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Complex Lie Semigroups, Hardy Spaces and the Gelfand-Gindikin Program

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Let G be the group of motions of a bounded symmetric domain and let G_C be its complexification. It is known that there exist open subsemigroups $\Gamma \subset G_C$ containing G. These semigroups Γ can be viewed as non-commutative analogues of Siegel domains, where $G \subset \Gamma$ plays the role of the skeleton. The aim of this paper is to construct analogues of Hardy spaces H^2 on semigroups Γ and to study their properties.

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§0. Introduction

The classical Hardy space H^2 in the halfplane [1] is the Hilbert space of all functions $f(z)$ which are holomorphic in the halfplane $\operatorname{Re} z > 0$ and satisfy

$$||f||^2 := \sup_{x>0} \int_{\mathbb{R}} |f(x+iy)|^2 dy < \infty.$$

The generalisation to several variables in which the halfplane is replaced by a tube domain in $\mathbb{C}^n$ of the form $i\mathbb{R} + C$, where $C \subseteq \mathbb{R}^n$ is a convex cone, is well known [2]. In the present paper a Hardy space is constructed on a noncommutative open subsemigroup of a complex Lie group. We give a detailed account of the paper and some motivation.

Let G be the group of motions of a bounded symmetric domain and $\mathfrak{g}$ its Lie algebra. The algebra $\mathfrak{g}$ has an important property: it contains non trivial $\operatorname{Ad} G$-invariant convex cones (whose complete description can be found in [4] and [5], see also [15]). As was shown in [4], one can associate a semigroup $\Gamma(C)$ with any such cone. The semigroup $\Gamma(C)$ is contained in G_C and of the form $\Gamma(C) = G\exp(iC)$. These semigroups turn out to be a noncommutative analogue of the tube domains. The paper gives a construction of Hardy spaces $H^2(C)$ on the interior $\Gamma^o(C)$ of the semigroup $\Gamma(C)$. On $H^2(C)$ one has a natural holomorphic representation of the semigroup $\Gamma(C)$ by contraction operators.

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Furthermore an operator $f \mapsto If$ is constructed using the boundary values of the function $f \in H^2(C)$. This operator yields an isometric G-equivariant embedding of $H^2(C)$ in $L^2(G)$. There will be given a description of the unitary representation of the group G in $H^2(C) \subseteq L^2(G)$. In the case of the minimal cone $C = C_{min}$ it is decomposed precisely into representations from the holomorphic discrete series. A Cauchy-Szegö kernel is considered which gives a projection $L^2(G) \to H^2(C)$. Finally a few generalisations and unsolved problems are mentioned. As far as generalisations are concerned we note the following specific property of the proof: Its long first part (Theorem A), which consists of checking the "Hilbert property" and the construction of I etc., has purely formal character. Thus it is easy to transfer it to different situations (e.g. $L^2(G)$ can be replaced by the L^2-space on a pseudo-Riemannian symmetric space, the group need not be semisimple etc.). But veryfying that $H^2(C)$ is nontrivial and studying its structure is another thing: here we have to use a certain nontrivial information about representations of our group G. In the context of the present work such an information is morally delivered by the Harish-Chandra theorem on the existence of square integrable highest weight representations.

The idea to study Hardy type spaces on certain open submanifolds of complex groups is due to Gelfand and Gindikin [6]. In their paper a remarkable program for a harmonic analysis on semisimple Lie groups and pseudo-Riemannian symmetric spaces is outlined. Hardy spaces of holomorphic functions are the first level of this program. For the higher levels one needs cohomology theory. It was the author's observation that the manifold defined in [6] coincides with the interior of the semigroup $\Gamma(C_{min})$. Thus the machinery of holomorphic representations of these semigroups as developed in [4] was quite useful. The discussions with S. G. Gindikin, to whom the author expresses his deep gratitude, helped a lot to clarify the connections between [4] and [6].

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§1. Preliminaries (cf. [3,4])

Let $\mathfrak{g}$ be a simple real noncompact Lie algebra and D the corresponding symmetric space. We assume that $\mathfrak{g}$ is hermitian, i.e. that D is hermitian. Thus $\mathfrak{g}$ is one of the following algebras:

$$\mathrm{su}(p,q), \mathrm{sp}(n,\mathbb{R}), \mathrm{so}^*(2n), \mathrm{so}(2,n), \mathrm{EIII}, \mathrm{EVII}.$$

Let G_C be a connected simply connected Lie group with Lie algebra $\mathfrak{g}_C$ and $G \subseteq G_C$ the subgroup corresponding to $\mathfrak{g}$. For our purposes it is convenient to consider invariant cones not in $\mathfrak{g}$ but in $i\mathfrak{g}$. Let Con be the set of all closed convex $\mathrm{Ad}\, G$- invariant cones in $i\mathfrak{g}$ which are different from $\{0\}$. Since $\mathfrak{g}$ is hermitian we know that Con is non empty. Each cone in Con contains no lines and has non empty interior C^o. The set Con has a maximal element C_{max} and a minimal element C_{min}, both unique up to multiplication by -1. These facts can be found in [7,3]. Except for the case $\mathfrak{g} = sp(n, \mathbb{R})$ in which $C_{min} = C_{max}$, there is a continuum of cones between C_{min} and C_{max} in Con. A complete description of the set Con is contained in [4,5]. For $C \in Con$ we set $\Gamma(C) = G \exp C$. In [4] it is proved that $\Gamma(C)$ is a subsemigroup of G_C. The interior $\Gamma^o(C) = G \exp(C^o)$ of the set $\Gamma(C)$ is a complex manifold. We denote by $g \mapsto g^\sharp$ the involutive antiholomorphic antiautomorphism of the group G_C which corresponds to the antiautomorphism $X + iY \mapsto (-X) + iY$ of the algebra $\mathfrak{g}_C = \mathfrak{g} + i\mathfrak{g}$. The map $^\sharp$ leaves the semigroup $\Gamma(C)$ invariant. Let $\mathcal{H}$ be a complex Hilbert space and Δ the multiplicative semigroup of operators on $\mathcal{H}$ of norm less or equal to one; Δ will be endowed with the weak operator topology. We note that in the point $1 \in \Delta$ this topology agrees with the strong operator topology.

1.1 [4]. Definition. A representation of the semigroup $\Gamma(C)$ in $\mathcal{H}$ is a continuous homomorphism $\mathcal{T}: \Gamma(C) \to \mathcal{C}$ such that $\mathcal{T}(1) = \mathbb{1}$ and $\mathcal{T}(\gamma^\sharp) = \mathcal{T}(\gamma)^*$ for all $\gamma \in \Gamma(C)$.

We note that the restriction $T = \mathcal{T}|_G$ of the representation $\mathcal{T}$ to G is a unitary representation of the group G.

1.2 [4]. Definition. A representation $\mathcal{T}$ of the semigroup $\Gamma(C)$ on the Hilbert space $\mathcal{H}$ is called holomorphic, if it is a holomorphic mapping from $\Gamma^o(C)$ to the Banach space of bounded operators on $\mathcal{H}$.

The following lemma is useful for checking the holomorphy of vector and operator valued functions:

1.3. Lemma.

(i) *Let $F(\cdot)$ be a function with values in $\mathcal{H}$ defined on a complex manifold. If F is bounded in norm and the scalar functions of the form $(F(\cdot), \xi)$ are holomorphic for all $\xi \in B$ where $B \subseteq \mathcal{H}$ is a dense subspace, then F is holomorphic.*

(ii) *The analogous statement holds for bounded operator valued functions, where we suppose that the functions of the form $(F(\cdot)\xi, \eta)$ are bounded for $\xi, \eta \in B$.*

For a proof, see [8]. ■

From this lemma we conclude that the "weak holomorphy" of the function F implies the norm continuity.

Let T be an arbitrary unitary representation of the group G on $\mathcal{H}$. To each $X \in i\mathfrak{g}$ we associate the operator $T(X)$ on $\mathcal{H}$ which is determined by the condition

$$T(\exp itX) = \mathrm{Exp}\, itT(X), \quad \forall t \in \mathbb{R}.$$

Note that in order to avoid confusion we denote the exponential of an operator by $\mathrm{Exp}(\cdot)$, the expontial mapping $\mathfrak{g}_C \to G_C$ by $\exp(\cdot)$ and the scalar exponential by $e^{(\cdot)}$. By $T(X) \leq 0$ we mean that the spectrum of the operator $T(X)$ is contained in the halfline $(-\infty, 0]$.

1.4. Definition. Let $C \in Con$. We call a representation T C-admissible if $T(X) \leq 0$ for all $X \in C$.

It really is only necessary to assume that $T(X) \leq 0$ for all $X \in C^o$.

1.5. Theorem. ([4], see also [14]). *Let $\mathcal{T}$ be a holomorphic representation of the semigroup $\Gamma(C)$ and $T = \mathcal{T}|_G$ then T is C-admissible. Conversely, any admissible representation T of G can in a natural way be extended to a holomorphic representation of $\Gamma(C)$.* ■

Thus the description of the holomorphic representations of the semigroup $\Gamma(C)$ is reduced to the description of the C-admissible representations of G. This problem has been solved in [4]. Here we say only so much that for $C = C_{min}$ the irreducible C-admissible representations are precisely the highest weight unitary representations. In the general case the supply of C-admissible representations gets bigger as the cone C gets smaller. Finally we note that an arbitrary C-admissible unitary representation is the integral of irreducible C-admissible representations.

A well known theorem due to Harish-Chandra gives a detailed description of those highest weight represenations which are square integrable, i.e., are discrete components of the regular representation on $L^2(G)$. These representations form the so called holomorphic discrete series. From the Harish-Chandra Theorem and the results of [4] we have

1.6. Corollary. *For any $C \in Con$ the set of C-admissible representations in the holomorphic discrete series is non empty. If $C = C_{min}$ then this set is precisely the whole holomorphic discrete series.* ■

§2. The Hardy space

We fix a cone $C \in Con$. In order to shorten notation we set $\Gamma = \Gamma(C)$, $\Gamma^o = \Gamma^o(C)$ and $L = L^2(G)$. If f is a function on Γ^o and $\gamma \in \Gamma^o$ then we denote by $\gamma \cdot f$ the function $g \mapsto f(g\gamma)$ on G (here we have to note that $g\gamma \in \Gamma^o$; more generally: $\Gamma\Gamma^o \subseteq \Gamma^o$).

2.1. Definition. The space of all holomorphic functions f on Γ^o which satisfy $||f||_H < \infty$ where

$$||f||_H := \sup_{\gamma \in \Gamma^o} ||\gamma \cdot f||_L$$

is denoted by $H^2(C)$ or simply H. For $f \in H$ and $\gamma \in \Gamma$ we define a function $\mathcal{T}(\gamma)f$ on Γ^o by the formula

$$\mathcal{T}(\gamma)f(\gamma_1) = f(\gamma_1\gamma).$$

It is obvious that $\mathcal{T}(\gamma)$ maps H into itself. For any $g \in G$ we denote by $R(g)$ the right translation on L, i.e. $R(g)\varphi(g_1) = \varphi(g_1 g)$.

2.2. Theorem. (Theorem A)

(i) *H is a Hilbert space w.r.t. the norm $||\cdot||_H$.*

(ii) *There exists an isometric embedding $I: H \to L$ such that for an arbitrary function $f \in H$ and an arbitrary sequence $\gamma_1, \gamma_2, \dots$ in Γ^o which converges to 1, the sequence $\{\gamma_j \cdot f\}$ converges to If w.r.t. the metric of L.*

(iii) *I commutes with right translations from G, i.e. $I\mathcal{T}(g) = R(g)I$.*
(iv) *$\mathcal{T}(\cdot)$ is a holomorphic representation of the semigroup $\Gamma(C)$ on H.*
(v) *$I(H)$ is the biggest $R(G)$-invariant subspace of L such that the corresponding unitary representation is C-admissible.* ■

The proof will be given in §4.

§3. The Main Lemma

Let $t \mapsto U(t)$ be a unitary representation of the additive group $\mathbb{R}$ on a Hilbert space $\mathcal{H}$. According to Stone's Theorem we have $U(t) = \mathrm{Exp}\,itA$ where A is a self adjoint operator on $\mathcal{H}$. Consider the projection valued spectral measure $dP(s)$ associated to A. Then

$$A = \int_{\mathbb{R}} s\,dP(s), \quad U(t) = \int_{\mathbb{R}} e^{its}dP(s), \quad \forall t \in \mathbb{R}.$$

For any interval (more generally, for any Borel set) $a \subseteq \mathbb{R}$ we have a projection $P(a)$ which is the value of the measure on a. We set $\mathcal{H}(a) = P(a)\mathcal{H}$ and $A(a) = A|_{\mathcal{H}(a)}$. In the case $a = (-\infty, 0]$ we replace this notation by $\mathcal{H}_-$ and A_- respectively. Note that $A_- \leq 0$, thus the operator $\mathrm{Exp}\,zA_-$ is defined for all z with $\mathrm{Re}\,z \geq 0$ and the norm of this operator is less or equal to one.

3.1. Lemma. (Main Lemma). *Let $F(z)$ be a holomorphic function, defined on the halfplane $\mathrm{Re}\,z > 0$ with values in $\mathcal{H}$ and satisfying the following two conditions:*
(i) *$F(z + it) = U(t)F(z)$ for $\mathrm{Re}\,z > 0, t \in \mathbb{R}$.*
(ii) *$\sup_{\mathrm{Re}\,z>0} ||F(z)|| < \infty$.*
Then there exists a unique vector $\xi \in \mathcal{H}$ such that $F(z) = (\mathrm{Exp}\,zA_-)\xi$ for $\mathrm{Re}\,z > 0$. Moreover, if $z \to 0$ ($\mathrm{Re}\,z > 0$) then we have $F(z) \to \xi$ in $\mathcal{H}$. Finally we have $\sup ||F(z)|| = ||\xi||$.

Proof. Let $a = [\alpha_1, \alpha_2]$ be a finite interval in $\mathbb{R}$. Then the function $F_a(z) = P(a)F(z)$ has all the properties of the function F. On the other hand the operator $A(a)$ is bounded, so that the operator $\mathrm{Exp}\,zA(a)$ makes sense for all $z \in \mathbb{C}$.

let us remark now that

$$(1) \qquad F_a(z_1 + z_2) = \mathrm{Exp}(z_2 A(a))F_a(z_1) \;\; \forall\, \mathrm{Re}\,z_1 > 0, \mathrm{Re}\,z_2 \geq 0.$$

In fact the equation (1) for $z_2 \in i\mathbb{R}$ is a consequence of (i) and the general case follows by analytic continuation. Moreover it is clear that

$$(2) \qquad ||(\mathrm{Exp}\,zA(a))\eta|| \geq e^{\mathrm{Re}\,z\alpha_1}||\eta|| \;\; \forall \eta \in \mathcal{H}.$$

If we assume that $a \subseteq (0, \infty)$, i.e. $\alpha_1 > 0$, then we get $\mathrm{Re}\,z\alpha_1 \to +\infty$ for $\mathrm{Re}\,z \to +\infty$. Comparing (1), (2) and (i) we find that $F_a(z) \equiv 0$. Thus we have $F(z) \in \mathcal{H}_-$ for all z

with $\mathrm{Re}\, z > 0$. Now we assume that $a \subseteq (-\infty, 0]$. Then it follows from (1) that the vector $((\mathrm{Exp} zA(a))\, F_a(z)$ does not depend on z. We call this vector ξ_a. It is obvious that

$$(3) \qquad ||\xi_a|| = \sup_{\mathrm{Re}\, z>0} ||F_a(z)|| \leq \sup_{\mathrm{Re}\, z>0} ||F(z)||.$$

Moreover, if $a \subseteq b \subseteq (-\infty, 0]$, then $\xi_a = P(a)\xi_b$. Now (3) implies the existence of the limit $\xi = \lim_a \xi_a$ where $a = [\alpha, 0]$ with $\alpha \to -\infty$. It is now easy to verify that ξ has all the desired properties. ■

§4. Proof of Theorem A

Step 1: For any $f \in H$ the map $\gamma \mapsto \gamma \cdot f$ is a holomorphic (in particular continuous) function on Γ^o with values in L.

In fact for an arbitrary integrable and bounded function $\varphi \in L$ the scalar function $\gamma \mapsto (\gamma \cdot f, \varphi)_L = \int_G f(g\gamma)\overline{\varphi(g)}dg$ is holomorphic in Γ^o. Now it only remains to apply the Lemma 1.3.

Step 2: If $f \in H$ and $X \in C^o$ then there exists the limit (in the norm of L)

$$\lim(\exp zX \cdot f) = I_X f \in L,$$

where $z \to 0$ and $\mathrm{Re}\, z > 0$. (We note here that $\exp zX \in \Gamma^o$ for all z with $\mathrm{Re}\, z > 0$; but if $z \in i\mathbb{R}$ then $\exp zX \in G$.)

To prove Step 2 it suffices to apply the Main Lemma with $\mathcal{H} = L$, $U(t) = R(\exp itX)$ and $F(z) = (\exp zX) \cdot f$. The fact that the function is holomorphic is verified using Step 1.

Step 3: Let $f \in H$ and $\gamma \in \Gamma^o$. For arbitrary $X \in C^o$ we have

$$I_X(\mathcal{T}(\gamma)f) = \gamma \cdot f.$$

In fact it is clear that $\gamma_1 \cdot (\mathcal{T}(\gamma)f) = \gamma_1\gamma \cdot f$ for all $\gamma_1 \in \Gamma^o$. Set $\mathrm{Re}\, z > 0$ and let z tend to 0. The limit on the left is by definition equal to $I_X(\mathcal{T}(\gamma)f)$. On the other hand the term on the right tends to $\gamma \cdot f$ by Step 1.

Step 4: Let $f \in H$ and $\gamma_1, \gamma_2 \in \Gamma^o$. Then $||(\gamma_1\gamma_2) \cdot f||_L \leq ||\gamma_2 \cdot f||_L$.

In fact, we write γ_1 in the form $g \exp X$ with $g \in G$ and $X \in C^o$. Now we note that

$$||(\gamma_1\gamma_2) \cdot f||_L = ||R(g)((\exp X\, \gamma_2) \cdot f)||_L = ||(\exp X\, \gamma_2) \cdot f||_L.$$

Thus we may assume without loss of generality that $g = 1$. Further we have for arbitrary $z, \mathrm{Re}\, z > 0$, and $f_1 \in H$

$$||\exp zX \cdot f_1||_L \leq ||I_X f_1||_L.$$

If we set $z = 1$ and $f_1 = \mathcal{T}(\gamma_2)f$ we obtain the result from Step 3.

Step 5: Let $f \in H$ and $X \in C^o$. Then $||I_X f||_L = ||f||_H$.

In fact, we have

$$||I_X f||_L = \lim_{z\to 0} || \exp zX \cdot f||_L \leq \sup_{\gamma\in\Gamma^o} ||\gamma\cdot f||_L = ||f||_H.$$

For the proof of the converse inequality we will show that for an arbitrary $\varepsilon > 0$ we have $||I_X f||_L \geq ||f||_H - \varepsilon$. Let us choose $\gamma \in \Gamma^o$ in such a way that $||\gamma\cdot f||_L \geq ||f||_H - \varepsilon$. If z is sufficiently small (and $\mathrm{Re}\, z > 0$) then $\gamma \exp(-zX) \in \Gamma^o$. Apply Step 4 to $\gamma_1 = \gamma \exp(-zX)$ and $\gamma_2 = \exp zX$. Then we obtain $||\gamma\cdot f||_L \leq || \exp zX \cdot f||_L$. Letting z tend to zero we obtain the desired estimate.

Thus for any $X \in C^o$ the operator I_X is an isometric embedding of H into L. In particular H is a pre-Hilbert space w.r.t. the norm $||\cdot||_H$.

Step 6: Let B be an arbitrary compact subset of Γ^o. Then

$$|f(\gamma)| \leq \text{const}\cdot||f||_H \text{ for all } f \in H \text{ and } \gamma \in B,$$

where the constant depends only on B.

This can be proved just as Lemma 2.12 from chapter III in [2].

Step 7: Let $f_1, f_2, \ldots$ be a Cauchy sequence in H. There exists an $f \in H$ such that for arbitrary $\gamma \in \Gamma^o$ we have $\gamma\cdot f_j \to \gamma\cdot f$ in the weak topology of the space L.

In fact, by Step 6 the sequence $\{f_j\}$ converges uniformly on compact subsets to a function f which is holomorphic on Γ^o. For arbitrary $\gamma \in \Gamma^o$ the integral of the function $|\gamma\cdot f|^2$ on a an arbitrary compactum in G is no bigger than $\lim||f_j||^2_H$. This implies that $f \in H$. In the same way it is easy to show that $\{\gamma\cdot f_j\}$ converges weakly to $\gamma\cdot f$.

Step 8: Let $\{f_j\}$ be a Cauchy sequence in H and $f \in H$ be the same as in Step 7. Then $f_j \to f$ in the weak topology of H.

In fact, let us fix an arbitrary $X \in C^o$. It suffices to show that $\{I_X f_j\}$ converges weakly to $I_X f$ in L. Let us set $\mathcal{H} = L$ and $U(t) = R(\exp itX)$. Let $\mathcal{H}_-$ and A_- be defined as in §3. Then we have $I_X(H) \subseteq \mathcal{H}_-$ (cf. Step 2) and for any z, $\mathrm{Re}\, z > 0$, and $h \in H$

$$(\mathrm{Exp} zA_-) I_X h = \exp zX \cdot h. \tag{4}$$

Note that the operators in $\mathcal{H}_-$ of the form $\mathrm{Exp}zA_-$ converge weakly to $\mathbf{1}$ as $z \to 0$, $\mathrm{Re}\, z > 0$, and note also that $(\mathrm{Exp}zA_-)^* = (\mathrm{Exp}\bar{z}A_-)$. Thus for the proof of our claim it suffices now to verify that $(\mathrm{Exp}zA_-)I_X f_j \to (\mathrm{Exp}zA_-)I_X f$ in the weak topology of L . To do this we have to apply (4) to $h = f, f_1, f_2, \ldots$ and use the result of Step 7.

Step 9: H is a Hilbert space.

In fact each Cauchy sequence in H has a weak limit (Step 8). Thus H is complete.

Thus we have proved claim (i) of Theorem A.

Step 10: The set of functions of the form $\mathcal{T}(\gamma)f$ with $f \in H$ and $\gamma \in \Gamma^o$ is dense in H.

In fact, let $X \in C^o$ be fixed and let A_- be the same operator as in Step 8. Set $\gamma_\varepsilon = \exp \varepsilon X$ with $\varepsilon > 0$. Then we have for arbitrary $f \in H$ that

$$I_X \mathcal{T}(\gamma_\varepsilon) f = (\mathrm{Exp}\varepsilon A_-) I_X f \to I_X f$$

for $\varepsilon \to +0$. Therefore we have also $\mathcal{T}(\gamma_\varepsilon)f \to f$ for $\varepsilon \to +0$.

Step 11: The isometric embedding $I_X: H \to L$ does not depend on the choice of $X \in C^o$.

In fact, this follows easily from the results of Step 10 and Step 3.

Now we can write I instead of I_X.

Step 12: Let $\{\gamma_j\}$ be a sequence in Γ^o which converges to $\mathbf{1}$ and let $f \in H$. Then $\gamma_j \cdot f \to If$ in L.

In fact, consider the sequence of operators $\{B_j: H \to L\}$ defined by $B_j f = \gamma_j \cdot f$ $(j = 1, 2, \ldots)$. The results of the Steps 1, 3 and 10 show that the sequence $B_1, B_2, \ldots$ converges to the operator I on a dense subset of vectors in H. But $\|B_j\| \leq 1$ hence $\{B_j\}$ converges strongly to I.

Now we have proved claim (ii) of Theorem A.

Step 13: The operator I commutes with the right translations by elements of G.

In fact, for any $g \in G$ and $X \in C^o$ we have

$$R(g) I \mathcal{T}(g)^{-1} = R(g) I_X \mathcal{T}(g)^{-1} = I_{\mathrm{Ad}\, g \cdot X} = I.$$

Thus we have proved claim (iii) of Theorem A.

Step 14: $\mathcal{T}(\cdot)$ is a holomorphic representation of the semigroup Γ on H.

In fact, it is obvious that $\mathcal{T}(\gamma_1\gamma_2) = \mathcal{T}(\gamma_1)\mathcal{T}(\gamma_2)$ and $\|\mathcal{T}(\gamma)\| \leq 1$ for all $\gamma_1, \gamma_2, \gamma \in \Gamma$. Further for any $\gamma \in \Gamma^o$ the linear functional $f \mapsto f(\gamma)$ is continuous on H (Step 6). Therefore there exists a vector $f_\gamma \in H$ such that $(f, f_\gamma)_H = f(\gamma)$ for all $\gamma \in \Gamma^o$ and $f \in H$. The set of vectors of the form f_γ is total in H (i.e. its linear hull is dense in H) since $(f, f_\gamma)_H \equiv_\gamma 0$ implies that $f = 0$.

Futher, for arbitrary $\gamma \in \Gamma^o$ and $f \in H$ the scalar function $\gamma_1 \mapsto (\mathcal{T}(\gamma_1)f, f_\gamma) = f(\gamma\gamma_1)$ is continuous on Γ and holomorphic on Γ^o. Since $\|\mathcal{T}(\cdot)\| \leq 1$ it follows that $\mathcal{T}$ is weakly continuous on Γ and holomorphic on Γ^o.

It remains to verify that

$$\mathcal{T}(\gamma^\sharp) = \mathcal{T}(\gamma)^*. \tag{5}$$

To this end we note that (5) holds for all $\gamma \in G$ since the restriction of $\mathcal{T}$ to G is obviously unitary. On the other hand are both parts of (5) antiholomorphic functions on Γ^o (cf. [4]). Thus (5) holds for all $\gamma \in \Gamma$.

Now we have proved claim (iv) of Theorem A.

Step 15: For any $X \in C^o$ let us denote by $\mathcal{H}_-(X)$ the subspace of L which in the beginning of §3 we have denoted by $\mathcal{H}_-$. Let us set $\widetilde{H} = \bigcap \mathcal{H}_-(X)$ (the intersection over all $X \in C^o$). Obviously $\widetilde{H}$ is closed and $R(G)$-invariant and moreover it is the biggest invariant subspace of L for which the corresponding unitary representation of the group G is C-admissible.

Step 16: Let us denote by $\widetilde{\mathcal{T}}$ the restriction of the representation R of the group G to the subspace $\widetilde{H} \subseteq L$. According to the Theorem 1.5 one can extend $\widetilde{\mathcal{T}}$ to a certain holomorphic representation $\widetilde{\mathcal{T}}$ of the semigroup Γ on $\widetilde{H}$. Let $\varphi \in \widetilde{H}$ and $\gamma \in \Gamma^o$. Then we claim that $\widetilde{\mathcal{T}}(\gamma)\varphi$ is a smooth function on G.

In fact, it follows from the fact that $\widetilde{\mathcal{T}}$ is holomorphic that $\widetilde{\mathcal{T}}(\gamma)\varphi$ is an analytic vector (hence smooth) vector for the regular representation R. But it is well known that any smooth vector of the regular representation is a smooth function (this follows from the Sobolev embedding theorem).

Step 17: There exist functions $\psi_1, \dots, \psi_n \in L$ and left invariant differential operators $D_1, \dots, D_n$ on G such that, for an arbitrary function $\psi \in L$ which is a C^∞-vector, we have

$$\psi(1) = ((D_1\psi, \psi)_L + \cdots + (D_n\psi, \psi_n)_L.$$

In fact, this follows easily from the fact that the order of the singularity of the δ-function on G is finite.

Step 18: Let us associate a function f on Γ^o to any $\varphi \in \widetilde{H}$ as follows: $f(\gamma) = (\widetilde{\mathcal{T}}(\gamma)\varphi)(1)$. Then $f \in H$.

In fact, applying the result of Step 17 to $\psi = \widetilde{\mathcal{T}}(\gamma)\varphi$, we see that f is holomorphic on Γ^o. Moreover it follows easily from the definition of f that $\gamma.f = \widetilde{\mathcal{T}}(\gamma)\varphi$. Since $||\widetilde{\mathcal{T}}(\gamma)|| \leq 1$ this proves that $f \in H$.

Step 19: We have $\widetilde{H} = I(H)$.

In fact, we know that $\widetilde{\mathcal{T}}(\cdot)$ is strongly continuous in 1. Therefore for $\gamma \to 1$ in Γ^o we have

$$If = \lim \widetilde{\mathcal{T}}(\gamma)\varphi = \varphi.$$

Therefore $\widetilde{H} \subseteq I(H)$. The converse is obvious.

This proves claim (v) of Theorem A and hence Theorem A is completely proved.■

§5. The Theorem B

5.1. Theorem. (Theorem B) *The Hardy space $H = H^2(C)$ is nontrivial for any $C \in Con$. The representation of the group G in H can be decomposed into a direct sum of irreducible unitary representations of the group G with finite multiplicities. The components of this decomposition are precisely all the holomorphic discrete series representations which are C-admissible.*

Proof. This follows easily from claim (v) of Theorem A and the Corollary 1.6. ■

5.2. Remark.

(1) $H^2(C_{min})$ decomposes precisely on the representations of the holomorphic discrete series. Thus $\Gamma^o(C_{min})$ is just the same the complex manifold which is discussed in [6].

(2) Theorem B can be viewed as an analog of the Paley-Wiener Theorem.

§6. The Cauchy-Szegö kernel

Let $C \in Con$. For $\gamma_1, \gamma_2 \in \Gamma^o(C)$ we set $K(\gamma_1, \gamma_2) = f_{\gamma_2}(\gamma_1)$ (where f_γ is defined as in Step 14).

6.1. Theorem. (Theorem C) *Let $C \in Con$. There exists a holomorphic function $K(\gamma)$ on $\Gamma^o(C)$ such that*

(i) *$K(\gamma_1, \gamma_2) = K(\gamma_1 \gamma_2^\sharp)$ for all $\gamma_1, \gamma_2 \in \Gamma^o(C)$.*

(ii) *Let P denote the composition of the orthogonal projection $L^2(G) \to I(H^2(C))$ and the operator $I^{-1}: I(H^2(C)) \to H^2(C)$. Then we have*

$$P\varphi(\gamma) = \int_G \varphi(g) K(g\gamma^\sharp) dg.$$

We omit the simple proof. ■

It is natural to call the function $K(\cdot,\cdot)$ the *Cauchy-Szegö kernel.*

6.2. Theorem. (Theorem D) *Let $C \in Con$, T an arbitrary irreducible C-admissible representation of the group G and $\mathcal{T}$ the corresponding holomorphic representation of the semigroup $\Gamma(C)$. For any $\gamma \in \Gamma^o(C)$ the operator $\mathcal{T}(\gamma)$ is a nuclear operator and so has a trace.*

The proof follows easily from the results of [4]. ■

We note that the first result of such a type was obtained by Graev [9].

6.3. Theorem. (Theorem E) *For any $C \in Con$, the corresponding function $K(\gamma)$, where $\gamma \in \Gamma^o(C)$, can be written as sum*

$$K(\gamma) = \sum \dim T \cdot \mathrm{Tr}\,\mathcal{T}(\gamma)$$

where the sum is taken over all C-admissible representations T of the holomorphic discrete series, $\dim T$ *denotes the formal dimension and $\mathcal{T}$ is the corresponding representation of the semigroup $\Gamma^(C)$. Moreover the series on the right side converges absolutely and uniformly on compact subsets in $\Gamma^o(C)$.*

We omit the proof which consists of a simple estimate of $\dim T$ and $\mathrm{Tr}\,\mathcal{T}(\gamma)$. ■

6.4. Remark. It follows from Theorem E that $K(\gamma^\sharp) = \overline{K(\gamma)}$. Therefore we can replace $K(g\gamma^\sharp)$ in the formula for the projection P (Theorem C) by $K(\gamma g^{-1})$.

§7. Some generalisations

Let $M \subseteq G$ be a closed subgroup such that $M\backslash G$ has an invariant measure. Let us suppose that there exists a complex Lie subgroup $M_C \subseteq G_C$ such that $M_C \cap G = M$. Let $S = M\backslash G$, $S_C = M_C\backslash G_C$ and let $\pi: G_C \to S_C$ denote the canonical projection. Let us fix $C \in Con$ and set $\Omega = \pi(\Gamma^o(C)) = M_C\backslash M_C\Gamma^o(C) \subseteq S_C$. It is important to note that Ω is a complex manifold on which the semigroup $\Gamma(C)$ operates on the right.

The manifold S is contained in the boundary $\partial\Omega = \overline{\Omega}\backslash\Omega$ and its translates S_γ with $\gamma \in \Gamma^o(C)$ are contained in the interior of Ω. Now we can define the Hardy space as in §2. For this it is only necessary to replace $L^2(G)$ by $L^2(S)$. Theorem A can be easily transferred to this case. In particular we find that $H^2(\Omega)$ is that part of $L^2(S)$ which decomposes into C-admissible irreducible representations (however in this case they no longer have to be in the holomorphic discrete series).

For example if M is a discrete subgroup one can take $M_C = M$. Another interesting case is the following: Let M be the subgroup of fixed points of an involutive automorphism of G; then S is a pseudo-Riemannian symmetric space (cf. [6]). It would be very interesting to study the Cauchy-Szegö kernel in this case. (Note that Hardy spaces on pseudo-Riemannian symmetric spaces are studied in the recent paper [13].)

Note also that the Theorems A–E can easily be transferred to the universal covering group $\widetilde{G}$ of G (the point is that our linear group $G \subseteq G_C$ always has a covering with infinitely many sheets, which is specific for the hermitian algebra $\mathfrak{g}$). In such a generalization we have to replace the semigroup $\Gamma(C)$ by its universal covering semigroup.

§8. Non-semisimple groups and the orbit method

As was first noted in [4] there are non-semisimple Lie algebras for which one can find invariant cones and semigroups such that the relation between C-admissible representations of the group and holomorphic representations of the semigroup are still valid. The simplest example is the solvable Lie algebra $\mathfrak{l}$ with basis $\{T, X, Y, Z\}$ and the non-zero commutators $[X,Y] = Z$, $[T,X] = Y$, $[T,Y] = -X$. This Lie algebra, which is called the oscillator algebra, is well known in representation theory (cf. [10]). It contains a continuum of invariant cones whose "bases" are the interiors of paraboloids of revolution. It is easy to transfer all the results of this paper to the algebra $\mathfrak{l}$. It seems very probable that all the results of this paper can be generalized to a wide class of Lie groups. In fact almost all in this direction has been made, see [12] and [14].

Let us remark at this point that there is an obvious connection with Kirillov's orbit method [11]. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that there exist nontrivial invariant convex cones in $i\mathfrak{g}$. Let C be one of them. Let us further suppose that the semigroup $\Gamma(C) = G \exp C$ exists (see [4] for sufficient conditions). Finally we shall assume that G is such that the the orbit method can be applied to it.

Now let T be an irreducible unitary representation of such a group G and Ω denote the corresponding coadjoint orbit. It follows from the functoriality of the correspondence between orbits and representations [11, §15.5] that T is C-admissible if and only if

$$\langle F, iX\rangle \leq 0 \;\; \forall F \in \Omega, \forall X \in C^o.$$

Assuming that Theorem D holds we deduce from the Kirillov universal formula

$$\text{(6)} \qquad \operatorname{Tr}\mathcal{T}(\exp X) = \operatorname{Tr}\big(\operatorname{Exp}\mathcal{T}(X)\big) = p_\Omega^{-1}(X) \int_\Omega e^{2\pi i \langle F,X\rangle} d\beta_\Omega(F) \quad \forall X \in C^o$$

where β_Ω is the canonical measure on the orbit and p_Ω a function which is 1 in 0, cf. [11].

It seems that the right hand side can be analytically continued from $\exp C^o$ to $\Gamma^o(C)$. On the other hand we also note that for hermitian algebras the conclusion of Theorem D has broader applications, namely: The operator $\operatorname{Exp}\mathcal{T}(X)$ is a nuclear operator for all $X \in C^o_{max}$. Similar fact has to be valid for other, non necessarily hermitean, Lie algebras as well.

Comparing (6) with Theorem E and the way of obtaining the Plancherel formula via the orbit method we come to the following conjecture:

Conjecture. *Under previous assumptions, let us suppose in addition that the algebra* $\mathfrak{g}$ *is solvable and that the group* G *is connected and simply connected. Then for the function* $K(\cdot)$ (cf. §6) *we have the following formula* ($X \in C^o$):

$$K(\exp X) = \det\Big(\frac{\operatorname{ad}\frac{X}{2}}{\operatorname{sh}(\operatorname{ad}\frac{X}{2})}\Big) \int_{F\in\mathfrak{g}^*, \langle F, iY\rangle \le 0\ \forall Y\in C} e^{2\pi i\langle F,X\rangle} dF.$$

The assumptions of solvability and simple connectedness have only been made to make sure that there corresponds a representation to any orbit.

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