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Near-perfect matrices

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Near-Perfect Matrices

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Abstract. A $0,1$ matrix A is *near-perfect* if the integer hull of the polyhedron $\{x \geq 0 : Ax \leq \bar{1}\}$ can be obtained by adding one extra (rank) constraint. We show that in general, such matrices arise as the clique-node incidence matrices of graphs. We give a colouring (that is, nonpolyhedral) characterization of the corresponding class of near-perfect graphs and make the following conjecture: a graph is near-perfect if and only if sequentially lifting any rank inequality associated with a minimally imperfect graph results in the rank inequality for the whole graph. We show that the conjecture is implied by the Strong Perfect Graph Conjecture. It is also shown to hold for small graphs (no stable set of size eleven). Our results are used to strengthen (and give a new proof of) a theorem of Padberg and give a new characterization of minimally imperfect graphs: a graph is minimally imperfect if and only if both it and its complement are near-perfect.

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1 Introduction

A $0,1$-matrix A (whose columns are indexed by V say), is *perfect* if the polyhedron

$$(1) \qquad P(A) = \{x \in \mathbb{Q}^V : A \cdot x \leq \bar{1}, \quad x \geq 0\}$$

is integral.

The notion of a perfect graph was introduced Berge in 1959. A graph is *perfect* if each of its induced subgraphs H has chromatic number, denoted by χ_H, no greater than the size, ω_H, of a maximum clique in H. In 1975 Chvátal noted that results of Lovász imply a polyhedral characterization of such graphs: a graph is perfect if and only if any nontrivial facet of its stable set polytope is induced by a clique inequality. (The *stable set polytope* of a graph is the convex hull of incidence vectors of its stable sets.) This result of Chvátal and a result of Padberg (see [17]) show that perfect matrices are essentially equivalent to clique matrices of perfect graphs.

Theorem 1.1 *A matrix A is perfect if and only if there is a perfect graph G such that the incidence vectors of the maximal cliques of G are exactly the maximal rows of A.*

In particular, the graph whose existence is asserted in the theorem is the *derived graph* of A which we denote by $G(A)$. This is the graph whose nodes correspond to the columns of A and two nodes are adjacent in $G(A)$ if some row of A has a one in each of their components. This theorem shows that we lose no generality by restricting ourselves to studying perfect

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graphs instead of perfect matrices, i.e., by studying stable set polyhedra instead of the polyhedron (1).

Note that even if $P(A)$ is not integral, its integer hull, denoted by $P(A)_I$, can be described in terms of the derived graph $G(A)$, of A.

Proposition 1.2 *For any* 0, 1*-matrix* A, $P(A)_I = P(G(A))$.

In [18] Padberg defines a polyhedron P to be *almost integral* if P is not integral but for each $v \in V$, $P \cap \{x \in \mathbb{Q}^V : x_v = 0\}$ is integral. He proves the following surprising result.

Theorem 1.3 (Padberg [18]) *If P is almost integral, then P has a unique fractional vertex $\bar{x}$. Furthermore, $\bar{x}$ is adjacent to exactly $|V|$ vertices $v_1, \ldots v_{|V|}$ of P and for $i = 1, \ldots |V|$, v_i is integral and $\bar{1} \cdot v_i = \alpha_P$.*

Here, we use α_P to denote the value $\max\{\bar{1} \cdot x : x \in P_I\}$. This yields a full description of the integer hull of $P(A)$.

Corollary 1.4 (Padberg) *If $P(A)$ is almost integral, then $P(A)_I$ is given by*

$$\{x \in \mathbb{Q}^V : x \geq 0,\ A \cdot x \leq \bar{1},\ \ \bar{1} \cdot x \leq \alpha_{P(A)}\}. \tag{2}$$

This leads to the definition of a near-perfect matrix: a 0, 1-matrix A is *near-perfect* if the polyhedron (2) is integral. We will see that there are many near-perfect matrices A for which $P(A)$ has a large number of fractional vertices and hence is not almost integral. We return to near-perfect matrices but first we discuss their graphical counterparts.

It can be shown that if G is minimally imperfect, then $P(G)$ is almost integral. Thus for such graphs we have the following.

Theorem 1.5 (Padberg) *If G is minimally imperfect, then*

$$P(G) = \left\{ x \in \mathbb{R}^V : \begin{array}{ll} (i) & x \geq 0 \\ (ii) & x(K) \leq 1 \ \textit{for each clique } K \\ (iii) & x(V) \leq \alpha \end{array} \right\}. \tag{3}$$

We call a graph *near-perfect* if its stable set polytope is defined by the inequalities (i)-(iii) of (3). It follows from a result of Chvátal (see Theorem 2.4) that the inequalities of (3) are also sufficient to define the stable set polytope of any replication of a minimally imperfect graph, i.e., a graph obtained by 'expanding' nodes into cliques. These are not however, the only graphs with this property. Figure 1 gives some small examples of other such graphs.

We know that the clique-node incidence matrices of near-perfect graphs form one class of near-perfect matrices. Theorem 1.1 shows that the concepts of perfect graphs and matrices are essentially equivalent; the same is not quite true for near-perfection. The matrix $J - I$ is near-perfect but is not obtained from the cliques of any graph. The derived graph of A, in fact, is a clique! A near-perfect matrix A, is said to be *graph-representable* if the set of maximal rows of A is exactly the set of incidence vectors of maximal cliques of $G(A)$. It is easy to see that this is equivalent to stating that the incidence vector of each maximal clique of $G(A)$ is a row of A. For suppose that some maximal row χ^K say, of A is not the incidence vector of a maximal clique in $G(A)$. Hence there is some other clique K' which contains K. By maximality of χ^K, $\chi^{K'}$ does not appear as a row of A. The next theorem shows that the near-perfect matrices which are not graph-representable form a very restricted class.

Theorem 1.6 *If A is a near-perfect matrix, then either A is graph-representable or $G(A)$ is a clique.*

Proof: Suppose that A is not representable. By the preceding comments, there is some maximal clique K of $G(A)$ for which χ^K is not a row of A. Proposition 1.2 yields that K gives a facet-inducing inequality of $P(A)_I$. Thus $\chi^K \cdot x \leq 1$ must appear in a defining system of $P(A)_I$. Since A is near-perfect, this implies that $\chi^K = \bar{1}$. Thus $G(A)$ is a clique.
□

Hence for the remainder of this paper we focus our attention on the class of near-perfect graphs.

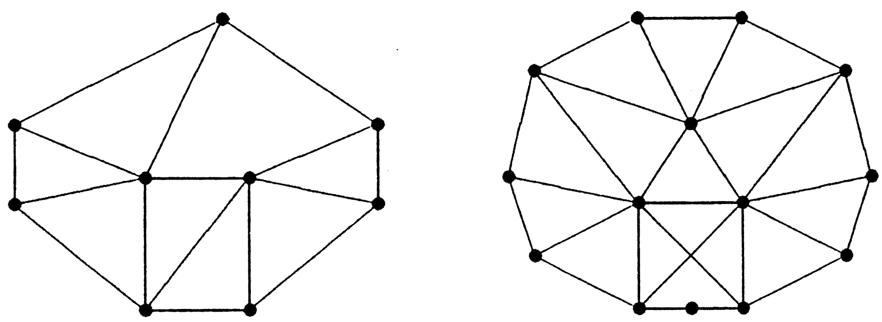

Figure 1:

The definition of near-perfection is given in terms of a graph's stable set polytope. Conversely, perfect graphs were defined in terms of a colouring property. It was over a decade after their introduction that the polyhedral characterization of perfect graphs was found. Sections 4.1-4.4 are devoted to finding a colouring-like characterization of near-perfect graphs. Such a result should somehow characterize the structure of bad subgraphs in a near-perfect graph (a graph H is *bad* if $\chi_H > \omega_H$). This approach leads to the following conjecture:

Conjecture 4.10 *A graph is near-perfect if and only if each lifting of a rank facet corresponding to a minimally imperfect induced subgraph yields the constraint* $\bar{1} \cdot x \leq \alpha$. (We define the lifting operation in Section 2.) We show that a minimal counterexample to the conjecture must satisfy several stringent conditions. We use these to show that if the Strong Perfect Graph Conjecture is true, then so is Conjecture 4.10. We also show that any counterexample to Conjecture 4.10 must have a stable set of size at least 11.

In Section 4.5 we discuss the complements of near-perfect graphs. Clearly any perfect graph is also near-perfect. In contrast to the Perfect Graph Theorem however, the complements of near-perfect graphs need not be near-perfect. For example, the graph of Figure 2 is a replication of an odd hole and hence near-perfect. The inequality $x_1 + \ldots + x_5 \leq 2$ is an odd hole inequality for the stable set polytope of the complement of this graph. It can be seen to be facet-inducing by lifting, and so the complement is not near-perfect.

We use some of our earlier results to give a new polyhedral characterization of minimally imperfect graphs:

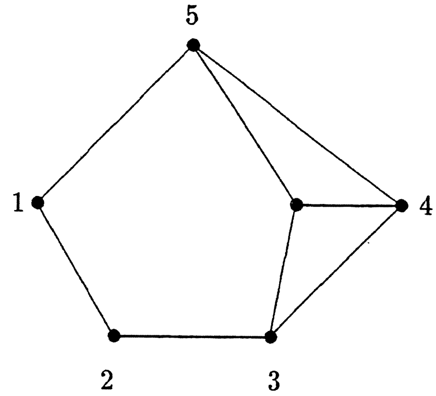

Figure 2:

Theorem 4.35 *An imperfect graph is minimally imperfect if and only if both it and its complement are near-perfect.*

(We give a different proof of the only if part which is different from [18].) Section 4.6 contains some conjectures about the structure of the complements of near-perfect graphs. In Section 4.7 we discuss briefly the problem of recognizing a near-perfect graph. It is shown that this problem is in **coNP** and that if it is in **NP**, then so too is the problem of recognizing a perfect graph. The rest of this section is dedicated to frequently used definitions and notations.

1.1 Definitions and Notation

We follow the text [4] for terms which we have not defined below. A *graph* G, is an ordered pair (V, E) consisting of a *node set* V and *edge set* E. The edges are a subset of $\{\{u, v\} : u, v \in V, u \neq v\}$. (Note that by the definition there are no multiple edges.) We denote a set $\{u, v\}$ simply by uv. If $uv \in E$, the nodes u and v are said to be *adjacent.* The *neighbourhood* of a node v, denoted by $N(v)$, is the set $\{u \in V : u, v \text{ are adjacent}\}$. The *closed neighbourhood* of v, denoted by $N[v]$, is the set $N(v) \cup \{v\}$. A *stable set* of G is either the ϕ or a set of mutually nonadjacent nodes of V. A *clique* of G is a subset of V which is a stable set in $\bar{G}$. The collection of all stable sets (respectively cliques) of G is denoted by $\mathcal{S}(G)$ (respectively $\mathcal{K}(G)$). The *stability number* (respectively *clique number*) of G, denoted by α_G, or simply α, (respectively ω_G, or ω), is the size of a maximum stable set (clique) of G. A stable set S (respectively clique), is *universal* if each maximum clique (respectively stable set) contains a node of S. For an integer k, a *k-clique* of G is a clique with k elements. Similarly we define a *stable k-set*. A *colouring* of G is a partition of V into stable sets; the size of a colouring is the number of sets in the partition. A colouring is *proper* if none of the stable sets is ϕ. The *chromatic number* of G, denoted by χ_G, or simply χ, is the minimum size of a colouring of G. A *clique cover* of G is a partition of V into cliques. A *clique k-cover* is a clique cover of size k. The size of a minimum clique cover is denoted θ_G.

For $X \subset V$, the *subgraph* of G *induced by* X (or simply the graph induced by X), denoted by G_X, is the graph $(X, \{uv : u, v \in X,\ uv \in E\})$. Such a graph is called an *induced subgraph* of G. The node set, edge set, stability number, clique number and chromatic number of G_X are denoted by $V_X, E_X, \alpha_X, \omega_X, \chi_X$ respectively. For a graph H, we say G *contains* H, if there is $X \subseteq V$, such that $H \simeq G_X$.

A *cycle*, C, of G is a sequence of distinct nodes $v_0, v_1, \ldots v_{k-1}$ such that for each $i = 0, \ldots k-1$, $v_i v_{i+1} \in E$ (using modulo k arithmetic). A *chord* of C is any edge $v_i v_j$ of G with $(|i-j| \bmod k) > 1$. A *path* P is defined similarly, except that $v_0 v_{k-1}$ is not an edge. The nodes $\{v_0, v_{k-1}\}$ are called the *endpoints* of P and P is called a (v_0, v_{k-1})-path. The *internal nodes* of the path P are the nodes $v_1, \ldots v_{k-2}$.

For $X \subseteq V$ we denote by $\bar{X}$, the set $V - X$. For $X \subseteq V$, the notation $G - X$ may be used to denote $G_{\bar{X}}$. Similarly for $E' \subseteq E$, $G - E'$ denotes the graph $(V, E - E')$.

The graph obtained from G by *replicating* a node v, $k \geq 1$ times, is the graph with node set

$$(V - \{v\}) \cup \{v^1, \ldots v^k\}$$

and edge set

$$(E - \{uv : u \in N(v)\}) \cup \{v^i v^j : 1 \leq i, j \leq k\} \cup \{uv^i : 1 \leq i \leq k, u \in N(v)\}.$$

where $v^1, \ldots v^k$ are new, distinct nodes. A *replication* of G is a graph which is obtainable from G by replicating a sequence of nodes. *Stable replicating* is analogous to replicating except that the new nodes $v^1, \ldots v^k$ form a stable set instead of a clique. For $w \in \mathbb{Z}^V$, we denote by $G[w]$ the graph obtained from G by deleting each node v if w_v is nonpositive and replicating each node v, w_v times otherwise. We define $G(w)$ analogously for stable replication.

2 Stable Set Polyhedra

For a graph G, the *stable set polytope* of G, denoted by $P(G)$, is $conv(\{\chi^S : S$ is a stable set of $G\})$. The vertices of $P(G)$ are the integral vectors in:

$$(4) \qquad \left\{ x \in \mathbb{Q}^V : \begin{array}{ll} x_v \geq 0 & \text{for each node } v \\ x_u + x_v \leq 1 & \text{for each edge } uv \in E \end{array} \right\}.$$

Since $P(G)$ is full dimensional, there is a unique, up to scalar multiplication, facet-inducing inequality corresponding to each facet of $P(G)$. An obvious family of valid inequalities is the class of *trivial inequalities*: $x_v \geq 0$, for each $v \in V$. The corresponding face is called a *trivial facet*. A valid supporting inequality for $P(G)$ is called *nontrivial* if it does not induce a trivial face. There is the following well known fact.

Proposition 2.1 *Let G be a graph. Suppose that $a \cdot x \leq 1$ is a nontrivial facet-inducing inequality for $P(G)$. Then $a \geq 0$.*

Let A^G be a matrix whose rows consist of all $a \in \mathbb{Q}^V$, such that $a \cdot x \leq 1$ induces a nontrivial facet of $P(G)$, i.e., $P(G) = \{x \in \mathbb{Q}^V : A^G \cdot x \leq \bar{1},\ x \geq 0\}$. The next result shows that a defining linear system for the stable set polytope of a graph is inherited by its induced subgraphs.

Proposition 2.2 *For any subset X of V, $P(G_X) = \{x \in \mathbb{Q}^X : x \geq 0, A^G_X \cdot x \leq 1\}$.*

Here A^G_X denotes the matrix obtained by restricting to the columns X.

We now give a procedure due to Padberg [16], called *sequential lifting*, which is used to build facet-inducing inequalities from those for induced subgraphs. Consider $X \subseteq V$ and $a \cdot x \leq 1$, a valid inequality for $P(G_X)$. Suppose $v \in V - X$ and let $\gamma = 1 - \max\{a \cdot \chi^S : S \in \mathcal{S}(G_{X-N(v)})\}$. The *lift of $a \cdot x \leq 1$ to $X \cup v$* is the inequality $\gamma x_v + a \cdot x \leq 1$. The next theorem shows that this operation can be repeated to obtain a facet-inducing inequality for $P(G)$.

Theorem 2.3 (Padberg [16]) *Let G be an arbitrary graph and $X \subseteq V$. If $a \cdot x \leq 1$ is facet-inducing for $P(G_X)$, $v \in V - X$ and $\gamma = 1 - \max\{a \cdot \chi^S : S \in \mathcal{S}(G_{X-N(v)})\}$, then $\gamma x_v + a \cdot x \leq 1$ is facet-inducing for $P(G_{X \cup x})$.*

We consider the substitution operation. Consider two node-disjoint graphs G and H. The *substitution of H for the node v* (in G), denoted by $G_{v \to H}$, is the graph obtained from $(G - v) \cup H$ by joining each node of H to each node in $N(v)$. Chvátal [6] has shown that a defining system of inequalities for $P(G_{v \to H})$ can be described simply, in terms of the inequalities for $P(G)$ and $P(H)$. Cunningham showed [10] that each of the inequalities described by Chvátal is facet-inducing.

Theorem 2.4 (Chvátal,Cunningham) *Let G and H be graphs and v a node of G. Then a nontrivial inequality is facet-inducing for $P(G_{v \to H})$ if and only if it can be scaled to be in the form*

$$\sum_{y \in V - \{v\}} a^G_y x_y + a^G_v \Big(\sum_{z \in V_H} a^H_z x_z \Big) \leq 1, \tag{5}$$

where a^G and a^H are, respectively, rows of A^G and A^H.

The following is an immediate consequence.

Corollary 2.5 *If G' is obtained from G by replicating a node v, k times, then $A^{G'}$ can be obtained from A^G by adding $k - 1$ copies of the column corresponding to v.*

We denote by $\mathcal{G}_2$, the class of graphs G, with $\alpha = 2$. Note that the weighted stable set problem is easy for this class of graphs as one need only check at most $|V|^2$ subsets of the nodes. A description of a defining family of inequalities for $\mathcal{G}_2$ was first given by Cook [9]. Knowing such a family for $\mathcal{G}_2$ provides a useful testing ground for conjectures about general stable set polyhedra. A proof of this result is given in [21] which also shows how to assign the integral dual variables for the associated LP. This also shows that the system given by Cook is **TDI**. We use the following notation: for a graph G and $X \subseteq V$ we denote by $\tilde{N}(X)$ the set of all nodes v for which $X \subseteq N(v)$ if $X \neq \phi$, otherwise $\tilde{N}(X) = V$.

Theorem 2.6 (Cook [9], Shepherd [21]) *If $G \in \mathcal{G}_2$, then the following system is* **TDI***:*

$$x \geq 0 \tag{6}$$

$$2x(K) + x(\tilde{N}(K)) \leq 2 \qquad \textit{for each clique } K. \tag{7}$$

The next theorem tells us exactly which inequalities are facet-inducing for graphs in $\mathcal{G}_2$ (a proof may be found in [21]).

Theorem 2.7 *If $G \in \mathcal{G}_2$ and K is a clique of G, then K's inequality is facet-inducing for $P(G)$ if and only if no component of $\bar{G}_{\tilde{N}(K)}$ is bipartite.*

This describes the unique minimal defining system. We now describe the unique minimal integral **TDI** defining system. For this theorem we let $\mathcal{K}^*$ denote the set of all maximal cliques and cliques K for which $\bar{G}_{\tilde{N}(K)}$ is nonbipartite and does not contain any isolated nodes.

Theorem 2.8 (Shepherd [21]) *For $G \in \mathcal{G}_2$, the following is the minimal integral* **TDI** *system for $P(G)$*

$$(8) \qquad \begin{cases} x \geq 0 \\ 2x(K) + x(\tilde{N}(K)) \leq 2 & \text{for each } K \in \mathcal{K}^*. \end{cases}$$

The reader is referred to [19] and [20] for further background in polyhedral combinatorics.

3 Perfect Graphs

In any colouring of a graph G, each node in a clique must have a distinct colour, hence $\chi \geq \omega$. A graph is *perfect* if *every* induced subgraph H satisfies $\chi_H = \omega_H$. This class of graphs was first defined by Berge; he made two conjectures (see [2]) which have since attracted much attention. The first was known as the Weak Perfect Graph Conjecture and was resolved by Lovász [14] in 1971. This result is known as the Perfect Graph Theorem:

Theorem 3.1 (Perfect Graph Theorem) *A graph G is perfect if and only if $\bar{G}$ is perfect.*

The smallest example of an imperfect graph is a chordless cycle of length five. Note that the chromatic number of this graph is 3 although the size of the largest clique is 2. An *odd hole* is any odd length (chordless) cycle of length at least five. The same reasoning shows that odd holes are imperfect. It is also easy to see that the complement of a hole with $2k+1$ nodes has chromatic number $k+1$ and maximum clique size k. Hence such graphs, called *odd antiholes*, are also imperfect. The second conjecture made by Berge, which remains unsolved, asserts that graphs without odd holes or antiholes are perfect. It is called the Strong Perfect Graph Conjecture because it immediately implies Theorem 3.1.

Conjecture 3.2 (Strong Perfect Graph Conjecture) *A graph G is perfect if and only if neither G nor $\bar{G}$ contain an odd hole.*

A graph is *minimally imperfect* if it is imperfect and each proper induced subgraph is perfect. The Strong Perfect Graph Conjecture is equivalent to stating that the only minimally imperfect graphs are the odd holes and antiholes.

We now examine some results on perfect graphs which we will need later. Our attention focuses on results relating to stable set polyhedra.

3.1 Characterizations of Perfect Graphs

Fulkerson [11] used anti-blocking theory to reduce the Weak Perfect Graph Conjecture to the following statement:

Lemma 3.3 (Replication Lemma) *If G is perfect, then so is any replication of G.*

In addition, he called a graph *pluperfect* if it had this property.

Independently of Fulkerson's work, Lovász [14] settled the Weak Perfect Graph Conjecture. His proof is based on the following theorem.

Theorem 3.4 (Lovász [14]) *If G and H are perfect graphs, then substituting the graph H for any node of G results in a perfect graph.*

It follows that every perfect graph is pluperfect. The following is also immediate.

Corollary 3.5 *If G is minimally imperfect, then G does not contain a pair of replicated nodes.*

Lovász later gave an even stronger characterization of perfect graphs.

Theorem 3.6 (Lovász [13]) *A graph G is perfect if and only if for each subset S of V, $|S| \leq \omega_S \alpha_S$.*

Note that if $|S| > \omega_S \alpha_S$, for some $S \subseteq V$, then the graph G_S could not possibly be ω_S-colourable since each colour can be used for at most α_S nodes of S. This characterization leads to another useful fact about minimally imperfect graphs.

Theorem 3.7 (Lovász) *If G is minimally imperfect, then $|V| = \alpha\omega + 1$.*

About the same time, Chvátal noted that the results of Lovász imply a characterization of a different nature.

Theorem 3.8 (see [6]) *A graph G is perfect if and only if*

$$P(G) = \left\{ x \in \mathbb{R}^V : \begin{array}{ll} (1) & x \geq 0 \\ (2) & x(K) \leq 1 \text{ for each clique } K \end{array} \right\}.$$

Note that Theorem 3.8 is equivalent to having for each $w \in \mathbb{Q}_+^V$, an integral optimum of maximize $w \cdot x$, subject to the constraints (1) and (2) of Theorem 3.8. If G is perfect, then for 0, 1-valued vectors w this is just a restatement of the definition of a perfect graph. Chvátal appeals to the Replication Lemma to exhibit an integral optimum for any integral weight vector w.

3.2 Minimally Imperfect and Partitionable Graphs

For $p, q \geq 2$, a graph G is an (p,q)*-graph* if $|V| = pq + 1$ and for each node v, $G - v$ can be partitioned into q stable sets of size p and p cliques of size q. The following is immediate:

Remark 3.8.1 *If G is an (p,q)-graph, then $\alpha = p, \omega = q, \chi = \omega + 1$ and $\bar{\chi} = \alpha + 1$.*

In light of this remark we refer to such graphs as (α, ω)-graphs. We call a graph G *partitionable* if it is an (α, ω)-graph. Note that Remark 3.8.1 implies that each partitionable graph is imperfect. It is easy to check that each odd hole and antihole is partitionable. In fact it follows from the Perfect Graph Theorem and Theorem 3.7 that:

Theorem 3.9 *Every minimally imperfect graph is partitionable.*

Other examples of partitionable graphs have been constructed in [7] and [8]. Indeed every known example of a partitionable graph has been shown to contain either an odd hole or antihole. In [15] Lovász states:

> ...it seems that virtually all structural results which we know for minimally imperfect graphs also follow for (α, ω)-graphs. (This indicates the main difficulty in the proof of the Strong Perfect Graph Conjecture - it is difficult to determine that an (α, ω)-graph is not minimally imperfect.)

This suggests that the partitionable graphs act as imposters of the minimally imperfect graphs.

The next theorem shows that partitionable graphs have some interesting and apparently strong properties. These properties were shown to hold first for minimally imperfect graphs by Padberg [17] and later for all partitionable graphs by Bland, Huang and Trotter [3]. For an (α, ω)-graph G and each node $v \in V$, arbitrarily choose a partition $K_1^v, \ldots K_\alpha^v$ of $G - v$ into ω-cliques and similarly choose a colouring $S_1^v, \ldots S_\omega^v$ of $G - v$. In fact, the following theorem implies that these partitions are unique.

Theorem 3.10 (Padberg [17];Bland,Huang,Trotter [3]) *If G is a partitionable graph, then G has the following properties:*

(1) *G has exactly $|V|$ ω-cliques; in fact, $\{\chi^K : K$ is a maximum clique $\}$ is linearly independent,*

(2) *G has exactly $|V|$ stable α-sets; in fact, $\{\chi^S : S$ is a maximum stable set $\}$ is linearly independent,*

(3) *each node is in exactly ω maximum cliques,*

(4) *each node is in exactly α maximum stable sets,*

(5) *each maximum clique is disjoint from exactly one maximum stable set,*

(6) *each maximum stable set is disjoint from exactly one maximum clique,*

(7) *for any ω-clique K, $(\{K\} \cup (\cup_{v \in K} \{K_i^v\}_{i=1}^{\alpha})$ is the set of all maximum cliques in G,*

(8) *for any α-set S, $(\{S\} \cup (\cup_{v \in S} \{S_i^v\}_{i=1}^{\omega})$ is the set of all maximum stable sets in G.*

4 Near-Perfection

4.1 Some Properties of Near-Perfect Graphs

A graph H is said to be *bad* if $\chi_H > \omega_H$. Perfect graphs are originally defined in terms of the structure of their bad subgraphs, namely, that they do not have any such induced subgraphs. In contrast, near-perfect graphs are defined in terms of a polyhedral property. We find a *colouring* characterization for the class of near-perfect graphs. This also leads to two conjectures about characterizations of a different type: a so-called *Near-Perfect Conjecture* and a *Weak Near-Perfect Conjecture* (given in Section 4.2). We begin by examining some of the properties implied by near-perfection.

Proposition 2.2 shows that a defining linear system for the stable set polytope of a graph is inherited by its induced subgraphs. Hence we have:

Proposition 4.1 *If G is near-perfect and S is a subset of V, then G_S is near-perfect.*

A subset S of V is a *bad set* of G if $\chi_S > \omega_S$. Evidently, a graph is perfect if and only if it has no bad sets. We now describe three properties which we show are possessed by near-perfect graphs.

P_1: S is a bad set implies $\alpha_S = \alpha$, for all $S \subseteq V$,

P_2: S is a bad set implies $\alpha_{S-N[v]} = \alpha_S - 1$, for all $S \subseteq V$ and $v \in V$,

P_3: S is a bad set implies $|S| > \omega_S \alpha_S$, for all $S \subseteq V$.

Proposition 4.2 *If G is near-perfect, then G has property P_1.*

Proof: Suppose S is a bad set of G. By Corollary 4.1, G_S is near-perfect. Since G_S is imperfect, Theorem 3.8 implies that $x(S) \leq \alpha_S$ is facet-inducing for $P(G_S)$. Since $\alpha_S \geq 2$, Proposition 2.2 implies that $\alpha_S = \alpha$. □

Clearly, a graph with property P_1 need not be near-perfect. For example, the 5-wheel has property P_1 yet lifting the odd hole inequality yields a non-rank inequality. This graph does not however, have property P_2.

Proposition 4.3 *If G is near-perfect, then G has property P_2*

Proof: Suppose S is a bad set and v is some node. Since $G_{S\cup\{v\}}$ is near-perfect and imperfect, $x(S \cup \{v\}) \leq \alpha_{S\cup\{v\}}$ must be facet-inducing. In particular, we deduce that $\alpha_{S\cup\{v\}} = \alpha_S$, i.e., $\alpha_{S-N[v]} \leq \alpha_S - 1$. Otherwise $x(S \cup \{v\}) \leq \alpha_{S\cup\{v\}}$ is the addition of G_S's rank inequality and the clique inequality for $\{v\}$, a contradiction. Furthermore there is a set $\mathcal{L}$, of $|S \cup \{v\}|$ linearly independent incidence vectors of stable $\alpha_{S\cup\{v\}}$-sets in $G_{S\cup\{v\}}$. The linear independence of $\mathcal{L}$ implies that v must be in some maximum stable set of $G_{S\cup\{v\}}$. Hence $\alpha_{S-N[v]} \geq \alpha_S - 1$. Thus $\alpha_{S-N[v]} = \alpha_S - 1$. □

We now show that near-perfect graphs must have property P_3.

Proposition 4.4 *If G is near-perfect, then G has property P_3.*

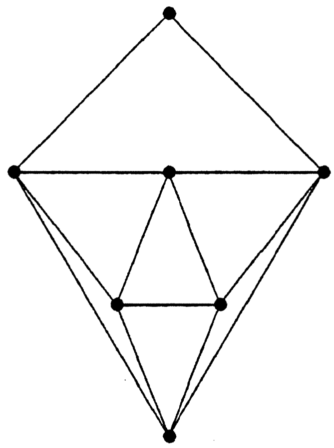

Figure 3:

Proof: The proof is by induction on ω, the case $\omega = 1$ being trivial. We may assume that the bad set which violates the definition of P_3 is V. So suppose G is near-perfect such that $|V| \leq \omega\alpha$ and $\omega > 1$. The vector $(1/\omega)\cdot\bar{1}$ satisfies the inequalities in (3) and so is in $P(G)$. Thus for some $\lambda \in \mathbb{R}^{\mathcal{S}(G)}$ satisfying $\bar{1}\cdot\lambda = 1$ we have $(1/\omega)\bar{1} = \sum_{S\in\mathcal{S}(G)} \lambda_S\, \chi^S$. Let k be an integer such that $k\omega\lambda_S \in \mathbb{Z}$ for each $S \in \mathcal{S}(G)$; set $k_S = k\omega\lambda_S$. Then $k\cdot\bar{1} = \sum_{S\in\mathcal{S}(G)} k_S\, \chi^S$. Let G' be the graph obtained from G by replicating each node k times. Clearly $\omega_{G'} = k\omega$. Also λ gives rise to a colouring of G' with $\sum_{S\in\mathcal{S}(G)} k_S = k\omega \sum_{S\in\mathcal{S}(G)} \lambda_S = k\omega = \omega_{G'}$ stable sets. Let $S_1, \ldots S_{\omega_{G'}}$ be such a colouring of G'. Since this is an $\omega_{G'}$-colouring, each S_i is a universal stable set of G'. Each such set has a natural correspondence with a universal stable set of G. Let $r = |V| - (\omega-1)\alpha$. Now since $\sum_{i=1}^{\omega_{G'}} |S_i| = |V_{G'}| = k|V|$, one of the stable sets must have cardinality at least $k|V|/\omega_{G'} = |V|/\omega \geq (\omega\alpha - \alpha + r)/\omega = \alpha - (\alpha - r)/\omega$, which is at least r since $\omega > 1$ and $r < \omega$. Thus G has a universal stable set S such that $|V - S| \leq (\omega - 1)\alpha$. Now if $G - S$ is perfect, then clearly it is $(\omega-1)$-colourable. Otherwise, since G has property P_1, $\alpha_{G-S} = \alpha$ and so by the induction hypothesis and the fact that $G - S$ is near-perfect, $G - S$ is $(\omega - 1)$-colourable. Hence G is ω-colourable. □

Figure 3 shows a graph with property P_2 but not P_1. This graph and the 5-wheel together show that P_1 and P_2 are independent. This may not be true for the third property P_3. (We discuss this further in the next section.)

We complete this section by noting how the properties we have discussed are affected by the replication of nodes. It follows from Corollary 2.5 that near-perfection is closed under performing replications.

Remark 4.4.1 *The replication of a near-perfect graph is near-perfect.*

We also have the following.

Remark 4.4.2 *The replication of a graph with property P_1 has property P_1,*

Proof: Suppose G has property P_1. Let G' be a replication of G and S be any bad set of G'. Since G'_S is imperfect it must contain an induced minimally imperfect subgraph H.

Corollary 3.5 states that H' cannot contain a pair of replicated nodes. Thus G contains an induced subgraph H isomorphic to H'. We have $\alpha_H = \alpha$ $(= \alpha_{G'})$ and since $\alpha_H = \alpha_{H'} \leq \alpha_S$ we must have $\alpha_S = \alpha_{G'}$. Hence G' has property P_1. □

The following is proved in a similar fashion.

Remark 4.4.3 *The replication of a graph with property P_2 has property P_2.*

This does not hold for property P_3. Figure 4 shows a graph which has property P_3 but not P_3^* since replicating the node v yields a graph G' with 12 nodes and $\omega_{G'}\alpha_{G'} = 12$. It is, however, straightforward to check that $\chi_{G'} > 3 = \omega_{G'}$ and so G' does not have property P_3.

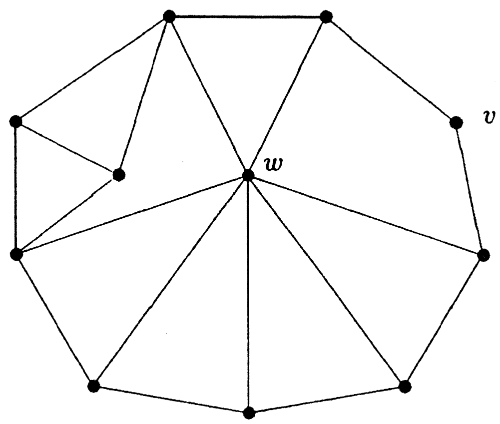

Figure 4: A graph with property P_3 but not P_3^*.

We consider one more property that a graph G may have:

P_3^*: each replication of G has property P_3.

4.2 Two Conjectures and a Characterization

In this section we give a characterization of near-perfect graphs. We also make two conjectures of alternative characterizations: a strong and a weak version.

First, we show that graphs with properties P_1 and P_3 have a strong colouring property.

Proposition 4.5 *If G has properties P_1 and P_3, then $\chi(G) = \max\{\omega, \lceil \frac{|V|}{\alpha} \rceil\}$.*

Proof: The proof is by induction on $|V|$, the base case being trivial. Now if G satisfies the hypotheses and $|V| \leq \omega\alpha$, then certainly the proposition holds. So suppose $|V| = m\alpha + r$, $m \geq \omega$, $1 \leq r < \alpha$. Let S be a maximum stable set of G. If $\alpha_{G-S} < \alpha$, then since G has property P_1, $G - S$ is ω_{G-S}-colourable. Otherwise, $m = \lceil \frac{|V-S|}{\alpha_{G-S}} \rceil$ and so by the induction hypothesis is m-colourable. In either case we can colour G with $m + 1$ colours. □

We now give the characterization. The proof of this result also shows that if G is near-perfect, then $P(G)$ has the integer decomposition property (see [1]).

Theorem 4.6 *A graph is near-perfect if and only if it has properties P_1 and P_3^*.*

Proof: First suppose G is near-perfect. Then Remark 4.4.1 states that any replication of G is near-perfect. Hence Propositions 4.2 and 4.4 imply that G has properties P_1 and P_3^*.

Conversely, suppose G has property P_1 and P_3^*. Let x be a rational vector in the polyhedron defined by (3) and let k be an integer such that $kx \in \mathbb{Z}^V$. Let G' be obtained from G by replicating each node v, $(kx)_v$ times. Remark 4.4.2 and our hypothesis then imply that G' has properties P_1 and P_3. Equation 3 implies that $kx(K) \leq k$ for each clique K of G. Hence $\omega_{G'} \leq k$. Also x satisfies $|V_{G'}| = kx(V) \leq k\alpha_G$ and so $\lceil |V_{G'}|/\alpha \rceil \leq k$. Thus if $\alpha_{G'} = \alpha$, then by Proposition 4.5, G' is k-colourable. Otherwise $\alpha_{G'} < \alpha$ and hence must be a replication of an induced subgraph H of G with $\alpha_H < \alpha$. Thus H is perfect and so by Theorem 3.4, G' is also perfect and hence $\alpha_{G'}$-colourable. In either case G' is k-colourable and so kx is the sum of k vertices of $P(G)$. Hence x is a convex combination of vertices of $P(G)$. It follows that $P(G)$ is given by (3). □

This theorem is the best possible in the sense that we cannot relax either of the conditions. It is clear that we cannot eliminate the condition of having property P_1 but neither can we relax the condition of P_3^*. For example, Figure 4 shows a graph with properties P_1 and P_3 (but not P_3^*). This graph is not near-perfect since the node w together with the bad set forming the odd cycle of length nine, violate the requirement in the definition of P_2.

It would be desirable to have a characterization which did not require a property to hold for each replication of a graph. We do not know of a graph which has properties P_1,P_2 and P_3 but which does not have property P_3^*. We conjecture the following:

Conjecture 4.7 (Weak Near-Perfect Conjecture) *A graph is near-perfect if and only if it satisfies properties P_1, P_2 and P_3.*

As mentioned in the preceding section, we do not even know if property P_3 is independent of P_1 and P_2. In light of Remarks 4.4.2 and 4.4.3 and Theorem 4.6, the following conjecture is even stronger than the Weak Near-Perfect Conjecture.

Conjecture 4.8 *If a graph has properties P_1 and P_2, then it has property P_3.*

Of course if Conjecture 4.8 holds, then using Remarks 4.4.2, 4.4.3 and Theorem 4.6 we could also prove the following:

Conjecture 4.9 (Near-Perfect Conjecture) *A graph is near-perfect if and only if it has properties P_1 and P_2.*

An equivalent form of this conjecture is the following:

Conjecture 4.10 *Given a graph G, exactly one of the following statements is true.*

- *G is near-perfect*
- *G contains a minimally imperfect graph I, such that the inequality $x(I) \leq \alpha_I$ can be lifted to V to obtain an inequality other than $x(V) \leq \alpha$.*

Let us examine the equivalence of the two conjectures. Suppose G is a graph which has properties P_1 and P_2. Let H (not a clique) be an induced subgraph such that $x(V_H) \leq \alpha_H$ is facet-inducing for $P(H)$. Suppose $v \in V - V_H$, then since H contains a bad set, lifting results in a coefficient of 1 for the new node v.

Conversely, suppose that G is a graph such that lifting a non-clique rank inequality which is facet-inducing for an induced subgraph of G results in a rank inequality for a larger subgraph. Now suppose H is a minimally imperfect subgraph of G. If $v \in V - V_H$, then since lifting the inequality $x(V_H) \leq \alpha_H$ yields a rank inequality, we must have $\alpha_{V_H - N(v)} = \alpha_H - 1$. It now follows that G has property P_2. Repeating this lifting process we obtain the inequality $x(V) \leq \alpha_H$. Thus $\alpha_H = \alpha$, and so G has property P_1.

We end this section by noting that the Near-Perfect Conjecture holds for graphs G, with $\alpha = 2$.

Theorem 4.11 *For a graph G with $\alpha = 2$, the following are equivalent:*

(1) *G is near-perfect.*

(2) *G has properties P_1 and P_2.*

(3) *For each node v, $G_{N(v)}$ is perfect.*

Proof: We already know from Propositions 4.2 and 4.3 that (1) implies (2). Now suppose G is a graph with properties P_1 and P_2 and $v \in V$. If $G_{N(v)}$ is not perfect, then it contains an induced minimally imperfect subgraph, H say. But then $\alpha_{H-N(v)} = 0 \neq \alpha_H - 1$, a contradiction. Hence G must also satisfy (3).

We now show that (3) implies (1). This follows from Theorem 2.7 which states that any facet-inducing inequality of $P(G)$ can be scaled to be in the form $2x(K) + x(\Gamma(K)) \leq 2$, where K is a clique such that $\bar{G}_{\Gamma(K)}$ is nonbipartite (i.e., $G_{\Gamma(K)}$ is imperfect). Thus G is near-perfect if and only if $\Gamma(K)$ is perfect for each nonempty clique K, or equivalently $G_{N(v)}$ is perfect for each node v. □

4.3 Towards the Near-Perfect Conjecture

As noted previously, the Near-Perfect Conjecture is equivalent to Conjecture 4.8. We now study the structure of a minimal node counterexample to this latter conjecture. These are graphs which have properties P_1, P_2 and not P_3 but for which every proper induced subgraph has property P_3. We show that the node set of any such graph can be partitioned into two sets Q and $\bar{Q}$ which satisfy:

- Q is a universal stable set of G of size at most $\alpha - 1$,
- $\bar{Q}$ induces a minimally imperfect subgraph.

Any graph with properties P_1 and P_2 which can be partitioned in this fashion is called *decomposable relative to the set* Q and (G, Q) is called a *decomposition* (of G). The pair (G, Q) is called a *strong decomposition* if it is a node minimal counterexample to Conjecture 4.8.

We need the following fact.

Lemma 4.12 *If A, B are $m \times n$ matrices and $m > n$, then $A \cdot B^T$ is singular.*

Proof: Let $a_1, \ldots a_m$ be the rows of A. Since $m > n$ we may assume that there exist real numbers $\lambda_2, \ldots \lambda_m$ such that $a_1 = \sum_{i=2}^m \lambda_i a_i$. Thus $a_1 \cdot B^T = \sum_{i=2}^m \lambda_i (a_i \cdot B^T)$, that is, the first row of $A \cdot B^T$ is a linear combination of the last $m - 1$ rows. □

We use this lemma to show that we can find a universal stable set in a minimum counterexample to Conjecture 4.8.

Theorem 4.13 *If $|V_G| \leq \omega\alpha$ and $G - v$ is ω-colourable for each node v in some maximum stable set S, then G has a universal stable set of size at least $r = max\{1, |V| - (\omega - 1)\alpha - 1\}$.*

Proof: Let S_0 be a maximum stable set of G. For each $v \in S_0$, let $S_1^v, \ldots S_\omega^v$ be a colouring of $G - v$. Then $\mathcal{S} = S_0 \cup (\cup_{v \in S_0} \{S_i^v\}_{i=1}^{\omega})$ is a collection of $\alpha\omega + 1$ stable sets of size at least r. Note that for each maximum clique K, if $v \in S_0 - K$, then K must intersect each of $S_1^v, \ldots S_\omega^v$. Thus we deduce:

(9) each ω-clique is disjoint from at most one member of $\mathcal{S}$.

Let A be a matrix whose rows are incidence vectors of the stable sets in $\mathcal{S}$. If no stable set in $\mathcal{S}$ is universal, then for each $S \in \mathcal{S}$ we can choose an ω-clique K_S, such that $S \cap K_S = \phi$. Let B be an $m \times n$ matrix such that for $i = 1, \ldots m$, the i^{th} row of B is χ^{K_S} if the i^{th} row of A is χ^S. Then $A \cdot B^T = J - I$ which is nonsingular, contradicting Lemma 4.12. □

The idea of constructing the collection $\mathcal{S}$ as defined in the previous proof was first used by Bland, Huang and Trotter [3] to prove Theorem 3.10. The construction is used again in the next theorem (the first part of the proof is nearly identical). We show that, in the definition of an (α, ω)-graph, we can remove the condition of $G - v$ being clique α-coverable for each node v if we insist that there are no universal stable α-sets.

Theorem 4.14 *A graph G is partitionable if and only if for some $p, q \geq 2$ such that $|V| = pq + 1$,*

- *G has a family of $|V|$ stable p-sets, $\mathcal{S}$, such that each node is in exactly p of the sets in $\mathcal{S}$.*
- *G has no stable p-set which intersects every q-clique.*

Proof: Let A be a matrix whose rows are incidence vectors of the sets in $\mathcal{S}$. By hypothesis, for each $S \in \mathcal{S}$ we can choose an n-clique K_S such that $S \cap K_S = \phi$. Let B be a matrix whose i^{th} row is χ^{K_S} if the i^{th} row of A is χ^S. Then

$$\bar{1} \cdot A \cdot B^T \cdot \bar{1} = q(\bar{1} \cdot B^T \cdot \bar{1}) = q(|V|p).$$

Hence $A \cdot B^T$ has exactly $|V|(|V| - 1)$ ones and $|V|$ zeros, that is each column has exactly one zero and so $A \cdot B^T = J - I$. Since $J - I$ is nonsingular, each of A and B^T is a nonsingular $|V| \times |V|$ matrix. Thus for each $v \in V$, there is a unique solution to:

(10) $$B^T \cdot x = \bar{1} - \chi^{\{v\}}.$$

Furthermore, the unique solution to (10) must also be the unique solution to

$$A \cdot B^T \cdot x = A \cdot (\bar{1} - \chi^{\{v\}}).$$

But the v^{th} column of A satisfies this last equation. Hence the solution to (10) is $(0,1)$-valued. It follows that for each node v, $G - v$ can be partitioned into p q-cliques. Similarly, $G - v$ can be partitioned into q stable sets of size p. It is straightforward to check now that $p = \alpha$, $q = \omega$ and so G *is* an (α, ω)-graph. □

We have the following consequence.

Theorem 4.15 *If* $|V| = \alpha\omega + 1$, *G has no universal stable* α*-set, and for some stable* α*-set* S_0, $G - v$ *is* ω*-colourable for each node* $v \in S_0$, *then G is an* (α, ω)*-graph.*

The next theorem is helpful in describing the structure of (α,ω)-graphs as induced subgraphs.

Theorem 4.16 *If G is an* (α,ω)*-graph and H is a proper induced subgraph which is a partitionable graph, then* $\alpha_H < \alpha$ *and* $\omega_H < \omega$.

Proof: Suppose H is a proper induced subgraph of G which is an (α_H, ω_H)-graph. Without loss of generality, $\omega_H < \omega$. Now suppose $\alpha_H = \alpha$. Consider the LP

$$(1)\qquad \begin{array}{rl} \min & \bar{1} \cdot x \\ x(K) & \geq \omega - \omega_H \quad \text{for each } \omega\text{-clique } K, \\ x & \geq 0. \end{array}$$

Clearly χ^{V-V_H} is a solution to (1) and $\bar{1} \cdot \chi^{V-V_H} = \omega\alpha + 1 - \omega_H\alpha_H - 1 = (\omega - \omega_H)\alpha$. But the dual of (1) is:

$$(2)\qquad \begin{array}{rl} \max & (\omega - \omega_H)(\bar{1} \cdot y) \\ \sum_{K: v \in K} y_K & \leq 1 \text{ for each node } v, \\ y & \geq 0. \end{array}$$

Theorem 3.10 implies that setting $y_K = \frac{1}{\omega}$ for each ω-clique K yields a feasible solution, y, to (2) such that $\bar{1} \cdot y = (\omega - \omega_H)(\alpha + \frac{1}{\omega})$. This contradicts weak LP duality, therefore $\alpha_H < \alpha$. □

We use this theorem to prove the following result.

Corollary 4.17 *If G has property* P_1 *and H is an induced subgraph which is a partitionable graph, then H is minimally imperfect.*

Proof: Suppose H is an induced subgraph which is a partitionable graph. If H is not minimally imperfect, then it contains some induced subgraph H' which *is* minimally imperfect. Theorem 4.16 implies that $\alpha_{H'} < \alpha_H$ which contradicts the fact that G has property P_1. □

We can now prove the main structural result.

Theorem 4.18 *If G is a minimal counterexample (with respect to the node set) to Conjecture 4.8, then G is decomposable.*

Proof: Suppose that G is a minimum counterexample. Then $|V| \leq \omega\alpha$ and $\chi > \omega$. Set $r = |V| - (\omega - 1)\alpha - 1$. By minimality and Proposition 4.5 we have

(11) G has no universal stable set of size greater than r.

Since G is imperfect and has property P_2, each node is in a stable α-set. Thus $\alpha_{G-v} = \alpha$ for each node v. Also $\omega_{G-v} = \omega$ for each node v (otherwise v would be in a universal stable α-set). Thus by minimality, $G - v$ is ω-colourable for each node v. Hence by (11) and Lemma 4.13, G has a universal stable set, S, of size r. In particular, note that r is positive. Now $\omega_{\bar{S}} = \omega - 1$ and $|\bar{S}| = \omega_{\bar{S}}\alpha + 1$. Thus $G_{\bar{S}}$ is imperfect and so $\alpha_{\bar{S}} = \alpha$. Since any universal stable set of $G_{\bar{S}}$ is also universal for G, $G_{\bar{S}}$ cannot have a universal stable α-set.

It follows that $\omega_{\bar{S}_{-v}} = \omega_{\bar{S}}$ for each node v (otherwise v would be in a universal stable α-set) and since each node $v \in \bar{S}$ is in a stable α-set of $G_{\bar{S}}$ (by property P_2), $\alpha_{\bar{S}_{-v}} = \alpha$. Hence $|\bar{S} - v| \leq \omega_{\bar{S}_{-v}}\alpha_{\bar{S}_{-v}}$ for each node v and so $G_{\bar{S}_{-v}}$ is $\omega_{\bar{S}_{-v}}$-colourable. Theorem 4.14 now implies that $G_{\bar{S}}$ is an $(\alpha, \omega-1)$-graph. Hence by Corollary 4.17, $G_{\bar{S}}$ is minimally imperfect. □

For a decomposition (G, Q), we denote by $\mathcal{M}(\bar{Q})$ the collection of maximum cliques in $G_{\bar{Q}}$. We say a clique K in $\mathcal{M}(\bar{Q})$, is *straddled* by a node $v \in Q$, if $K \subseteq N(v)$. We denote by $\mathcal{H}_Q(v)$ (or $\mathcal{H}(v)$ if the context is clear) the collection of all cliques in $\mathcal{M}(\bar{Q})$ which are straddled by v. We now bound the number of cliques straddled by a node in Q.

Lemma 4.19 *Suppose G has property P_2. If (G, Q) is a decomposition and $v \in Q$, then $|\mathcal{H}_Q(v)| \leq \omega$.*

Proof: Suppose $K \in \mathcal{H}_Q(v)$. Theorem 3.10 states that $\{K\} \cup (\cup_{x\in K}\{K_i^x\}_{i=1}^{\alpha_Q})$ is exactly the set $\mathcal{M}(\bar{Q})$. Since $\alpha_{\bar{Q}-N(v)} = \alpha_{\bar{Q}} - 1$ it follows that for each $x \in K$, at most one of the cliques in the partition $K_1^x, \ldots K_{\alpha_Q}^x$ is straddled by v. Hence $|\mathcal{H}_Q(v)| \leq |K| + 1$. □

In light of Lemma 4.19, for $k = 1, \ldots \omega$ we call v a *k-node* if $|\mathcal{H}_Q(v)| = k$. Recall that a decomposition (G, Q) for which G is a minimal counterexample to Conjecture 4.8 is called a *strong decomposition*. The next lemma shows an even stronger condition which must be possessed by these decompositions.

Lemma 4.20 *If (G, Q) is a strong decomposition, then each maximum clique of $G_{\bar{Q}}$ is straddled by some node.*

Proof: Suppose that K is in $\mathcal{M}(\bar{Q})$. By Theorem 3.10 there is a stable α-set S of $G_{\bar{Q}}$ which intersects every maximum clique of $\mathcal{M}(\bar{Q})$ except K. Since S is not a universal stable set of G (by (11)), K must be contained in a maximum clique of G. □

Lemma 4.20 and Theorem 3.10 imply the following.

Corollary 4.21 *For a strong decomposition (G, Q) we have*

$$\text{(12)} \qquad \sum_{v\in Q} |\mathcal{H}(v)| \geq \alpha(\omega - 1) + 1.$$

The right hand side of (12) may be even larger when there are cliques of $\mathcal{M}(\bar{Q})$ which are straddled by more than one node. We now show that such cliques must exist.

The *straddle intersection graph* of (G, Q), denoted G^Q, is a bipartite graph with bipartition $(Q, \mathcal{M}(\bar{Q}))$; there is an edge between a node v and clique K if $K \in \mathcal{H}(v)$. Using this terminology, Lemma 4.20 states that each node in $\mathcal{M}(\bar{Q})$ has degree at least one, or:

$$\text{(13)} \qquad \sum_{v\in Q} |\mathcal{H}(v)| = \sum_{K\in\mathcal{M}(\bar{Q})} d_{G^Q}(K).$$

The next lemma proves that the right hand side of (13) is larger than $|\mathcal{M}(\bar{Q})|$ by at least one half the number of ω-nodes in Q.

Lemma 4.22 *If (G, Q) is a strong decomposition and v is an ω-node, then there is some other node u, such that $\mathcal{H}(v) \cap \mathcal{H}(u) \neq \phi$.*

Proof: Let v be an ω-node. Since G has property P_2, there is some stable α-set, S, such that $S \cap Q = \{v\}$. Let $K \in \mathcal{H}(v)$. Then without loss of generality, for each $x \in K$, S intersects $K_2^x, \dots K_\alpha^x$. Thus $\mathcal{H}(v) = \{K\} \cup (\cup_{x \in K}\{K_1^x\})$. Hence S intersects each clique in $\mathcal{M}(\bar{Q})$ except those in $\mathcal{H}(v)$. Since S is not universal, some other node must straddle one of the cliques in $\mathcal{M}(\bar{Q})$. □

We use this lemma to enlarge the class of graphs for which we know Conjecture 4.8 holds. This new class is considerably more complex than the graphs with stability number two.

Theorem 4.23 *If G has properties P_1, P_2 and $\alpha \leq 6$, then G has property P_3.*

Proof: Suppose (G, Q) is a strong decomposition. Let m be the number of ω-nodes in Q. Then by Lemma 4.22 and (13)

$$\sum_{v \in Q} |\mathcal{H}(v)| \geq \alpha(\omega - 1) + 1 + m/2.$$

But also

$$\sum_{v \in Q} |\mathcal{H}(v)| \leq |Q|(\omega - 1) + m \leq (\alpha - 1)(\omega - 1) + m.$$

Combining these two inequalities yields $m \geq 2\omega$. Now $m \leq \alpha - 1$ and clearly $\omega_Q \geq 2$, hence $\alpha \geq 7$. □

This also implies the Near-Perfect Conjecture for the same class of graphs.

Corollary 4.24 *For a graph G with $\alpha \leq 6$, the following are equivalent:*

(1) *G is near-perfect,*

(2) *G has properties P_1 and P_2.*

We show in the next section that we may improve the bound of 6 in Theorem 4.23.

We now give the main result of this section which is a *finite* characterization of near-perfect graphs; that is, we do not require a property to hold for every replication. The theorem "almost" coincides with the Weak Near-Perfect Conjecture. A graph G has property P_3^2 if for any stable set S, the graph obtained by replicating once, each node in S, has property P_3 (or equivalently is not a strong decomposition).

Theorem 4.25 *A graph G is near-perfect if and only if it has properties P_1, P_2 and P_3^2.*

Proof: Theorem 4.6 implies that we need only show that if G has properties P_1, P_2 and P_3^2, then it has property P_3^*. If this is not the case, then some replication $G[w]$ of G gives rise to a strong decomposition $(G[w], Q)$. By Corollary 3.5 $G[w]_Q$ contains no replicated nodes. Hence any new nodes may be presumed to be in Q, but this contradicts G having property P_3^2. □

4.4 Assuming the Strong Perfect Graph Conjecture

We now show that the Near-Perfect Conjecture follows if the Strong Perfect Graph Conjecture holds. Our approach is to examine how the ω-nodes in a decomposition interact, i.e., how they jointly straddle cliques. In this section, all graphs are assumed to have both properties P_1 and P_2.

Lemma 4.26 *If $(G, \{v\})$ is a decomposition of G such that v is an ω-node, then:*

1. *there is a unique stable α-set of G which contains v,*
2. *for $K \in \mathcal{M}(G - v) - \mathcal{H}(v)$, v is not contained in a stable α-set of $G - K$.*

Proof: We first show 1. Suppose that S is a stable α-set containing v and $x \in S - v$ is a node which is not contained in every such stable set. Now let K be an $(\omega - 1)$-clique in $G - v$ which contains x. Clearly, $K \notin \mathcal{H}(v)$. Thus for some node $z \in K$, two of the cliques in the minimum clique cover of $G - z$ must be straddled by v (otherwise $|\mathcal{H}(v)| \leq |K|$). This implies that z is in every stable α-set containing v, but $zx \in E$, a contradiction.

Now suppose that $K \in \mathcal{M}(G - v) - \mathcal{H}(v)$ and $K' \in \mathcal{H}(v)$. We have seen that v straddles exactly one of the cliques in the clique cover of $G - x$ for each $x \in K'$. Furthermore, one of these clique covers, for $z \in K'$ say, contains the clique K as well. Since $zv \in E$, it follows that each stable α-set containing v must intersect K. □

A graph G is called *sparse* if it is triangle-free and has a subgraph which is an odd cycle of length $|V|$. We show that sparse graphs are imperfect.

Lemma 4.27 *If G is sparse, then it is imperfect.*

Proof: Since G contains a spanning odd cycle, we have that $\alpha \leq |V|/2$. Hence, since $|V|$ is odd we have that $|V| > 2\alpha = \omega\alpha$. □

We call a decomposition (G, Q) for which G_Q is an odd hole, a *hole-decomposition.* For the next few paragraphs we let $(G, \{v\})$ be a hole-decomposition and v be an ω-node. Note that v is of one of three types. By our assumption, $G - v$ is simply an odd cycle on $2\alpha + 1$ nodes. Hence each clique straddled by v is an edge. We say that v is a *type i* node if the length of a longest path in $G_{N(v)}$ is i. Figure 5 shows a type one node and a type two node. Let T be the neighbours of v which are contained in a triangle of G. It follows that if v is of type i, then $G - (v \cup T)$ has exactly $4 - i$ components (each of which is a path). A *segment* of v, is a subset of the nodes of the form $U \cup \{u, v\}$, where G_U is a component of $G - (v \cup T)$ and $\{x, y\}$ are the two nodes of T which are adjacent to U (i.e., to some node of U). Note that for any segment X, $G_{X \cup \{v\}}$ is triangle-free and so Lemma 4.27 implies the following.

Lemma 4.28 *If X is a segment of v, then $|X|$ is odd.*

We are now ready to prove our main lemma.

Lemma 4.29 *If $(G, \{u, v\})$ is a hole-decomposition and u, v are ω-nodes, then $\mathcal{H}(u) \cap \mathcal{H}(v) \neq \phi$.*

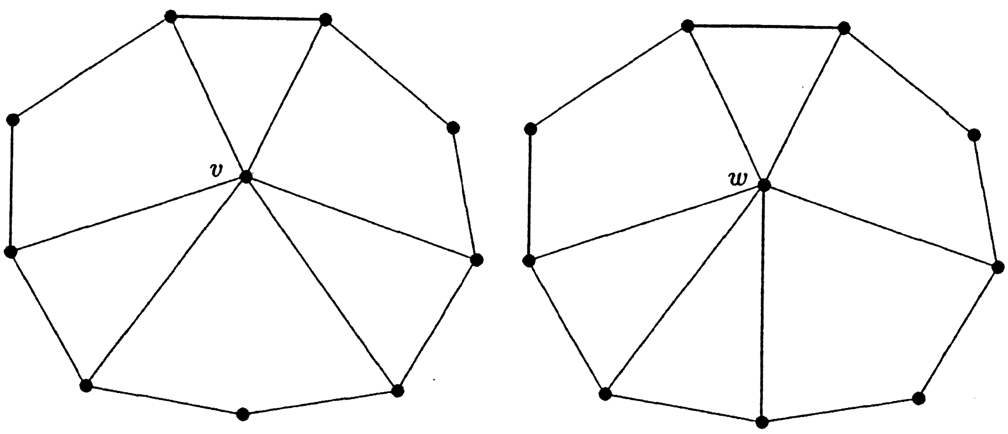

Figure 5: Example of a type one node (v) and type two node (w).

Proof: Suppose that the statement is false and let u, v satisfy the hypotheses but $\mathcal{H}(u) \cap \mathcal{H}(v) = \phi$. We consider three cases.

Case 1. *At least one of the nodes, say u, is of type three.*

Let the edges straddled by u be x_0x_1, x_1x_2, x_2x_3. It follows that these edges lie in the graph induced by some segment, X say, of v. Hence $G' = G_{\{u,v\} \cup (X - \{x_1, x_2\})}$ is triangle-free. Furthermore, by Lemma 4.28, G' has an odd number of nodes. Hence by Lemma 4.27, G' is imperfect, contradicting the fact that G has property P_1.

Case 2. *At least one of the nodes, say u, is of type two.*

Let u' be the node of $N(u)$ which is contained in two of the triangles containing u.

Case 2a. *A segment, X, of u is contained in some segment Y, of v.*

Since $|X|, |Y|$ are odd it follows that $G_{\{u,v\} \cup (Y - (X \cup u'))}$ has an odd number of nodes, and is hence sparse. Thus by Lemma 4.27 this contradicts the fact that G has property P_1.

Case 2b. *There are distinct segments Y_1, Y_2 of v, which contain edges straddled by u.*

Let Y_1 be the segment of v which contains u'. Let P be a (u, v)-path whose internal nodes are contained in $Y_1 - N(Y_2)$. Now since $|Y_2|$ is odd, there is another (u, v)-path in $G_{Y_2 \cup \{u,v\}}$ whose length is of different parity from P. The union of the nodes of the two paths induces a sparse graph, a contradiction.

Case 3. *Both u and v are type one.*

Case 3b. *Two segments X_1, X_2 of one of the nodes, u say, is contained in a single segment, X, of v.*

In this case $(X \cup \{u, v\}) - (X_1 \cup X_2)$ induces a sparse graph, a contradiction.

Case 3b. *There is some segment X, of u say, which contains exactly one edge straddled by v.*

As in Case 2b, there are (u, v)-paths in $G_{X \cup \{u,v\}}$ whose lengths are of different parity. We obtain an induced sparse subgraph by taking the union of one of these paths with an appropriate (u, v)-path which passes through one of the other segments of u. □

We now prove the main result of this section.

Theorem 4.30 *No hole-decomposition is a strong decomposition.*

Proof: Suppose that (G, Q) is a hole-decomposition. By Lemmas 4.19 and 4.29 we have

$$\sum_{v \in Q} |\mathcal{H}(v)| \leq |Q|(\omega - 1) + 1.$$

If (G, Q) is also a strong decomposition, then by Corollary 4.21 we have

$$\sum_{v \in Q} |\mathcal{H}(v)| \geq \alpha(\omega - 1) + 1.$$

Thus $|Q| \geq \alpha$, contradicting the definition of a decomposition. □

The following is now immediate.

Theorem 4.31 *If the Strong Perfect Graph Conjecture is true, then so too is the Near-Perfect Conjecture.*

This result also implies that we may improve the bound in Theorem 4.23.

Theorem 4.32 *If G has properties P_1, P_2 and $\alpha \leq 10$, then G has property P_3.*

Proof: The proof is similar to Theorem 4.23 except that we may assume that the minimally imperfect graph $G_{\bar{Q}}$ is not an odd hole or antihole. Tucker [22] has shown that any such graph must have a clique of size 4. Hence the lower bound in the proof of Theorem 4.23 becomes $\alpha \geq 2\omega + 1 \geq 2 \cdot 5 + 1$. □

We immediately have the following.

Theorem 4.33 *For a graph G with $\alpha \leq 10$, the following are equivalent:*

(1) *G is near-perfect,*

(2) *G has properties P_1 and P_2.*

4.5 A Polyhedral Characterization of Minimally Imperfect Graphs

We have seen that the complement of a near-perfect graph need not be near-perfect. We show that the only imperfect near-perfect graphs for which the complement is near-perfect are the minimally imperfect graphs. In fact, we only require that both the graph and its complement have property P_1 (or P_2).

Lemma 4.34 *The following are equivalent for an imperfect graph G:*

(1) G is minimally imperfect,

(2) both G and $\bar{G}$ have property P_1,

(3) both G and $\bar{G}$ have property P_2.

Proof: Clearly (1) implies (2). Since each node of a minimally imperfect graph is in a maximum stable set *and* a maximum clique (1) also implies (3).

It is straightforward to check that (2) and (3) both imply that if S induces a minimally imperfect graph in G and $v \in V - S$, then $\omega_{S \cup \{v\}} = \omega_S$ and $\alpha_{S \cup \{v\}} = \alpha_S$. We use this fact to show that both (2) and (3) imply (1). For let S be a subset of V such that G_S is minimally imperfect and subject to this $\omega_S + \alpha_S$ is minimized. If $S \neq V$, then consider $v \in V - S$. Set $N = N(v) \cap S$, $\bar{N} = S - N$. Now consider $w \in S$ and set $G^w = G_{S \cup \{v\}} - w$. Note that G^w has the same number of nodes as S, that is, $\alpha_S \omega_S + 1$. Furthermore since G_S is minimally imperfect it is easy to show that $\alpha_{G^w} = \alpha_S$ and $\omega_{G^w} = \omega_S$ (i.e., deleting w does not destroy all of the maximum cliques or stable sets). Hence G^w has $\alpha_{G^w}\omega_{G^w} + 1$ nodes and so contains a minimally imperfect subgraph. But by our choice of S, G^w must itself be minimally imperfect. In particular, v must be in exactly ω_S cliques of size ω_S. Thus $N - w$ contains exactly ω_S cliques of size $\omega_S - 1$. Thus choosing $w \in \bar{N}$ implies that N contains exactly ω_S cliques of size $(\omega_S - 1)$ and choosing w to be some node in a maximum clique of G^w which contains v, implies that N contains at least $(\omega_S + 1)$ cliques of size $(\omega_S - 1)$, a contradiction. Thus S must be the whole node set V. □

We now give the promised polyhedral characterization and a new proof of Theorem 1.5.

Theorem 4.35 *An imperfect graph G is minimally imperfect if and only if both G and $\bar{G}$ are near-perfect.*

Proof: First suppose that G is minimaly imperfect. Clearly G has properties P_1, P_2 and P_3. Furthermore, replicating each node of a stable set S of G cannot result in a strong decomposition. Otherwise by Lemma 4.20 S would be a universal stable set of G. Hence by Theorem 4.25 G is near-perfect.

Conversely, if G and $\bar{G}$ are near-perfect, then they both have property P_2 by Proposition 4.3 and so by Lemma 4.34, G is minimally imperfect. □

4.6 Complements of Near-Perfect Graphs

A graph G is called *ω-separable* if there is a nonempty, proper subset S of V such that $\omega = \omega_S + \omega_{\bar{S}}$. A graph which is not ω-separable is called *ω-nonseparable.* We have seen that if G is near-perfect, then

- if $|V| \leq \alpha\omega$, then G is ω-colourable, and hence ω-separable,
- if $|V| = \alpha\omega + 1$, then G is minimally imperfect, and hence ω-nonseparable.

We do not know what happens when $|V| > \omega\alpha + 1$; could it be that all such near-perfect graphs are ω-separable?

Question 4.36 *Are the minimally imperfect graphs the only ω-nonseparable near-perfect graphs?*

We consider a consequence of this question, but first we state a well known lemma.

Lemma 4.37 *If G is a graph such that $x(V) \leq \omega$ is facet-inducing for $P(\bar{G})$, then G is ω-nonseparable.*

We now show that an affirmative answer to Question 4.36 would yield a nice description for the stable set polytope of the complement of a near-perfect graph.

Proposition 4.38 *If the answer to Question 4.36 is yes, then for any near-perfect graph G:*

$$P(\bar{G}) = \left\{ x \in \mathbb{R}^V : \begin{array}{l} x \geq 0 \\ x(S) \leq 1 \text{ for each stable set } S \\ x(I) \leq \omega_I \text{ for each minimally imperfect induced subgraph } I \end{array} \right\}.$$

Proof: Suppose that the answer to Question 4.36 is yes. Let G be near-perfect and $a \cdot x \leq \gamma$ be a nontrivial facet-inducing inequality for $P(\bar{G})$. Without loss of generality a and γ are integral and $gcd(\{\gamma\} \cup \{a_v\}_{v \in V}) = 1$. Let G' be the graph obtained by making a_v copies of each node v. Note that G' is also near-perfect. If a is 0-1 valued, then $G_{\{v:a_v=1\}} = G'$ and so by Lemma 4.37 is ω-nonseparable and hence minimally imperfect. Otherwise, G' contains a pair of replicated nodes and so by Corollary 3.5 is not minimally imperfect. By assumption, there is a nonempty, proper subset Z, of $V_{G'}$, such that $\gamma = \omega_{G'} = \omega_Z + \omega_{\bar{Z}}$. For each node $v \in V$ let a'_v be the number of copies of v in G' that are contained in Z. Evidently $a' \cdot x \leq \omega_Z$ is a valid inequality for $P(\bar{G})$. Furthermore, if Q is a clique with $a \cdot \chi^Q = \gamma$, then Q also satisfies $a' \cdot \chi^Q = \omega_Z$. Thus the two inequalities define the same facet. This implies that $a = \frac{\gamma}{\omega_Z} a'$. Hence by our choice of a, $\gamma = \omega_Z$. Thus $\bar{Z} = \phi$, a contradiction. □ Note that the statement about the linear description in Proposition 4.38 is equivalent to stating that each fractional vertex of the fractional stable set polytope is of the form $\frac{1}{\omega_I}\chi^I$ for some minimally imperfect G_I. It can be shown that the Near-Perfect Conjecture is implied by this property. (B. Reed has independently shown this.)

Proposition 4.38 and Theorem 4.11 together imply the following.

Corollary 4.39 *Suppose the answer to Question 4.36 is yes. Then any triangle-free graph G, such that $G - N(v)$ is bipartite for each node v, is t-perfect.*

4.7 The Recognition Problem

The recognition problem associated with a class $\mathcal{P}$ of graphs is a decision problem which takes a graph G as input and outputs *YES* if $G \in \mathcal{P}$ and *NO* otherwise. We denote by *PERFECT*, *MINIMPR* and *NEARPERF* the recognition problems associated with classes of perfect, minimally imperfect and near-perfect graphs respectively. At present, none of these problems is known to be polynomially solvable. Grötschel et al. [12] and Cameron [5] have shown that *PERFECT* is in **coNP**.

Theorem 4.40 *The problem* PERFECT *is in* **coNP**.

Furthermore it is easy to show:

Theorem 4.41 MINIMPR *is in* **coNP**.

On the other hand the following problems are still open.

Conjecture 4.42 PERFECT *is in* **NP**.

Conjecture 4.43 MINIMPR *is in* **NP**.

Note that the first conjecture implies the second. Conjecture 4.42 is stronger also in the sense that an affirmative answer to the Strong Perfect Graph Conjecture bears would immediately imply a polynomial time algorithm for *MINIMPR* whereas it is not clear how this would bear on Conjecture 4.42.

We now outline a proof to show that *NEARPERF* is in **coNP** but first we need one fact. Suppose G is near-perfect and $v \in V$. Since G has property P_2, $\alpha_{G-N[v]} < \alpha$ and since G has property P_1 we deduce:

Remark 4.43.1 *If G is near-perfect, then for each node $v \in V$, $G - N[v]$ is perfect.*

Now suppose G is a graph which is not near-perfect. If there is some node v such that $G - N[v]$ is imperfect, then we need only display an induced partitionable graph in $G - N[v]$. So assume that no such node exists. To show that G is not near-perfect it is enough to show that there is some nontrivial facet-inducing inequality of $P(G)$ which is not a constant multiple of any of the inequalities in (3). Note that it is easy to check that a is not a constant multiple of $\bar{1}$ or χ^K for some clique K. Suppose $a \cdot x \leq \gamma$ is such an inequality. We can verify that this is valid for $P(G)$ simply by showing for each node v that $\max\{a \cdot \chi^S : S \in \mathcal{S}(G - N[v])\} \leq \gamma - a_v$. Since $G - N[v]$ is perfect, this can be done by displaying an appropriate clique cover of $G - N[v]$. Finally, to see that our chosen inequality is facet-inducing we must exhibit $|V|$ linearly independent incidence vectors of stable sets which satisfy the inequality with equality. Thus we have:

Theorem 4.44 *The problem* NEARPERF *is in* **coNP**.

We close this section with a remark on how near-perfect graph recognition relates to perfect graph recognition.

Remark 4.44.1 *If* NEARPERF *is in* **NP**, *then* PERFECT *is in* **NP**.

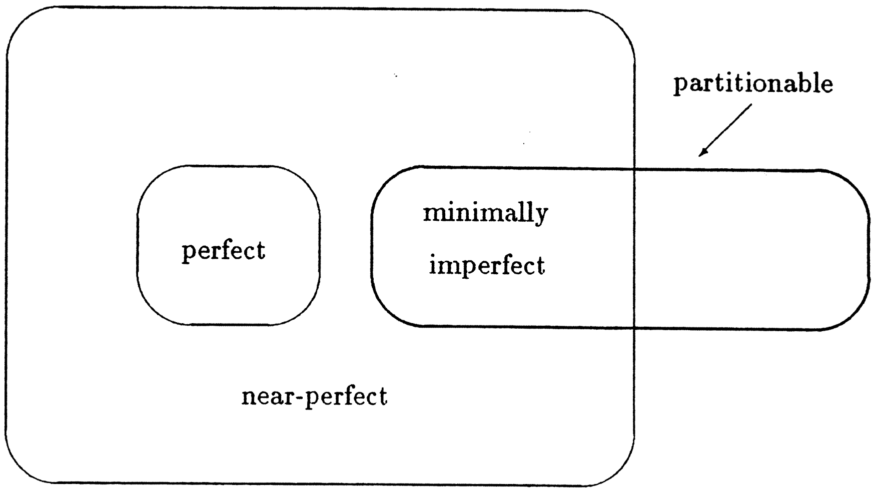

Figure 6:

This is easy to see, for suppose G is a perfect graph. If *NEARPERF* is in **NP**, then we can give a certificate to show that G is near-perfect. In order to show that G is perfect we need only show that $x(V) \leq \alpha$ is not facet-inducing for $P(G)$ (since this implies that $P(G)$ is given by the clique inequalities). This can be done by exhibiting a stable set and a clique cover of G with the same size (i.e., of size α). (Note that a near-perfect graph with $\theta_G = \alpha_G$ is perfect.)

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