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Transfinite reductions in orthogonal term rewriting systems (Extended abstract)

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Transfinite Reductions in Orthogonal Term Rewriting Systems

(Extended abstract)

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Abstract. We establish some fundamental facts for infinitary orthogonal term rewriting systems (OTRSs): for strongly convergent reductions we prove the Transfinite Parallel Moves Lemma and the Compressing Lemma. Strongness is necessary as shown by counterexamples. Normal forms (which we allow to be infinite) are unique, in contrast to ω-normal forms. Fair reductions result in ω-normal forms if they are converging, and in normal forms in case of strong convergence.

Rather surprisingly the infinite Church-Rosser Property fails for both converging reductions and strongly converging reductions in OTRSs. Extending the notions head normal form and Böhm tree from Lambda Calculus we prove the infinite Church-Rosser Property for non-unifiable OTRSs. The top-terminating OTRSs of Dershowitz c.s. are examples of non-unifiable OTRSs.

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1. INTRODUCTION

The theory of Orthogonal Term Rewrite Systems (TRS) is now well established within theoretical computer science. Comprehensive surveys have appeared recently in [Der90a, Klo91]. In this paper we consider extensions of the established theory to cover infinite terms and infinite reductions.

1.1. Motivation

At first sight, the motivation for such extensions might appear of theoretical interest only, with little practical relevance. However, it turns out that both infinite terms and infinite rewriting sequences do have practical relevance.

A practical motivation for studying infinite terms and term rewriting arises in the context of lazy functional languages such as Miranda [Tur85] and Haskell [Hud88]. In such languages it is possible to work with infinite terms, such as the list of all Fibonacci numbers or the list of all primes. This style of programming has been advocated by Turner [Tur85], Peyton-Jones [Pey87] and others. Of course the outcome of a particular computation must be finite, but it is pleasant to define such results as finite portions of an infinite term. It would be even more pleasant to know that nice properties (for example Church-Rosserness) hold for infinite as well as finite rewriting, but the standard theory does not tell us this. As we show below, Church-Rosserness is one of several standard results which does *not* hold for infinite rewriting in general, although it does hold for terms which have an infinite normal form (Theorem 4.1.3).

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A second practical motivation for considering infinite reduction sequences arises from the common graph-rewrite based implementations of functional languages. The correspondence between graph rewriting and term rewriting was studied in [Bar87] for acyclic graphs. When cyclic graphs are considered, the correspondence with term rewriting immediately requires consideration of infinite terms and infinite reductions. The correspondence with graphs is the motivation for [Far89].

1.2. Overview

With these motivations in mind, we set out to identify precise foundations for transfinite rewriting. A certain amount of care is needed to establish appropriate notions and we do this in Section 2. One can take a topological approach as in [Der89a,b&90] and consider infinite reduction sequences that are converging to a limit in the metric completion of the space of finite terms. However, converging reductions fail to satisfy some natural properties for orthogonal TRSs. Instead we concentrate on strongly converging reductions as introduced by [Far89], which turn out to be better behaved.

Basic facts for infinitary orthogonal term rewrite systems

	converging reductions	*strongly converging reductions*
Transf. Parallel Moves Lemma	NO (3.1.3)	YES (3.1.2)
Inf. Church-Rosser Property	NO (4.1.1)	NO (4.1.1)
Unique ω–normal forms	NO (4.1.1)	NO (4.1.1)
Unique normal forms	YES (3.3.6)	YES (3.3.6)
Compressing Lemma	NO [Far89], (3.2.1)	YES (3.2.5) partial result in [Far89]
Fair reductions result in	ω-normal forms [Der90b], (3.4.2.i)	normal forms (3.4.2.ii)

(Table 1.1)

In Section 3 we prove the fundamental results for infinitary orthogonal rewrite systems, as summarized in Table 1.1. Then in Section 4 we show the failure of the infinite Church-Rosser Property for both converging and strongly converging reductions. Introducing ideas from Lambda Calculus we eliminate the subterms that have no head normal form by reducing them to $\perp$. The new reduction $\rightarrow_\perp$ has the infinite Church Rosser Property for converging reductions. Normal forms for $\rightarrow_\perp$-reduction are the so called Böhm Trees: they are unique. Finally we show that orthogonal TRSs in which there are no rule in which a left hand side of a rule can be unified with the right hand side have the infinite Church-Rosser Property. This class of orthogonal TRSs includes the top-terminating orthogonal TRSs of Dershowitz c.s.

The present paper is an extended abstract of a longer paper by the same authors [Ken90a]. The full paper contains also an extension of the theory of needed redexes to infinitary orthogonal term rewriting systems and unravels the precise connections between graph rewriting and infinitary term rewriting.

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2. INFINITARY ORTHOGONAL TERM REWRITING SYSTEMS

We briefly recall the definition of a finitary term rewriting system, before we define infinitary orthogonal term rewriting systems involving both finite and infinite terms. For more details the reader is referred to [Der90a] and [Klo91]

2.1. Finitary term rewriting systems

A *finitary term rewriting system* over a signature Σ is a pair (Ter(Σ),R) consisting of the set Ter(Σ) of finite terms over the signature Σ and a set of rewrite rules $R \subseteq \mathrm{Ter}(\Sigma)\times\mathrm{Ter}(\Sigma)$.

The *signature* Σ consists of a countably infinite set Var_Σ of variables (x,y,z,...) and a non-empty set of function symbols (A,B,C,...,F,G,...) of various finite arities ≥ 0. Constants are function symbols with arity 0. The set Ter(Σ) of *finite* terms (t,s,...) over Σ can be defined as usual: the smallest set containing the variables and closed under function application.

The set O(t) of *occurrences* (or positions) in t is defined by induction to the structure of t as follows: $O(t) = \{< >\}$ if t is a variable and $O(t) = \{< >\} \cup \{<i,u> \mid 1\le i\le n \text{ and } <u>\in O(t_i)\}$ if t is of the form $F(t_1,...,t_n)$. If $u\in O(t)$ then the subterm t/u at occurrence u is defined as follows: $t/< > = t$ and $F(t_1,...,t_n)/<i,u> = t_i/u$. The *depth* of a subterm of t at occurrence u is the length of u.

Contexts are terms in $\mathrm{Ter}(\Sigma\cup\{\square\})$, in which the special constant □, denoting an empty place, occurs exactly once. Contexts are denoted by C[ ] and the result of substituting a term t in place of □ is $C[t]\in \mathrm{Ter}(\Sigma)$. A *proper* context is a context not equal to □.

Substitutions are maps $\sigma:\mathrm{Var}_\Sigma\rightarrow\mathrm{Ter}(\Sigma)$ satisfying $\sigma(F(t_1,...,t_n)) = F(\sigma(t_1),...,\sigma(t_n))$.

The set R of *rewrite rules* contains pairs (l,r) of terms in Ter(Σ), written as $l \rightarrow r$, such that the left-hand side l is not a variable and the variables of the right-hand side r are contained in l. The result l^σ of the application of the substitution of σ to the term l is called an instance of l. A *redex* (reducible expression) is an instance of a left-hand side of a rewrite rule. A *reduction step* $t \rightarrow s$ is a pair of terms of the form $C[l^\sigma] \rightarrow C[r^\sigma]$, where $l \rightarrow r$ is a rewrite rule in R. Concatenating reduction steps we get a *finite reduction sequence* $t_0 \rightarrow t_1 \rightarrow ... \rightarrow t_n$, which we also denote by $t_0 \rightarrow_n t_n$, or an infinite reduction sequence $t_0 \rightarrow t_1 \rightarrow ...$.

2.2. Infinitary orthogonal term rewriting systems

An *infinitary term rewriting system* over a signature Σ is a pair $(\mathrm{Ter}^\infty(\Sigma),R)$ consisting of the set $\mathrm{Ter}^\infty(\Sigma)$ of finite and infinite terms over the signature Σ and a set of rewrite rules $R\subseteq\mathrm{Ter}(\Sigma)\times\mathrm{Ter}^\infty(\Sigma)$. We don't consider rewrite rules with infinite left handsides, but right hand sides may be infinite in order to be able to interpret various liberal forms of graph rewriting in infinitary term rewriting. In [Der90b] only finite left and right hand sides are considered.

It takes some elaboration to define the set $\mathrm{Ter}^\infty(\Sigma)$ of *finite and infinite terms*. Finite terms may be represented as finite trees, well-labelled with variables and function symbols. Well-labelled means that a node with n ≥ 1 successors is labelled with a function symbol of arity n and that a node with no successors is labelled either with a constant or a variable. Now *infinite terms* are infinite well-labelled trees with nodes at finite distance to the root. Substitutions, contexts and reduction steps generalize trivially to the set of infinitary terms $\mathrm{Ter}^\infty(\Sigma)$.

To introduce the *prefix ordering* $\leq$ on terms we extend the signature Σ with a fresh symbol Ω. The prefix ordering $\leq$ on $Ter^{\infty}(\Sigma \cup \{\Omega\})$ is defined inductively: $x \leq x$ for any variable x, $\Omega \leq t$ for any term t and if $t_1 \leq s_1, ..., t_n \leq s_n$ then $F(t_1,...,t_n) \leq F(s_1,...,s_n)$.

If all function symbols of Σ occur in R we will write just R for $(Ter^{\infty}(\Sigma),R)$. The usual properties for finitary TRSs extend verbatim to infinitary TRSs:

2.2.1. DEFINITION. Let R be an infinitary TRS.

(i) R is *left-linear* if no variable occurs more than once in a left-hand side of R's rewrite rules;

(ii) (informally) R is *non-overlapping* (or non-ambiguous) if non-variable parts of different rewrite rules don't overlap and non-variable parts of the same rewrite rule overlap only entirely:

(ii') (formally) R is *non-overlapping* if for any two left hand sides s and t, any occurrence u in t, and any substitutions σ and $\tau: Var_{\Sigma} \rightarrow Ter(\Sigma)$ it holds that if $(t/u)^{\sigma} = s^{\tau}$ then either t/u is a variable or t and s are left hand sides of the same rewrite rule and u is the empty occurrence $< >$, the position of the root.

(iii) R is *orthogonal* if R is both left-linear and non-overlapping.

It is well-known (cf. [Ros73], [Klo91]) that finitary orthogonal TRSs satisfy the finitary Church-Rosser property, i.e., ${}^*\!\leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^*\!\leftarrow$, where $\rightarrow^*$ is the transitive, reflexive closure of the relation $\rightarrow$. It is obvious that infinitary orthogonal TRSs inherit this finitary property.

In the present infinitary context it is natural to define that a term is a *normal form* if it contains no redexes, just like in the finitary context. A term t has a normal form s if there is a reduction $t \rightarrow_{\alpha} s$. Dershowitz, Kaplan and Plaisted [Der89a, Der89b and Der90b] consider a weaker, more liberal notion of normal form: the ω-*normal forms*. An ω-normal form is a term such that if this term can reduce, then it reduces in one step to itself. One sees easily that restricted to finite terms normal forms and ω-normal forms are already different concepts: in the TRS with rule $A \rightarrow A$ the term A is an ω-normal form, but not a normal form.

2.3. Converging and strongly converging transfinite reductions

Generalizing the finite situation we would like to express that there is a reduction of length $\alpha+1$ that transforms t_0 into t_{α}, where α may be any ordinal. Compare the following three reductions of length ω, the corresponding TRSs are easy to imagine:

(i) $A \rightarrow B \rightarrow A \rightarrow B \rightarrow ...,$

(ii) $C \rightarrow S(C) \rightarrow S(S(C)) \rightarrow ...,$

(iii) $D(E) \rightarrow D(S(E)) \rightarrow D(S(S(E))) \rightarrow ... .$

Clearly in the first reduction A will not be transformed in the limit to anything fixed, in contrast to C and D(E) in the second and third reduction. It is tempting to say that the limit of C will be S^{ω}, an infinite reduction of S (plus all the necessary brackets), and similar D(E) should have as limit $D(S^{\omega})$. Cauchy convergence is the natural formalism in which to express all this.

The set $Ter(\Sigma)$ of finite terms for a signature Σ can be provided with an ultra-metric d: $Ter(\Sigma) \times Ter(\Sigma) \rightarrow [0,1]$ (cf. e.g. [Arn80]). The distance d(t,s) of two terms t and s is 0 if t and s are equal, and otherwise 2^{-k}, where $k \in N$ is the largest number such that the labels of all nodes of s and t at depth less than or equal to k are equally labelled. The metric completion of $Ter(\Sigma)$ is isomorphic to the set of infinitary terms $Ter^{\infty}(\Sigma)$ (cf. [Arn80])

In the complete metric space $Ter^\infty(\Sigma)$ all Cauchy sequences of ordinal length α have a limit. We will now recall the transfinite converging reductions by Dershowitz, Kaplan and Plaisted [Der90b].

2.3.1. DEFINITION. A *sequence* of length α is a set of elements indexed by some ordinal $\alpha \geq 1$: notation $(t_\beta)_{\beta<\alpha}$. Instead of $(t_\beta)_{\beta<\alpha+1}$ we often write $(t_\beta)_{\beta\leq\alpha}$.

2.3.2. DEFINITION. By induction to the ordinal α we define when a sequence $(t_\beta)_{\beta\leq\alpha}$ is a *converging sequence* towards its limit t_α (notation: $t_0 \overset{c}{\rightarrow}_{\alpha} t_\alpha$):

(i) $t_0 \overset{c}{\rightarrow}_{0} t_0$;
(ii) $t_0 \overset{c}{\rightarrow}_{\beta+1} t_{\beta+1}$ if $t_0 \overset{c}{\rightarrow}_{\beta} t_\beta$;
(iii) $t_0 \overset{c}{\rightarrow}_{\lambda} t_\lambda$ if $t_0 \overset{c}{\rightarrow}_{\beta} t_\beta$ for all $\beta<\lambda$ and $\forall\varepsilon>0\, \exists\beta<\lambda\, \forall\gamma\, (\beta<\gamma<\lambda \rightarrow d(t_\gamma,t_\lambda) < \varepsilon)$.

This definition of transfinite convergence is an instance of the so-called Moore-Smith convergence over nets (cf. for instance [Kel55]). Limits are unique: if the topological space is a Hausdorff space then each net in the space converges to at most one point; the spaces $Ter(\Sigma)$ and $Ter^\infty(\Sigma)$ are Hausdorff spaces.

2.3.3. DEFINITION. A *reduction of length* $\alpha \geq 1$ is a sequence $(t_\beta)_{\beta<\alpha}$ such that $t_\beta \rightarrow t_{\beta+1}$ for all β such that $\beta+1 < \alpha$. The redex contracted $t_\beta \rightarrow t_{\beta+1}$ will be denoted by R_β, its depth as subterm of t_β by d_β.

We will now define strong reductions as reductions in which the depth of the reduced redexes tends to infinity. We present the definition for reductions of arbitrary transfinite length.

2.3.4. DEFINITION. By induction to the ordinal $\alpha \geq 1$ we define when a reduction $(t_\beta)_{\beta<\alpha}$ is a *strong* reduction:

(i) $(t_\beta)_{\beta<1}$ is a strong reduction;
(ii) $(t_\gamma)_{\gamma<\beta+1}$ is a strong reduction if $(t_\gamma)_{\gamma<\beta}$ is a strong reduction;
(iii) $(t_\gamma)_{\gamma<\lambda}$ is a strong reduction if for all $\beta<\lambda$ the reduction $(t_\gamma)_{\gamma<\beta}$ is strong and $\forall d>0\, \exists\beta<\lambda\, \forall\gamma\, (\beta\leq\gamma<\lambda \rightarrow d_\gamma>d)$.

3.1.4. DEFINITION. A *strongly converging* reduction is a converging sequence that is a strong reduction.

Of importance for the theory of infinitary term rewriting are the strongly converging reductions. Therefore we denote a strongly converging reduction $(t_\beta)_{\beta\leq\alpha}$ by $t_0 \rightarrow_\alpha t_\alpha$. By $t \rightarrow_{\leq\alpha} s$ we denote the existence of a strong reduction of length less than or equal to α converging towards limit s. We use a similar notation $t \overset{c}{\rightarrow}_{\leq\alpha} s$ for converging reductions of length less than or equal to α.

The second example in the beginning of this section is an example of a strongly converging reduction. Other examples of strongly converging reductions are found in (3.2.1.ii) and (4.1.1).

2.4. Counting steps in reductions

Convergent transfinite reductions exist of any length. Consider for example the TRS with the single rule $A \rightarrow A$. Reductions of the form $A \overset{c}{\rightarrow}_{\alpha} A$ are converging for any ordinal α. However these sequences are not strongly convergent. The example $A \overset{c}{\rightarrow}_{\alpha} A$ shows also that in a converging reduction any number of reduction steps may be performed below some depth. For strongly converging reductions this is different:

2.4.1. THEOREM. *If* $t_0 \rightarrow_\lambda t_\lambda$ *is strongly convergent, then the number of steps in* $t_0 \rightarrow_\lambda t_\lambda$ *reducing a redex at depth* $\leq n$ *is finite.*

PROOF. Assume $t_0 \rightarrow_\lambda t_\lambda$ is strongly convergent. As this reduction is strong there is a last step $t_\alpha \rightarrow t_{\alpha+1}$ at which a redex is contracted at depth $\leq n$. Consider the initial segment $t_0 \rightarrow_\alpha t_\alpha$, and repeat the argument. By the well-ordering of the ordinals (no infinite descending chains of ordinals) this process stops in finitely many steps. □

We have the following corollary:

2.4.2. COROLARY. *A strongly converging transfinite reduction has countable length.*

PROOF. By the previous Theorem 2.4.1 a strongly convergent transfinite reduction can only perform finitely many reductions at any given depth $d \in \mathbb{N}$. □

For any countable ordinal α it is possible to construct a strongly converging reduction of length α. Exercise: construct such reductions in the Binary Tree TRS: $C \rightarrow B(C,C)$.

We have a similar theorem for the number of reduction steps that somehow have been relevant or have contributed to a particular occurrence in the final term of a reduction sequence. To this end we generalize Huet's and Lévy's notion [Hue79] of preservation of occurrences of a term to strongly convergent reductions in left-linear TRSs:

2.4.3. DEFINITION. Let $t_0 \rightarrow_\alpha t_\alpha$ be a strongly convergent reduction in a left-linear TRS.

(i) A strongly converging reduction $t_0 \rightarrow_\alpha t_\alpha$ *preserves* an occurrence u in t_0 if no reduction step of the reduction is performed at an occurrence which is a proper prefix of u.

(ii) Let u be an occurrence of t_α. The set of steps of $t_0 \rightarrow_\alpha t_\alpha$ which *contribute to* u is defined thus: If no step of $t_0 \rightarrow_\alpha t_\alpha$ is performed at an occurrence which is a prefix of u, then no step of $t_0 \rightarrow_\alpha t_\alpha$ contributes to u. Otherwise, since $t_0 \rightarrow_\alpha t_\alpha$ is strongly convergent, there must be a last step $t_\beta \rightarrow t_{\beta+1}$ reducing R_β at an occurrence v that is a prefix of u. Then $t_\beta \rightarrow t_{\beta+1}$ contributes to u, and every step of $t_0 \rightarrow_\beta t_\beta$ which contributes to v or to any node of t_β pattern-matched by R_β (the variable free part of the left hand side of the rule for which R_β is a redex) contributes to u.

(iii) The set of steps of $t_0 \rightarrow_\alpha t_\alpha$ which contribute to a set of occurrences U of t_α is the set of steps which contribute to any member of U.

2.4.4. THEOREM. *Let* $t_0 \rightarrow_\alpha t_\alpha$ *be a strongly convergent reduction in a left-linear TRS. For every finite prefix of* t_α, *there are only finitely many steps in* $t \rightarrow_\alpha t_\alpha$ *contributing to all occurrences of the prefix.*

PROOF. A variation on the proof of Theorem 2.4.2 works. The crucial step in repeating the proof of 2.4.2 is the insight that there is a last step $t_\beta \rightarrow t_{\beta+1}$ contributing to the prefix. □

3. FUNDAMENTAL FACTS OF INFINITARY TERM REWRITING

From now on we consider infinitary orthogonal term rewriting systems, except in 3.4.

3.1. The Transfinite Parallel Moves Lemma

In $t \to s$ let s be obtained by contraction of the redex S in t. Recall the notation $u\backslash S$ of the set descendants of a redex occurrence u of t in the contraction of S (cf. [Hue79]). Descendance can be extended to transfinite reductions:

3.1.1. DEFINITION. Let $t_0 \to_\alpha t_\alpha$ be a transfinite strongly converging reduction such that for all $\beta<\alpha$ t_β reduces to $t_{\beta+1}$ by contraction of the redex R_β. By induction to the ordinal α we define the set of descendants $u\backslash\alpha$ in t_α that descend from the redex occurrence u in t_0:

(i) $u\backslash 0 = \{u\}$

(ii) $u\backslash(\beta+1) = \bigcup\{v\backslash R_\beta \mid v \in u\backslash\beta\}$

(iii) $u\backslash\lambda = \{v \mid \exists\beta<\lambda\ \forall\gamma\,(\beta\leq\gamma<\lambda \to v\in u\backslash\gamma)\}$

3.1.2. TRANSFINITE PARALLEL MOVES LEMMA.

Let $t_0 \to_\alpha t_\alpha$ *be a strongly converging reduction sequence of* t_0 *with limit* t_α *and let* $t_0 \to s_0$ *be a reduction of a redex* S *of* t_0. *Then for each* $\beta \leq \alpha$ *a term* s_β *can be constructed by outermost contraction of all descendants of* S *in* t_β *such that* $s_\beta \to^* s_{\beta+1}$ *for each* $\beta \leq \alpha$ *and all these reductions together form a strongly converging reduction from* s_0 to s_α.

$$\begin{tikzcd}
t_0 \arrow[r] \arrow[d, "S"'] & t_1 \arrow[r] \arrow[d, "*"] & \cdots & t_\beta \arrow[r, "R_\beta"] \arrow[d, "\leq\omega"] & t_{\beta+1} \arrow[r] \arrow[d, "\leq\omega"] & \cdots & t_\alpha \arrow[d, "\leq\omega"] \\
s_0 \arrow[r, "*"] & s_1 \arrow[r, "*"] & \cdots & s_\beta \arrow[r, "*"] & s_{\beta+1} \arrow[r, "*"] & \cdots & s_\alpha
\end{tikzcd}$$

(Figure 3.1)

PROOF. First note that outermost reduction of a finite or an infinite number of disjoint redexes in some term gives a strongly converging reduction. hence all vertical reductions in Figure 3.1 are strongly converging.

We prove the lemma by induction to the ordinal α. The case with zero is easy. Next, let α be of the form $\beta+1$. This goes like the traditional proof, taking care of the possible infinite right hand sides. Finally, let α be a limit ordinal λ. Assume as induction hypothesis that we have the Transfinite Parallel Moves Lemma for $\beta < \lambda$. There are two possibilities: there exists a $\beta < \lambda$ such that the actual length of the reduction sequence $t_\beta \to_{\leq\omega} s_\beta$ is zero, that is there are no descendants of S in t_β, or there is no such β. The first possibility is easy: we find that $t_\gamma = s_\gamma$ for all γ with $\beta \leq \gamma < \lambda$. It follows that s_0 strongly converges to s_λ.

So let us pursue the second possibility and suppose there is no such β.

Let $(v_\beta)_{\beta\leq\mu}$ be the reduction of the bottom line of Figure 4.1 obtained by refining the sequence $(s_\beta)_{\beta\leq\lambda}$ with reductions $s_\beta \to_{\leq\omega} s_{\beta+1}$ for each $\beta < \alpha$.That such a μ exists follows by an exercise on well-orderings: refining a well-ordering with well-orderings gives again a well-ordering. In order to conclude $s_0 = v_0 \to_\mu v_\mu = s_\lambda$ we have to show: (i) the reduction $(v_\beta)_{\beta\leq\mu}$ is strong, (ii) the reduction $(v_\beta)_{\beta\leq\mu}$ is converging.

PROOF OF (i): By induction clause in the definition of strong sequence we only have to show $\forall d{>}0\ \exists\beta{<}\mu\ \forall\gamma\ (\beta{<}\gamma{<}\mu \to dv_\gamma{>}d)$ to conclude that $(v_\beta)_{\beta\leq\mu}$ is strong.

Observe that the depth of the redexes contracted in $s_\beta \to_{\leq\omega} s_{\beta+1}$ (the descendants of redex R_β under $t_\beta \to_{\leq\omega} s_\beta$) is at least $dt_\beta - h$, where dt_β is the depth of R_β in t_β and h is the maximal distance in the left hand side of the rule applied to R from its root to any variable. As the depth of the redexes R_β tends to infinity with β tending to μ we get $\forall d{>}0\ \exists\beta{<}\mu\ \forall\gamma\ (\beta{<}\gamma{<}\mu \to dv_\gamma{>}d)$.

PROOF OF (ii): By induction hypothesis it suffices to show that $\forall\varepsilon{>}0\ \exists\beta{<}\mu\ \forall\gamma\ (\beta{<}\gamma{<}\mu \to d(v_\gamma,s_\lambda) < \varepsilon)$. So, let $\varepsilon > 0$. Let $2^{-k} < \varepsilon$ for some natural number k.

Let $t_\lambda = r_0 \to r_1 \to \ldots \to_{\leq\omega} s_\mu$ be a (possible finite) reduction obtained by outermost contraction of the descendants of R in t_λ. Consider the rule $l \to r$ of which R is a redex. Let h be the maximum of the differences of the depth of a variable in r and the depth of the same variable in l.

For some N large enough we have $d(r_n,s_\lambda) \leq 2^{-k}$ for $n \geq N$. For some ξ large enough all the descendants of S in t_λ contracted in the reduction up to r_{N+1} are present in all t_γ for $\gamma \geq \xi$. For some ζ large enough the redexes reduced in t_γ for $\gamma \geq \zeta$ are at depth larger than k. Hence for $\gamma \geq \max(\zeta,\xi)$ the initial part of t_γ and t_λ up to level k+1 are equal.

If we now contract the (disjoint!) descendants of R in t_γ and in t_λ, and compare the result s_γ and s_λ, then we see that up to level (k+1)-h the terms s_γ and s_λ are equal. By (ii) we find that for η large enough the depth of the redexes contracted in $v_\gamma \to v_{\gamma+1}$ for $\gamma \geq \eta$ is at least k. So finally if we take $\beta = \max(\zeta,\xi,\eta)$ then up to level (k+1)-h the terms v_γ and s_λ are identical for $\gamma \geq \beta$.

Hence for any $\varepsilon > 0$ there is a β such that for $\beta \leq \gamma < \mu$ the distance of v_γ and s_λ is smaller than ε.

END PROOF OF (ii) □

It seems natural to ask whether a transfinite parallel moves lemma exists for the larger class of converging reductions. The following example shows that the construction embodied in the Transfinite Parallel Moves Lemma for strongly converging reductions does not generalize.

3.1.3. COUNTEREXAMPLE.

Rules: $A(x,y) \to A(y,x)$, $C \to D$

Sequences:

$$\begin{tikzcd}
A(C,C) \arrow[r] \arrow[d] & A(C,C) \arrow[r] \arrow[d] & A(C,C) \arrow[r] \arrow[d] & A(C,C) \arrow[r] \arrow[d] & \ldots \arrow[r, "c", "\omega"'] & A(C,C) \arrow[d, phantom, "\not\downarrow"] \\
A(C,D) \arrow[r] & A(D,C) \arrow[r] & A(C,D) \arrow[r] & A(D,C) \arrow[r] & \ldots & \text{NO LIMIT}
\end{tikzcd}$$

The bottom infinite reduction obtained by standard projection over the one step reduction $C \to D$ does not converge to any limit. □

However it seems possible that by altering the construction, perhaps by considering a more liberal notion of descendant, a parallel moves lemma does exist for transfinite converging reductions. After all, every term occurring in the counterexample can reduce to A(D,D).

3.2. The Compressing Lemma

In this section we will prove the Compressing Lemma for infinitary left-linear TRSs: if $t \rightarrow_{\alpha} s$ is strongly converging, then $t \rightarrow_{\leq\omega} s$. That is: any strongly converging reduction from t into s of length $\alpha+1$ can be compressed in a reduction of length lesser or equal than $\omega+1$.

The following Table (3.1) collects counterexamples against compressing lemmas with weaker conditions:

Validity of Compressing Lemma under various conditions:

	converging	**strongly converging**
left-linear	NO overlapping: [Far89], (3.2.1.i) non-overlapping: (3.2.1.i)	YES (3.2.5)
non-left-linear	NO [Der89a], (3.2.1.i)	NO [Der89a]
	[Der89a] is presented in (3.2.1.ii)	

(Table 3.1)

3.2.1. COUNTEREXAMPLES.

(i) Example against a compressing lemma for converging reductions in orthogonal TRSs.

Rules: $A(x) \rightarrow A(B(x))$, $B(x) \rightarrow E(x)$

Sequence: $A(C) \rightarrow_{\omega} A(B(B^{\omega})) \rightarrow A(E(B^{\omega}))$.

Note: A(C) cannot reduce to $A(E(B^{\omega}))$ in $\leq \omega$ steps. The reduction is converging but not strong.

(ii) Example of [Der89a] against a compressing lemma for strongly converging reductions in non-left-linear, non-overlapping TRSs.

Rules: $A \rightarrow S(A)$, $B \rightarrow S(B)$, $H(x,x) \rightarrow C$

Sequence: $H(A,B) \rightarrow^{*} H(S(A),S(B)) \rightarrow^{*} H(S(S(A)),S(S(B))) \rightarrow_{\omega} H(S^{\omega},S^{\omega}) \rightarrow C$

Note: The term H(A,B) of Dershowitz and Kaplan (cf. [Der89a]) can reduce via the limit $H(S^{\omega},S^{\omega})$ to C. But not $H(A,B) \rightarrow_{\leq\omega} C$. The sequence is strongly converging. □

The proof of the Compressing Lemma will go in two steps. First we compress the reduction sequence up to the last limit ordinal to a sequence of length $\leq \omega+1$. Then, if necessary, we apply the Compressing Lemma for $\omega+1$. The Compressing Lemma for $\omega+1$ is simple to prove:

3.2.2. COMPRESSING LEMMA for $\omega+1$. *If* $t \rightarrow_{\omega+1} s$ *is strongly converging, then* $t \rightarrow_{\leq\omega} s$.

PROOF. Suppose $t_0 \rightarrow_{\omega} t_{\omega}$ is strongly converging and $t_{\omega} \rightarrow s$. Let the redex R_s contracted in $t_{\omega} \rightarrow s$ have depth S. By strongness there exists an N such that for $n \geq N$ the depth of the redex R_n contracted in $t_n \rightarrow t_{n+1}$ is larger than S+h, where h is the height of the non-variable part of the redex R_s. The set of descendants in t_m of the copy of R_s in t_N is a singleton for all $m>N$. We will now construct a strongly converging reduction $t_0 \rightarrow_{\leq\omega} s$. For the first N steps we take $t_0 \rightarrow t_1 \rightarrow \ldots \rightarrow t_N$. Then we reduce $t_N \rightarrow s_N$ by contracting R_s in t_N. We apply the projection method of the Parallel Moves Lemma to $t_N \rightarrow s_N$ and $t_N \rightarrow_\omega t_\omega$. Thus we obtain a strongly converging reduction $t \rightarrow_{\leq\omega} s$. □

The proof of the Compressing Lemma for limit ordinals is more involved and needs some preliminary theory.

3.2.3. LEMMA. *Let* $t_0 \rightarrow_\lambda t_\lambda$ *be a strongly convergent reduction. Let* s *be a finite prefix of* t_ω. *Then the reduction* $t_0 \rightarrow_\lambda t_\lambda$ *can be factorized in a strongly convergent reduction* $t_0 \rightarrow^* t_1 \rightarrow_\gamma t_\lambda$ *such that all steps in* $t_0 \rightarrow^* t_1$ *contribute to the prefix* s *and there are no steps contributing to* s *in* $t_1 \rightarrow_\gamma t_\lambda$.

PROOF. By Theorem 2.4.4 there are finitely many steps that contribute to the prefix s. We will handle them one by one. Let R_0 be the contracted redex of the first of these finitely many steps, say in step $t_\beta \rightarrow t_{\beta+1}$. If R_0 is not a redex in t_0, then somewhere in the reduction R_0 has been constructed. But then the reduction step using R_0 was not the first reduction step contributing to the finite prefix s. Hence R_0 is a redex of t_0. In $t_0 \rightarrow_{\beta+1} t_{\beta+1}$ there are no terms containing multiple copies of R_0 in t_0: otherwise $t_\beta \rightarrow t_{\beta+1}$ would not have been the first step contributing to the finite s of t_ω. Also no terms contain no copy of R_0, for the same reason. So applying the projection method of the Transfinite Parallel Moves Lemma, we get a strongly converging reduction $r_0 \rightarrow^* r_1 \rightarrow^* r_2 \rightarrow \ldots r_\beta$, where each r_α is obtained from t_α ($0 \leq \alpha \leq \beta$) by reduction of the unique occurrence of the descendant of the redex R_0. By construction r_β equals $t_{\beta+1}$. Hence we have factorized $t_0 \rightarrow_\lambda t_\lambda$ in $t_0 \rightarrow r_0 \rightarrow_\delta t_\beta \rightarrow_\gamma t_\lambda$. Clearly the remaining n-1 steps contributing to the prefix s are performed beyond t_β, so that sufficient repetition of the construction yields the desired factorization. □

3.2.4. COMPRESSING LEMMA for limit ordinals. *If* $t_0 \rightarrow_\lambda t_\lambda$ *is strongly convergent, then there exists a strongly convergent reduction* $t_0 \rightarrow_{\leq\omega} t_\lambda$.

PROOF. Choose some depth n. Apply Theorem 2.4.4 to find the finitely many steps of $t_0 \rightarrow_\lambda t_\lambda$ contributing to occurrences of t_λ at depth $\leq n$. With an appeal to Lemma 3.2.3 perform the finitely many contributing steps first to find a strongly converging reduction $t_0 \rightarrow^* t_1 \rightarrow_\gamma t_\lambda$ where all steps in $t_0 \rightarrow^* t_1$ contribute to occurrences of t_λ at depth $\leq n$ and no steps contribute to occurrences of t_λ at depth $\leq n$ in $t_1 \rightarrow_\gamma t_\lambda$.

Now choose a bigger n and repeat the argument for $t_1 \rightarrow_\alpha t_\lambda$, getting a sequence $t_1 \rightarrow^* t_2 \rightarrow_\beta t_\lambda$ for some $\beta \leq \alpha$. Repeat ad infinitum: we obtain the sequence $t_0 \rightarrow^* t_1 \rightarrow^* t_2 \rightarrow^* \ldots$ which by construction is a strongly converging reduction to t_λ. □

3.2.5. COMPRESSING LEMMA. *For any ordinal* α *if* $t \rightarrow_\alpha t_\alpha$ *is strongly convergent, then there exists a strongly convergent reduction* $t \rightarrow_{\leq\omega} t_\alpha$.

PROOF. Together 3.2.4 and 3.2.2 establish the Compressing Lemma. Every infinite ordinal α has the form $\lambda+n$, for a limit ordinal λ and a finite n. For any strongly convergent sequence $t \rightarrow_{\lambda+n} t_\alpha$, we apply Theorem 3.3.4 to the first λ steps, to obtain a sequence $t \rightarrow_{\leq \omega+n} t_\alpha$, then apply Theorem 3.2.2 n times to obtain $t \rightarrow_{\leq\omega} t_\alpha$. □

3.3. The unique normal form property

We will show for infinitary orthogonal TRSs that each term has at most one normal form. This is not the case for ω-normal forms: Example 4.1.1 shows that the unique ω-normal form property does not hold in general. Yet, there is a certain parallel between normal forms and ω-normal forms: the limit of a fair converging reduction will be an ω-normal form and the limit of a fair, strongly converging reduction will be a normal form.

To obtain these results we introduce the notion of a stable reduction. Informally, an infinite reduction will be called *stable* if the sequence of stable prefixes of its terms converges to its limit: a stable prefix of a term t is a prefix of t such that no occurrence of that prefix can become an occurrence of a redex in any reduction sequence starting from t. Stable reductions will be strongly converging. The formal definition of stability requires some preliminaries.

3.3.1. DEFINITION. (i) A prefix $s \leq t$ is called *stable* with respect to a reduction if no occurrence of s becomes an occurrence of a redex during that reduction.

(ii) A prefix $s \leq t$ is called *stable* if s is stable for all possible reduction sequences from t.

3.3.2. PROPOSITION. *In an orthogonal TRS: If a prefix* t *of* t_0 *is stable with respect to a strong reduction from* t_0 *which converges to normal form, then it is stable.*

PROOF. Without loss of generality consider the prefix $F(\Omega,\ldots,\Omega)$ consisting only of the top symbol of $t_0 = F(t_1,\ldots,t_n)$. Assume $F(\Omega,\ldots,\Omega)$ is stable with respect to a strong reduction $\mathcal{B}$, which converges to normal form, say s, and not stable for some other $\mathcal{B}'$. Then at some position in $\mathcal{B}'$ the symbol F is reducible for the first time. Let $\mathcal{B}^*$ be the finite reduction up to this point. We apply the Transfinite Parallel Moves Lemma repeatedly to $\mathcal{B}$ and $\mathcal{B}^*$. We obtain a strongly convergent reduction of t_0 to the same normal form s, which does not reduce F, and in which the terms after $\mathcal{B}^*$ all have the prefix $F(\Omega,\ldots,\Omega)$. By orthogonality, the redex at the root of t_0 cannot be destroyed, the redex at F is still present in the normal form s of t_0. Contradiction. Hence such a $\mathcal{B}'$ does not exist. □

3.3.3. DEFINITION. Let $\Sigma(t)$ denotes the maximal stable prefix of t. A converging reduction $t_0 \rightarrow_{\leq\omega} t_\omega$ is called stable if $\forall d \exists N \forall k \geq N \ |\Sigma(t_k)| > d$, where $|t|$ denotes the minimal distance of an occurrence of Ω in t to the root, if there is any, otherwise $|t| = \infty$.

Stability is a very strong condition. The limit of an infinite stable reduction sequence is a normal form, from which it easily follows that stable reduction is Church-Rosser. The proof of the following lemma is routine and therefore omitted.

3.3.4. LEMMA. (i) *If* $t \rightarrow s$ *then* $\Sigma(t) \leq \Sigma(s)$.

(ii) *For reductions: stable* $\Rightarrow$ *strongly convergent* $\Rightarrow$ *convergent. But not conversely.*

(iii) *The limit of a stable reduction sequence is a normal form.* □

3.3.5. THEOREM. *The following are equivalent:*

(i) $t \rightarrow_{\leq\omega} s$ *is a converging reduction to normal form;*

(ii) $t \rightarrow_{\leq\omega} s$ *is a strong converging reduction to normal form;*

(iii) $t \rightarrow_{\leq\omega} s$ *is a stable reduction*

PROOF. It is trivial to see that (iii) ⇒ (ii) ⇒ (i).
(i) ⇒ (ii): Let $t \rightarrow_{\leq\omega} s$ be converging to normal form. Suppose it is not strongly converging to normal form. Then there must be some depth d such that from some t_i onwards, every term has a redex at depth d. Since arities are finite, this implies that at some occurrence u, infinitely many reductions are performed. But then convergence implies that u is also an occurrence of a redex in the limit, contrary to hypothesis. (In fact, the implication is still true when operators of infinite arity are allowed.)
(ii) ⇒ (iii): Let $t \rightarrow_{\leq\omega} s$ be a strongly converging reduction normal form. Let t'_i be the largest prefix of the i'th term t_i which is stable with respect to the remainder of the sequence. Then by Proposition 3.3.2, t'_i is equal to the largest stable prefix $\Sigma(t_i)$ of t_i. Since the sequence is strongly convergent, the depths of the prefixes t'_i grow without bound, hence the sequence $t \rightarrow_{\leq\omega} s$ is stable. □

3.3.6. UNIQUE NORMAL FORM PROPERTY. *Normal forms are unique in orthogonal TRSs.*

PROOF. Suppose a term t admits two converging reductions $t \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots \overset{c}{\rightarrow}_{\leq\omega} s$ and $t \rightarrow r_1 \rightarrow r_2 \rightarrow \ldots \overset{c}{\rightarrow}_{\leq\omega} r$ to normal form. By Theorem 3.3.5 these reductions are stable. By the finite Church-Rosser property, for each n there exists u_n such that $s_n \rightarrow^* u_n$ and $r_n \rightarrow^* u_n$. Hence we get a reduction $t \rightarrow^* u_1 \rightarrow^* u_2 \rightarrow^* \ldots$. Using Lemma 3.3.4 (i) the newly constructed reduction $(u_n)_{n\in\mathbf{N}}$ inherits its stableness from the stable reductions $(s_n)_{n\in\mathbf{N}}$ and $(r_n)_{n\in\mathbf{N}}$. Thus we see by Theorem 3.3.5 that the limit u of (u_n) is a normal form. Once more applying Lemma 3.3.4 (i) we see that $\Sigma(s_n) \leq \Sigma(u_n)$ and $\Sigma(r_n) \leq \Sigma(u_n)$. Hence $s \equiv \lim_{n\to\infty} \Sigma(s_n) \leq \lim_{n\to\infty} \Sigma(u_n) \equiv u \geq \lim_{n\to\infty} \Sigma(r_n) \equiv r$. Since normal forms are maximal in the prefix ordering (in contrast to ω-normal forms) s and r are equal. □

3.4. Fair reductions

Theorem 3.3.5 implies that stable converging reductions result in normal forms. If we add a fairness condition to strongly converging reductions, then their limits will also be normal forms. The same fairness condition added to converging reductions results in converging reductions to ω-normal form [Der89b]. Fairness of a reduction will express that, whenever a redex occurs in a term during this reduction, the redex itself or a term containing the redex will be reduced within a finite number of steps.

3.4.1. DEFINITION. (i) Let r be a redex of t at occurrence u. A reduction $t \rightarrow_{\leq\omega} t'$ *preserves* r if no step of this reduction performs a contraction at an occurrence ≤ u.
(ii) A reduction $t \rightarrow_{\leq\omega} t'$ is *fair* if for every term t" in the reduction, and every redex r of t" some finite part of this reduction starting at t" does not preserve r.

Note that a finite sequence is fair if and only if it ends in a normal form, and fair reductions don't need to be converging. Note also that orthogonality guarantees that if the reduction $t \rightarrow_{\leq\omega} t'$ preserves a redex in t of a certain rule, then t' contains a redex of the same rule.

3.4.2. THEOREM. (i) [Der89b] *The limit of a fair, converging reduction is an ω-normal form.*
(ii) *The limit of a fair, strongly converging reduction is a normal form.*

PROOF. By the previous remark we only have to consider sequences of length ω.

(i) Consider the limit of a fair, converging reduction. If it contains no redexes then the limit is a normal form and *a fortiori* an ω-normal form. So let us suppose the limit contains a redex. Assume that contraction of the redex results in a term that differs at depth n with the limit. By convergence there is a point in the reduction such that all later terms in the sequence have the same initial part upto depth n+1. By fairness, it follows that there will be a later in the reduction where the redex is contracted. At that point k we see that the initial part of the k-th term upto level n+1 is equal to the similar initial parts of further terms. Hence in the limit there can be no difference at depth n. Contradiction. Therefore contraction of the redex in the limit results in the limit itself.

(ii) Using (i): strong convergence and fairness rule out that the limit can reduce to itself. □

3.4.3. COROLLARY. *A reduction sequence is fair, strongly convergent if and only if it is stable.*

□

4. THE INFINITE CHURCH-ROSSER PROPERTY

4.1. Failure of the infinite Church-Rosser Property for orthogonal TRSs

In the standard theory of orthogonal TRSs one proves the finite Church-Rosser Property after establishing the Finite Parallel Moves Lemma. In the infinitary setting we would expect to be able to prove the infinite Church-Rosser Property for strongly converging reductions, since we proved the Transfinite Parallel Moves Lemma for strongly converging reductions: ${\leftarrow_{\omega}} \circ {}_{\omega}{\rightarrow} \subseteq {}_{\leq\omega}{\rightarrow} \circ {\leftarrow_{\leq\omega}}$.

However, the following counterexample shows that the infinite Church Rosser property does not hold for even strongly converging reductions of length ω+1.

4.1.1. COUNTEREXAMPLE.

Rules: $A(x) \rightarrow x,\ B(x) \rightarrow x,\ C \rightarrow A(B(x))$

Sequences: $C \rightarrow A(B(C)) \rightarrow A(C) \rightarrow A(A(B(C))) \rightarrow A(A(C)) \rightarrow_{\omega} A^{\omega}$

$C \rightarrow A(B(C)) \rightarrow B(C) \rightarrow B(A(B(C))) \rightarrow B(B(C)) \rightarrow_{\omega} B^{\omega}$

Hence $C \rightarrow_{\leq\omega} A^{\omega}$ as well as $C \rightarrow_{\leq\omega} B^{\omega}$. But there is no term t such that $A^{\omega} \rightarrow_{\leq\omega} t \leftarrow_{\leq\omega} B^{\omega}$ be it converging or strongly converging. □

Although Counterexample 4.1.1 implies that an infinitary version of the Church-Rosser Property for strongly convergent sequences does not hold in general, a weaker version with slightly strengthened hypotheses can be proved.

4.1.2. WEAKENED INFINITE CHURCH ROSSER PROPERTY. *If* t *has a normal form, then for all* s,r *there exists a term* u *such that if* $t \rightarrow_{\omega} s$ *and* $t \rightarrow_{\omega} r$ *then* $s \rightarrow_{\leq\omega} u$ *and* $r \rightarrow_{\leq\omega} u$.

PROOF. It suffices to show that any strongly convergent reduction from a term with a normal form can be extended to a strongly converging reduction ending in that normal form. This can be proved with help of the Transfinite Parallel Moves Lemma and the Compressing Lemma. □

4.2. Böhm trees

The counterexample and Theorem 4.1 suggest that terms having ω-normal forms that are no normal forms are blocking a proof of the Infinitary Church-Rosser Property for converging. From Lambda Calculus (cf. [Bar84]) we will borrow terminology, head normal form (hnf), for terms that cannot be reduced to a redex and the idea for a reduction relation $\rightarrow_{\perp}$ extending $\rightarrow$ with an extra rule: $t \rightarrow \perp$ if t has no hnf. $\perp$ is a fresh symbol that we add to the signature of the TRS. As in Lambda Calculus we call normal forms with respect to $\rightarrow_{\perp}$ *Böhm trees.*

We will prove starting from an orthogonal TRS that convergent rewriting with the reduction relation $\rightarrow_{\perp}$ has the infinite Church-Rosser Property and that each term has a (possibly infinite) unique Böhm tree or normal form with respect to $\rightarrow_{\perp}$. The idea of the proof is application of the Weakened Infinite Church-Rosser Property to an orthogonal subrelation $\rightarrow_{[\perp]}$ of $\rightarrow_{\perp}$.

4.2.1. DEFINITION. A term *is a head normal form* (hnf) if the term cannot be reduced to a redex, and a term *has a hnf* if it can be reduced to a hnf.

4.2.2. DEFINITION. (i) Let $\rightarrow_{\perp}$ be the reduction relation $\rightarrow$ extended by the rule: $t \rightarrow_{\perp} \perp$ if t is a redex for the given TRS and t has no hnf.

(ii) Let $\rightarrow_{[\perp]}$ be the rewrite relation generated by the rule: $t \rightarrow_{[\perp]} t'$ if either $t \rightarrow t'$ or $t \rightarrow_{\perp} t'$ by contraction of an $\rightarrow_{\perp}$-redex which is not a $\rightarrow$-redex.

4.2.3. LEMMA. (i) $\rightarrow_{\perp}$ *is finitely CR.*
(ii) $\rightarrow_{\perp}$ *and* $\rightarrow_{[\perp]}$ *have the same normal forms.*
(iii) *Every* $\rightarrow_{[\perp]}$ *reduction is strongly convergent.*
(iv) *Every term has a normal form with respect to* $\rightarrow_{[\perp]}$.
(v) $\rightarrow_{[\perp]}$ *has the infinite Church-Rosser Property for converging reductions.*
(vi) *If* $t \rightarrow_{\perp\alpha} t'$ *is a convergent reduction, then there is a* t'' *such that* $t' \rightarrow_{\perp\leq\omega} t''$ *and* $t \rightarrow_{[\perp]\leq\omega} t''$.

4.2.4. THEOREM. *Convergent* $\rightarrow_{\perp}$*-reduction satisfies the infinite Church-Rosser Property.*

PROOF. Suppose we have two convergent reductions $t \rightarrow_{\perp\alpha} t_1$ and $t \rightarrow_{\perp\alpha} t_2$. By Lemma 4.2.3 (vi) we obtain sequences $t_1 \rightarrow_{\perp\leq\omega} t'_1$ and $t_2 \rightarrow_{\perp\leq\omega} t'_2$, and $t \rightarrow_{[\perp]\leq\omega} t'_1$ and $t \rightarrow_{[\perp]\leq\omega} t'_2$. Apply Lemma 4.2.3 (v) to the last two sequences, to obtain $\rightarrow_{[\perp]}$ reductions of t'_1 and t'_2 to some t_3. We then have reductions $t_1 \rightarrow_{\perp\leq\omega} t'_1 \rightarrow_{[\perp]\leq\omega} t_3$ and $t_2 \rightarrow_{\perp\leq\omega} t'_2 \rightarrow_{[\perp]\leq\omega} t_3$. Since these are also $\rightarrow_{\perp}$ sequences, the theorem is proved. □

4.3. Non-unifiable orthogonal TRSs have the infinite Church-Rosser Property

From the work of Dershowitz, Plaisted and Kaplan on convergent reductions it follows that any left-linear, top-terminating and semi-ω-confluent TRS satisfies the infinite Church-Rosser property:

$${}_{\omega}^{c}\!\leftarrow \circ \rightarrow_{\omega}^{c} \subseteq \rightarrow_{\leq\omega}^{c} \circ {}_{\leq\omega}^{c}\!\leftarrow$$

(cf. [Der90b]: combine Theorem 1, Proposition 2 with Theorem 9.). A TRS is *top-terminating* if there are no top-terminating reductions of length ω, that is reductions with infinitely many rewrites at the root of the initial term of the reduction. Semi-ω-confluency, that is

$${}^{*}\!\leftarrow \circ \rightarrow_{\omega}^{c} \subseteq \rightarrow_{\leq\omega}^{c} \circ {}_{\leq\omega}^{c}\!\leftarrow$$

holds if the Transfinite Parallel Moves Lemma holds for converging reductions. On the assumption that we are in a orthogonal TRS in which all convergent reductions are strong the infinite Church-Rosser Property holds for this TRS. Top-termination implies this assumption.

Hence in top-terminating orthogonal TRSs the infinite Church-Rosser Property holds. We can improve this using the following syntactic equivalent of the previous assumption.

4.3.1. DEFINITION. A TRS is called *unifiable* if the TRS contains a *unifiable* rule, that is a rule $l \rightarrow r$ such that for some substitution σ with finite and infinite terms for variables $l^\sigma = r^\sigma$.

Note that unifiability in the space of finite and infinite terms means unifiability "without the occurs check": the terms I(x) and x are unifiable in this setting, and their most general unifier is the infinite term I^ω. Collapsing rules, i.e. rules which right hand side is a variable are unifiable.

4.3.2. LEMMA. *The following are equivalent for an orthogonal TRS:*

(i) *the TRS is non-unifiable,*

(ii) *all convergent reductions of the TRS are strong,*

(iii) *all convergent reductions are top-terminating.*

PROOF. (i) ⇒ (ii): If a convergent sequence were not strongly convergent, then there would be some redex in its limit which reduces to itself. But condition (i) rules this out. (ii) ⇒ (iii): By easy contraposition. (iii) ⇒ (i): If an orthogonal TRS is non-unifiable, then one can construct the infinite, convergent and not top-terminating reduction $l^\sigma \rightarrow r^\sigma = l^\sigma \rightarrow l^\sigma \rightarrow \ldots$. □

4.3.3. THEOREM. *Any non-unifiable orthogonal TRS has the infinite Church-Rosser Property for converging reductions.*

PROOF. We claim that in a non-unifiable orthogonal TRS there are no terms without hnf. Hence $\rightarrow_\perp$ and $\rightarrow$ are the same reduction. By Theorem 4.2.4 the reduction $\rightarrow_\perp$ has infinite Church-Rosser Property for converging reductions.

Sketch of the proof of the claim: if a term t has a hnf, then t can be reduced to hnf in finitely many steps via an application of 2.4.4 to $\rightarrow_\perp$. It now follows that the term t is top-terminating via an argument by contraposition based on the fact that projection by the infinite parallel moves lemma preserves non-toptermination. □

4.3.4. OPEN QUESTION. It is open whether the condition non-unifiable can be weakened to non-collapsing. In non-collapsing orthogonal TRSs (possibly extended with one collaps rule of the form I(x) → x) we can prove the infinite Church-Rosser Property for strongly converging reductions only (cf. [Ken90b]).

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