI/G/1 queueing model with the conditions:

i. $\mathrm{E}\{\mathbf{c}\} < \infty$,

ii. $\mathbf{c}$ has not a lattice distribution;

the stochastic mean value theorem, cf. [3], [4], applies, i.e. the $\mathbf{v}_t$-process has a unique stationary distribution and for $\mathbf{v}_\infty$ a stochastic variable having this distribution holds

$$\Pr\{\mathbf{v}_\infty < v\} = \frac{1}{\mathrm{E}\{\mathbf{c}\}} \mathrm{E}\{\int_0^{\mathbf{c}} (\mathbf{v}_t < v)\mathrm{d}t\}, \quad v \geqslant 0, \tag{10}$$

For the same conditions it is similarly shown that the $\boldsymbol{\eta}_t$-process possesses a stationary distribution and for $\boldsymbol{\eta}_\infty$ a stochastic variable with this distribution holds

$$\Pr\{\boldsymbol{\eta}_\infty < v\} = \frac{1}{\mathrm{E}\{\mathbf{c}\}} \mathrm{E}\{\int_0^{\mathbf{c}} (\boldsymbol{\eta}_t < v)\mathrm{d}t\}. \tag{11}$$

Consequently from (10) and (11),

$$\boldsymbol{\eta}_\infty \sim \mathbf{v}_\infty,$$

a result obtained in [5]. By using again the properties of up- and downcrossings it is readily shown that for the G/G/1 queue the limits for $T \to \infty$ of

$$\frac{1}{T}\int_0^T(\mathbf{v}_t<v)\mathrm{d}t \text{ and } \frac{1}{T}\int_0^T(\boldsymbol{\eta}_t<v)\mathrm{d}t,\quad v\geqslant 0,$$

both exists with probability one and are equal with probability one (note that the number of upcrossings and that of downcrossings in an internal $(0,T)$ differ by at most one).

REFERENCES

[1] SAKASEGAWA, H. and WOLFF, R.W., The equality of the virtual delay and attained waiting time distributions, Adv. Appl. Prob. **22** (1990) 257-259.

[2] COHEN, J.W., On up- and downcrossings. J. Appl. Prob. **14** (1977) 405-410.

[3] COHEN, J.W., The Single Server Queue, North-Holland Publ. Co, Amsterdam, 1982, revised edition.

[4] COHEN, J.W., On Regenerative processes in Queueing Theory, Lect. Notes, Econ. and Math. Syst. **121** Springer, New York, 1976.

[5] SENGUPTA, B., An invariance relationship for the G/G/1 queue, Adv. Appl. Prob. **21** (1989) 956-957.