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A. Schrijver

Tait's flyping conjecture for well-connected links

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Tait's Flyping Conjecture for Well-Connected Links

A. Schrijver

Centre for Mathematics and Computer Science
P.O. Box 4079
1009 AB Amsterdam
The Netherlands

and

Department of Mathematics
University of Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands

Abstract. Call a link *well-connected* if its diagram has no 2-edge cut sets, and the only 4-edge cut sets are those made by a crossing. We prove Tait's flyping conjecture for well-connected links, i.e., any two well-connected alternating links are equivalent (= ambient isotopic), if and only if their diagrams are the same (up to trivial operations).

1980 Mathematics Subject Classification: 57M25.
Key words: knot, link, alternating, Tait conjecture, flyping, well-connected.

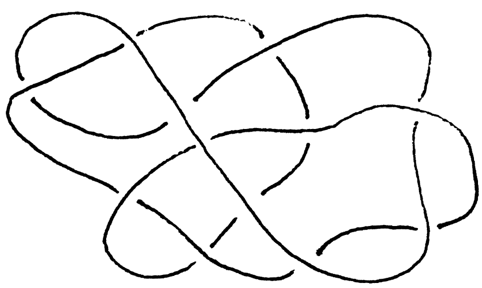
1. Introduction

A *knot* is a subset of $\mathbb{R}^3$ homeomorphic to the unit circle S_1. A *link* is a (nonempty) disjoint union of knots.

For the purpose of this paper we may assume that for each link K considered, the projection $\pi[K]$ of K to $\mathbb{R}^2$ is a 4-regular planar graph, with a finite set of vertices, edges, and faces. Here π denotes the projection from $\mathbb{R}^3$ onto $\mathbb{R}^2$ with $\pi(x_1, x_2, x_3) := (x_1, x_2)$. Throughout, by *projecting* we mean projecting by π.

We can associate with a link K the *diagram* of K that arises by projecting K to $\mathbb{R}^2$, indicating at each crossing which of the two curve segments goes over the other:

(1)



The link, or its diagram, is called *alternating* if, when following each component of the link in its diagram, we go alternatingly over and under, like in

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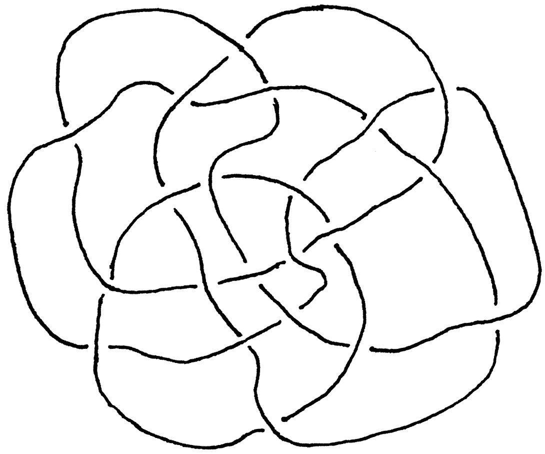
(2)

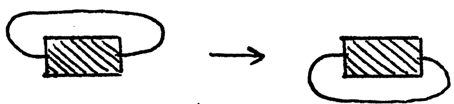


Two links K and K' are *equivalent* if there exists an isotopy of $\mathbb{R}^3$ bringing K to K'. (An *isotopy* of a topological space X is a continuous function $\Phi : [0,1] \times X \longrightarrow X$ such that $\Phi(0,u) = u$ for each $u \in X$, while for each fixed $t \in [0,1]$ the function $\Phi(t,.)$ is a homeomorphism of X. It *brings* Y *to* Y' if $\Phi(1,Y) = Y'$.)

Two link diagrams are called *equivalent* if one arises from the other by a series of the following operations:

(3) (i) turning the diagram upside down,

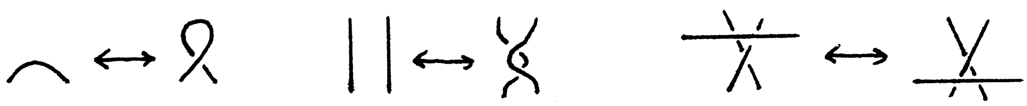
(ii) rerouting one of the edges of $\pi[K]$ through the unbounded face, as in



(The box denotes the rest of the diagram.)

Remark 1. It has been shown by Reidemeister [5] that if K and K' are equivalent, then their diagrams can be obtained from each other by the following operations:

(4)

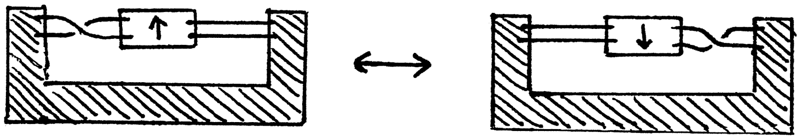


These operations are called the *Reidemeister moves.* In our proof below, we make no use of this result (except for deriving tameness of isotopies). ∎

Clearly, if two links have equivalent diagrams, they are equivalent. The converse need not hold in general. However, as we show in this paper, for well-connected alternating links the converse *does* hold. We call an alternating link K *well-connected* if the graph $\pi[K]$ has no 2-edge cut sets, and the only 4-edge cut sets are those determined by one vertex of $\pi[K]$.

Theorem. *Let K and K' be well-connected alternating links. If K and K' are equivalent, then their diagrams are equivalent.*

This is a special case of the *Tait flyping conjecture* [6], which does not require well-connectedness but the weaker *reducedness* instead (link K is *reduced* if the graph $\pi[K]$ has no loops and no cutpoints), while the operations (3) should be extended by *flyping*:

(5) 

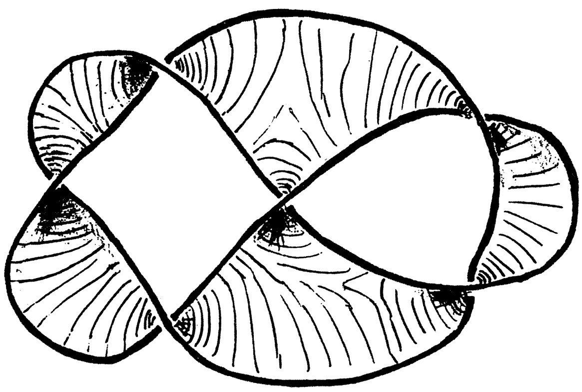
(up to exchanging up and down). Note that flypes are not possible for well-connected links. (Tait: 'The deformation process is, in fact, simply one of *flyping*, an excellent word, very inadequately represented by the nearest equivalent English phrase "turning outside in." '; 'When we *flype* a glove (as in taking it off when very wet, or as we skin a hare), we perform an operation which changes its character from a right-hand glove to a left.')

Remark 2. By an idea of Tait, any link K gives a planar graph H_K as follows. Color the faces of $\pi[K]$ black and white such that adjacent faces have different colors, and such that the unbounded face is colored white. Put a vertex in each black face, and for each crossing, make an edge connecting the vertices in the two black faces incident with the crossing (in such a way that the edge crosses the crossing).

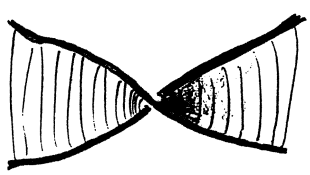
Now a link K is well-connected if and only if the graph H_K is 3-vertex-connected (i.e., has no vertex cut of less than three vertices and has no parallel edges (except if it has only two vertices connected by at most three parallel edges)). ∎

2. Proof of the theorem

We will associate with any link K a compact bordered surface Σ_K in $\mathbb{R}^3$, with $\mathrm{bd}(\Sigma_K) = K$. (bd denotes boundary.) A pictorial impression of Σ_K is given by

(6) 

Here any two black faces are connected at a crossing by a twist as in the Möbius strip:

(7) 

More precisely, Σ_K is defined as follows. For any link K, let V_K denote the set of vertices of $\pi[K]$, and let $v(K) := |V_K|$.

For each vertex v of the graph $\pi[K]$, let $p_v^{\uparrow}$ and $p_v^{\downarrow}$ be the two points in $K \cap \pi^{-1}(v)$, where $p_v^{\uparrow}$ is above $p_v^{\downarrow}$. (Here and below, *above* and *under* refer to larger and smaller x_3 coordinate.)

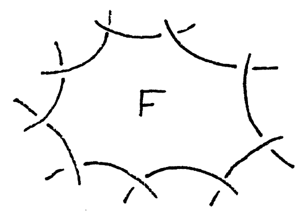
Moreover, let e_v be the open line segment in $\pi^{-1}(v)$ connecting $p_v^{\uparrow}$ and $p_v^{\downarrow}$. Define

(8) $$T := K \cup \bigcup_{v \in V_K} e_v.$$

So T forms a 3-regular graph embedded in $\mathbb{R}^3$, with $2v(K)$ vertices and $3v(K)$ edges.

Call a face F of $\pi[K]$ *even* if F is bounded and when following the boundary of F in clockwise orientation, we follow the edges from up to down, as in

(9)



or if F is unbounded and when following the boundary of F in clockwise orientation, we follow the edges from down to up. The other faces are called *odd*.

Note that of any two adjacent faces, one is even and the other is odd. So if the unbounded face is even, then the white faces are even, and the black faces are odd. If the unbounded face is odd, then the white faces are odd, and the black faces are even.

Note moreover that any link can be transformed to one in which the unbounded face is even, by (possibly) rerouting through the unbounded face (operation (3)(ii)). So putting the condition that the unbounded face be even, is not a restriction.

Let K be a link, such that the unbounded face of $\pi[K]$ is even. Let $\mathcal{B}$ denote the collection of odd faces. Consider an odd face F. The set $\pi^{-1}[\mathrm{bd}(F)] \cap T$ is a simple closed curve, consisting of parts of K and of the line segments e_v, for those vertices v of $\pi[K]$ that are incident with F. So it is the boundary of some open disk D_F so that π maps D_F one-to-one onto F. Fix for each odd face one such open disk D_F. Then we define:

(10) $$\Sigma_K := T \cup \bigcup_{F \in \mathcal{B}} D_F.$$

So Σ_K indeed is a compact bordered surface with boundary K.

Our proof is based on the following two theorems, which might be interesting in their own right:

Theorem A. *Let K and K' be well-connected alternating links such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If K and K' are equivalent, then there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$.*

(S^3 is the 3-dimensional sphere, considered as one-point compactification of $\mathbb{R}^3$.)

Theorem B. *Let K and K' be well-connected alternating links, such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$, then the diagrams of K and K' are equivalent.*

Theorems A and B clearly directly imply the theorem. Although Theorem A above holds in general, to avoid several technicalities, in this paper we prove Theorem A only under the condition that

(11) the unbounded face of $\pi[K]$ is bounded by at least four edges of $\pi[K]$.

This is enough to derive the theorem, since we may assume that either $\pi[K]$ or $\pi[K']$ has at least one face that is bounded by at least four edges. (If all faces of $\pi[K]$ and of $\pi[K']$ are bounded by at most three edges, then, by the well-connectedness of K and K', $\pi[K]$ and $\pi[K']$ have either at most three vertices or both are the octahedron, for which the theorem trivially holds.) Then by rerouting and, possibly, reflecting the diagram, we can obtain condition (11).

Remark 3. In fact a more general statement than Theorem A holds:

(12) Let K and K' be reduced alternating links such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If K and K' are equivalent, then there is an isotopy of S^3 bringing Σ_K to $\Sigma_{\tilde{K}'}$, where $\tilde{K}'$ is a link the diagram of which can be obtained from that of K' by a series of flypings.

∎

Remark 4. The following can be proved by methods similar to those used in this paper to show Theorem B:

(13) Let K and K' be reduced alternating links, such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$, then the cycle spaces of H_K and $H_{K'}$ are isomorphic.

Here the *cycle space* of a graph is the collection of cycles. A *cycle* is an edge-disjoint union of circuits.

Statements (12) and (13) imply that if K and K' are equivalent reduced alternating links such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even, then the cycle spaces of H_K and $H_{K'}$ are isomorphic. By a theorem of Whitney [10], (13) directly implies Theorem B. ∎

3. Preliminaries on links and surfaces

We give some preliminaries on links (cf. Kauffman [2]) and surfaces.

Kauffman [1], Murasugi [3], and Thistlethwaite [7] (cf. Turaev [9]) showed that if K and K' are equivalent reduced alternating links, then $v(K) = v(K')$. In fact they showed that any reduced alternating link K attains the minimum number of crossings (in its diagram) among all links equivalent to K.

A second invariant is obtained as follows. Give each component of K some orientation. This way we obtain an *oriented link*. Then there are two types of crossings, positive and negative:

(14)

The *writhe* $w(K)$ of K is the number of positive crossings minus the number of negative crossings. This number is not invariant under equivalence of links. However, Murasugi [4] and Thistlethwaite [8] showed that if K and K' are equivalent reduced alternating links, then $w(K) = w(K')$. Similarly, Murasugi and Thistlethwaite showed that the number $b(K)$ of odd faces is an invariant for reduced alternating links.

Let K_1 and K_2 be two disjoint oriented links. Consider the diagram made by $K_1 \cup K_2$. Define

(15) $$\mathrm{lk}(K_1, K_2) := \frac{1}{2}((\#\text{ positive } K_1 - K_2 \text{ crossings}) - (\#\text{ negative } K_1 - K_2 \text{ crossings})).$$

(A $K_1 - K_2$ *crossing* is a crossing of K_1 with K_2. # means 'number of'. Here no condition is put on which of K_1 and K_2 is above the other at the crossing.) This number is invariant under isotopy of S^3: if K_1' and K_2' are equivalent to K_1 and K_2, respectively, then $\mathrm{lk}(K_1', K_2') = \mathrm{lk}(K_1, K_2)$ (assuming that K_1' and K_2' are oriented as induced through the isotopy by the orientations of K_1 and K_2). This invariance of lk(.,.) follows directly by considering the Reidemeister moves.

Let K be an oriented link and let Σ be a disjoint union of compact bordered surfaces embedded in $\mathbb{R}^3$ and containing K. We define a number $\tau(K, \Sigma)$ as follows.

If each component of K is an orientation-preserving curve on Σ, we take for each component κ of K a curve $\tilde{\kappa}$ parallel on Σ to κ. The union of these $\tilde{\kappa}$ forms link $\tilde{K}$. Then $\tau(K, \Sigma) := 2\mathrm{lk}(K, \tilde{K})$, where we orient K and $\tilde{K}$ in the same direction.

If at least one component of K is orientation-reversing, we consider a link J homotopic on Σ to the set of closed curves that follow the components of K twice. So each component of J is orientation-preserving. We define $\tau(K, \Sigma) := \frac{1}{4}\tau(J, \Sigma)$.

Clearly, if K and K' are homotopic on Σ, then $\tau(K, \Sigma) = \tau(K', \Sigma)$. Moreover, if some isotopy of S^3 brings K, Σ to K', Σ', then $\tau(K, \Sigma) = \tau(K', \Sigma')$.

Direct calculation shows that for any alternating oriented link K for which the unbounded face of $\pi[K]$ is even:

(16) $$\tau(K, \Sigma_K) = 2(v(K) + w(K)) = 4(\#\text{ positive crossings of } K).$$

Finally, we note that the *Euler characteristic* $\chi(\Sigma_K)$ of Σ_K (= number of faces, minus number of edges, plus number of vertices) is equal to

(17) $$\chi(\Sigma_K) = b(K) - v(K),$$

where $b(K)$ denotes the number of odd faces of the diagram of K. (This follows from the facts that T has $2v(K)$ vertices and $3v(K)$ edges, and that $\Sigma_K \setminus T$ consists of $b(K)$ open disks.)

4. Theorem A

In this section we consider:

Theorem A. *Let K and K' be well-connected alternating links such that the unbounded faces of $\pi[K]$ and of $\pi[K']$ are even. If K and K' are equivalent, then there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$,*

We show Theorem A under the condition that the unbounded face of $\pi[K]$ is bounded by at least four edges of $\pi[K]$.

Proof. It suffices to show:

Lemma. *Let K be a well-connected alternating link such that the unbounded face of $\pi[K]$ is even. Let Σ be the disjoint union of compact bordered surfaces satisfying:*

$$\begin{aligned}(18)\qquad &\text{(i) } \mathrm{bd}(\Sigma) = K,\\ &\text{(ii) } \chi(\Sigma) \geq b(K) - v(K),\\ &\text{(iii) } \tau(K, \Sigma) = 2(v(K) + w(K)).\end{aligned}$$

Then there exists an isotopy of S^3 bringing Σ to Σ_K.

('Disjoint union of compact bordered surfaces' implies that each component of Σ has a nonempty border (being a nonempty disjoint union of closed curves). Observe that condition (18)(iii) is independent of the orientations chosen for K. The conclusion in the Lemma implies that Σ is connected and that equality holds in (18)(ii).)

We prove the Lemma under the condition that the unbounded face of K is bounded by at least four edges.

Remark 5. The Lemma also holds if this last condition is not satisfied. In fact, the Lemma can be extended to reduced, not necessarily well-connected links. In that case the conclusion is that there exists an isotopy of S^3 bringing Σ to $\Sigma_{\tilde{K}}$, where $\tilde{K}$ is some link the diagram of which is obtained from that of K by a series of flypings. ∎

To derive Theorem A, let K and K' be equivalent well-connected alternating links such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even, and such that the unbounded face of $\pi[K]$ is bounded by at least four edges.

Let Φ be an isotopy of S^3 bringing K' to K. Let $\psi(x) := \Phi(1, x)$ for all $x \in S^3$. So $K = \psi[K']$.

Applying the Lemma to $\Sigma := \psi[\Sigma_{K'}]$ gives Theorem A (since

$$(19)\qquad \tau(K, \psi[\Sigma_{K'}]) = \tau(\psi[K'], \psi[\Sigma_{K'}]) = \tau(K', \Sigma_{K'}) = v(K') + w(K') = v(K) + w(K)$$

and

$$(20)\qquad \chi(\psi[\Sigma_{K'}]) = \chi(\Sigma_{K'}) = b(K') - v(K') = b(K) - v(K)).$$

Proof of the Lemma.

Let

$$
\begin{aligned}
G &:= \pi[K], \\
V &:= V_K \\
P &:= \{p_v^{\uparrow} | v \in V\} \cup \{p_v^{\downarrow} | v \in V\},
\end{aligned}
\tag{21}
$$

Throughout we identify an embedded graph with its image. We consider edges as *open* curves, and faces as *open* regions.

In proving the Lemma, we make the assumption that Σ is tame and in general position with respect to the link K and the projection function π. In particular we assume that Σ has a simplicial decomposition into a finite number of vertices, edges, and faces, in such a way that each edge and each face projects one-to-one to $\mathbb{R}^2$. So the number:

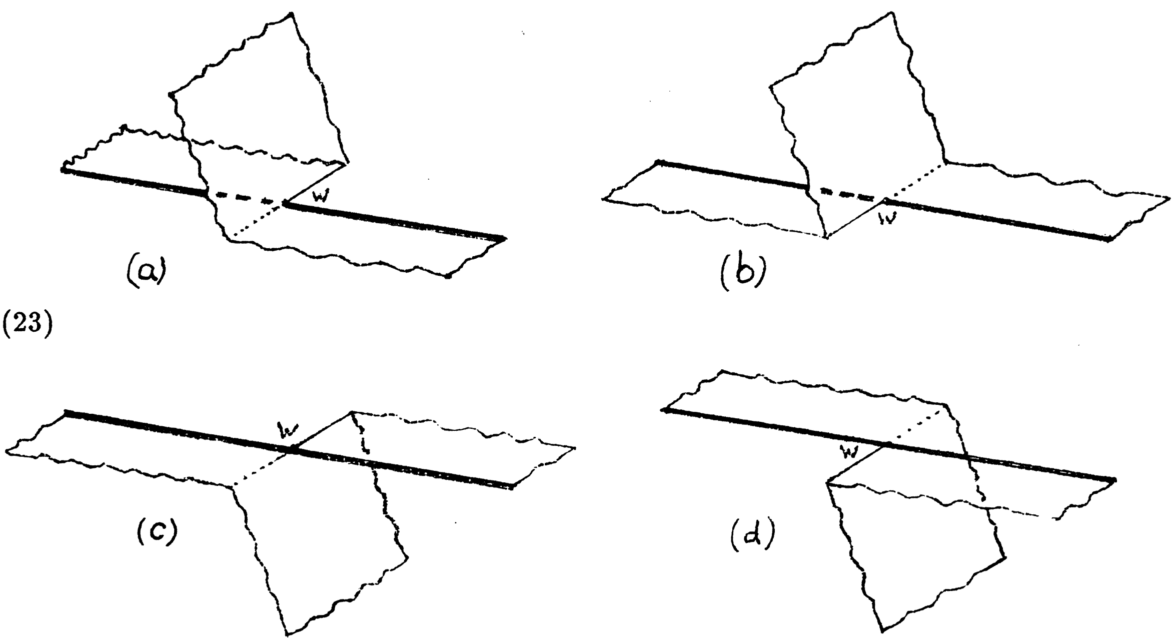
$$
\omega(x) := |\Sigma \cap \pi^{-1}(x)| \tag{22}
$$

is finite for each $x \in \mathbb{R}^2$.

Moreover, there exists a planar graph H in $\mathbb{R}^2$, with a finite number of vertices, edges, and faces, such that ω is constant on each edge and on each face of H. We may asume that $\omega(x) = 0$ in the unbounded face of H. (So Σ does not contain the point in $S^3 \setminus \mathbb{R}^3$. This is no restriction as we can easily shift Σ slightly.)

The simplicial decomposition of Σ implies that there exists a finite set W of points on K that do not have a neighbourhood in Σ that projects one-to-one to $\mathbb{R}^2$. We may assume that the neighbourhood of any point in W is like one in:

(23)



(In this and following pictures, the bold lines indicate parts of K or of $\pi[K]$. The wriggled lines give the cuts through Σ bounding the neighbourhood.)

We may assume that $P \cap W = \emptyset$. Moreover, we may assume that the projection of any vertex of the simplicial decomposition of Σ is not contained in the projection of any edge of Σ.

By the tameness and general position assumption,

(24) $$\Gamma := \Sigma \cap \pi^{-1}[G]$$

and

(25) $$\Delta := \{x \in \Sigma \mid x \text{ has no neighbourhood on } \Sigma \text{ that is an open disk and that projects one-to-one to } \mathbb{R}^2\}$$

are graphs (embedded in $\mathbb{R}^3$), with a finite number of vertices and edges.

The link K is contained both in Γ and in Δ. The graph Δ consists of K together with all 'fold' edges of Σ. The set W can be taken as the set of vertices of Δ of degree 3, all other vertices of Δ having degree 2. Note that we can take

(26) $$H = \pi[\Delta].$$

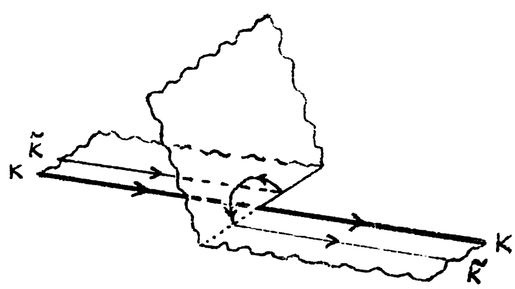
(It is not difficult to see that these assumptions can be satisfied. In fact, if we take $\Sigma = \psi[\Sigma_{K'}]$ as in Section 3 above, these assumptions are easily fulfilled, as the isotopy can be described by Reidemeister moves.)

We introduce some further notation and terminology. Let $W^{\uparrow}$ denote the set of points of type (a) or (b) in (23), and let $W^{\downarrow}$ denote the set of points of type (c) or (d) in (23). Let W^{+} denote the set of points of type (a) or (c) in (23), and let W^{-} denote the set of points of type (b) or (d) in (23). This notation is motivated by the fact that

(27) a link $\tilde{K}$ on Σ parallel and close to K, makes a positive crossing with K near any point $w \in W^{+}$, and a negative crossing with K near any point $w \in W^{-}$.

E.g., in (23)(a) a positive $K - \tilde{K}$ crossing can be seen:

(28)

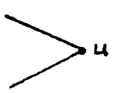


Let U be the set of points on 'fold' points in $\pi^{-1}[G]$. That is,

(29) $$U := \Delta \cap \pi^{-1}[G] \setminus K.$$

So U is the set of points u that have in $\Sigma \cap \pi^{-1}[G]$ a neighbourhood as in

(30)



Moreover, define

$$
(31) \qquad \begin{array}{rcl} VX & := & \text{set of vertices of } X, \\ EX & := & \text{set of edges of } X. \\ FX & := & \text{set of faces of } X, \\ \mathcal{C} & := & \text{set of components of } \Sigma \setminus \pi^{-1}[G], \\ F_0 & := & \text{unbounded face of } G. \end{array}
$$

Call a component of $K \setminus (P \cup W)$ (i.e., an edge of Γ on K) a *segment.*

By extension, define for any $x \in \mathbb{R}^3 : \omega(x) := \omega(\pi(x))$. Call a point x in $\mathbb{R}^2$ or $\mathbb{R}^3$ *even* or *odd* if $\omega(x)$ is even or odd. For any set X, X_{even} denotes the set of even points in X, and X_{odd} denotes the set of odd points in X.

For any nonempty subset X of $\mathbb{R}^2$ or $\mathbb{R}^3$ let

$$
(32) \qquad \mu(X) := \min\{\omega(x) \mid x \in X\}.
$$

Minimality of Σ.

Suppose Σ is a counterexample to the Lemma. We may assume that we have chosen Σ in such a way that:

(33)

(i) $\chi(\Sigma)$ is as large as possible;

(ii) $\sum_{v \in VG \cap \text{bd}(F_0)} \omega(v)$ is as small as possible;

(iii) $\sum_{v \in VG} \omega(v)$ is as small as possible;

(iv) $\sum_{\sigma \text{ segment}} \mu(\sigma)$ is as small as possible;

(v) $|U|$ is as small as possible;

(vi) $\sum_{w \in W} \omega(w)$ is as small as possible.

(In this order: (ii) should hold under condition (i), and so on.)

Σ is determined by Γ.

The surface Σ is determined by the graph Γ (up to inessential deformations). To see this, note that by our general position assumption, the boundary of any component $C \in \mathcal{C}$ is a disjoint union of simple closed curves. In fact it is only one closed curve:

Claim 1. *Each component $C \in \mathcal{C}$ is an open disk.*

Proof. Consider a face F of G. For any component $C \in \mathcal{C}$ contained in $\pi^{-1}[F]$, the boundary $\text{bd}(C)$ of C is a union of pairwise disjoint simple closed curves on $\text{bd}(\pi^{-1}[F])$.

Moreover, C is orientable, since we can extend C to a closed surface in $\mathbb{R}^3$ by adding disjoint closed disks to the boundary components of C ('outside' $\pi^{-1}[F]$).

Suppose $\pi^{-1}[F]$ contains a component in $\mathcal{C}$ that is not an open disk. Then we can choose a component $C \in \mathcal{C}$ contained in $\pi^{-1}[F]$ such that C is not an open disk and such that for one of its boundary components γ, one of the components of $\pi^{-1}[\text{bd}(F)] \setminus \gamma$ is minimal (inclusion-wise). (Minimal taken over all C that are not open disks, over all boundary components γ and over all components of $\pi^{-1}[\text{bd}(F)] \setminus \gamma$.)

By this minimality assumption, we know that we can replace C by two disjoint bordered surfaces C_1 and C_2 in $\pi^{-1}[F]$, disjoint from all other components, such that C_2 is an open disk with boundary γ, and such that:

$$(34) \qquad \mathrm{bd}(C_1) \cup \mathrm{bd}(C_2) = \mathrm{bd}(C) \text{ and } \chi(C_1) + \chi(C_2) = \chi(C) + 2.$$

Let Σ' be the manifold obtained from Σ by replacing C by C_1 and C_2. So $\chi(\Sigma') = \chi(\Sigma) + 2$.

Let Σ'' be the union of those components of Σ' that have a border (i.e., are not closed). So $\mathrm{bd}(\Sigma'') = K$. If $\chi(\Sigma'') > \chi(\Sigma)$, we would obtain a counterexample with larger Euler characteristic, contradicting our assumption (33)(i). (It is a counterexample, since $\chi(\Sigma'') > \chi(\Sigma) \geq b(K) - v(K) = \chi(\Sigma_K)$.)

So $\chi(\Sigma'') \leq \chi(\Sigma)$. Hence $\chi(\Sigma' \setminus \Sigma'') \geq 2$, and hence $\Sigma' \setminus \Sigma''$ is a 2-sphere S (as it is connected, since at most one component of Σ' is a closed surface, because each component of Σ has a nonempty border).

Then K is either enclosed by S or is contained in its exterior. (Indeed, K attains the minimum number of crossings among all links equivalent to K (cf. Section 3). Hence there cannot be a 2-sphere separating two components of K.)

By (possibly) applying an isotopy of S^3 we may assume that K is contained in the exterior of S.

It follows that there is an isotopy bringing $(S \setminus (C_1 \cup C_2)) \cup C$ to a bordered surface contained in $\pi^{-1}[F]$, fixing $\Sigma \setminus ((S \setminus (C_1 \cup C_2)) \cup C)$. Thereby we decrease $|U|$ or $\sum_{v \in VG} \omega(v)$, contradicting the minimality assumptions (33). ∎

It follows that, up to isotopy, we can reconstruct Σ from Γ. Note that at edges e of Γ not on K, the surface Σ is attached at both sides of $\pi^{-1}[\pi[e]]$. At each segment σ on K (= edge of Γ on K), Σ is attached at only one side. We can determine this side, as it is at the 'odd face side' if $\mu(\sigma)$ is odd, and at the 'even face side' if $\mu(\sigma)$ is even. ($\mu(\sigma)$ is determined by Γ.)

The graphs G and H.

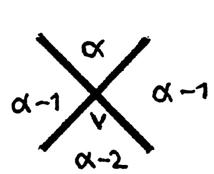
Note that $G = \pi[K]$ is a subgraph of H, and if $x \notin H$, that $\omega(x)$ is odd if x belongs to some odd face of $\pi[K]$, and $\omega(x)$ is even if x belongs to some even face of $\pi[K]$.

Note moreover that if e is an edge of H, and F and F' are the two faces of H incident with e, then $|\mu(F) - \mu(F')| = 1$ if e is part of G, and $|\mu(F) - \mu(F')| = 2$ otherwise.

H has three types of vertices: vertices that are also vertices of G, vertices that are on an edge of G, and vertices that are in a face of G. Consider a vertex v of H, and let $\alpha := \omega(v)$.

If v is also a vertex of G, it has degree 4 both in G and in H. Its neighbourhood is like that in

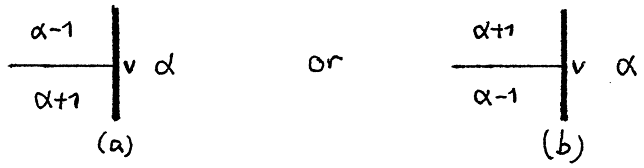
(35)



In this and in following figures, the numbers in the faces of H give their μ-values.

If v is on an edge of G, it has degree 3 or 4. If it has degree 3, it is the projection $\pi(w)$ of some point w in W, and its neighbourhood is as in

(36)

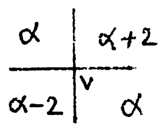


If v has degree 4, it is the projection $\pi(u)$ of some point u in U, and its neighbourhood is as in

(37)

If v is in a face of G, it has degree 2 or 4 in H. If it has degree 4, its neighbourhood is as in

(38)



Sometimes, we will indicate by a little arrow crossing any edge e of H which of the two faces incident with e has highest μ-value:

(39)

Moreover, we orient each edge e of H so that the face of H with highest μ-value is at the right hand side of e:

(40)

The set W.

For any $w \in W$, let ε_w be the (unique) edge of H incident with $\pi(w)$ but not being part of G. Note that

(41) w belongs to W^+ if either $w \in W^\uparrow$ and ε_w is oriented towards $\pi(w)$, or $w \in W^\downarrow$ and ε_w is oriented away from $\pi(w)$.
Similarly, w belongs to W^- if either $w \in W^\uparrow$ and ε_w is oriented away from $\pi(w)$, or $w \in W^\downarrow$ and ε_w is oriented towards $\pi(w)$.

We show:

Claim 2. *Let w and w' be two points in W connected by a segment. Then one of w and w' belongs to $W^\uparrow$, the other to $W^\downarrow$.*

Proof. Suppose to the contrary that both w and w' belong to $W^\uparrow$, say. Let e be the edge of G containing $\pi[\sigma]$. Then locally in $\pi^{-1}[e]$, Γ looks like one of the configurations in

(42)

Replace Γ locally by the corresponding configuration in

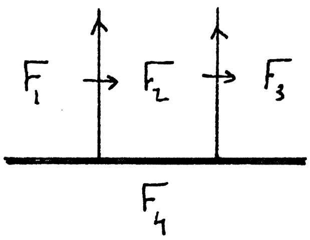
(43) 

As before, the new graph Γ' uniquely determines a surface Σ' (up to vertical shifts). It is not difficult to see that Σ' arises by an isotopy from Σ. Moreover, for Γ' the number of points of degree 3 along K is smaller than for Γ. Thus we would have a smaller counterexample, contradicting the minimality assumptions (33). ∎

Moreover: 

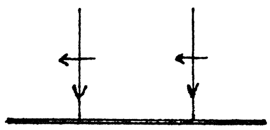
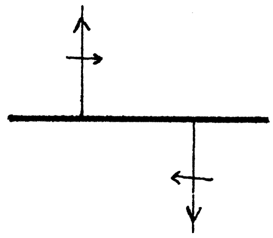
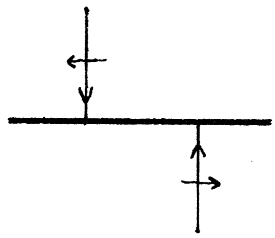
Claim 3. *Let w and w' be two points in W connected by a segment. Then one of $\varepsilon_w, \varepsilon_{w'}$ is oriented towards w or w', the other one away from w or w'.*

Proof. Suppose

(44) 

would occur. Then $\mu(F_3) = \mu(F_2) + 2 = \mu(F_1) + 4$. However, $\mu(F_4)$ differs by at most one from both $\mu(F_1)$ and $\mu(F_3)$, a contradiction.

Similarly

(45)  ,  and 

lead to a contradiction. ∎

As a direct corollary we have:

Claim 4. *For each edge e of G, either all points $w \in W$ with $\pi(w) \in e$ belong to W^+, or all belong to W^-.*

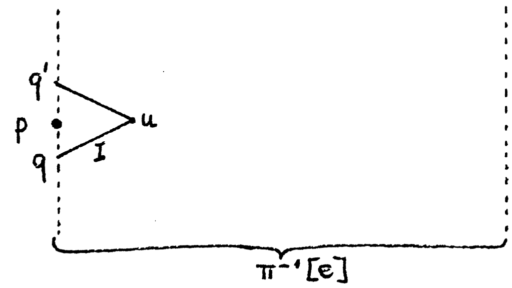
Proof. Directly from Claims 2 and 3 (cf.(41)). ∎

The set U.

Consider an edge e of G, connecting vertices v and v' of G. Consider the intersection $J := \Sigma \cap \pi^{-1}[\overline{e}]$.

The set J forms a graph with vertices of degree 1 on the boundary of $\pi^{-1}[\overline{e}]$ and vertices of degree 3 in each point in $W \cap \pi^{-1}[\overline{e}]$. Moreover, one of $p_v^{\uparrow}, p_v^{\downarrow}$ and one of $p_{v'}^{\uparrow}, p_{v'}^{\downarrow}$ might be an isolated vertex of J. All other vertices of J have degree 2.

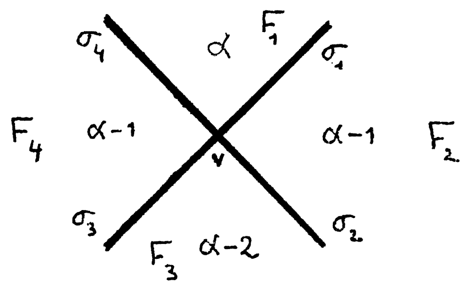
By the minimality of $|U|, \Sigma_{v\in VG}\omega(v)$ and $\sum_{w\in W}\omega(w)$, we know that (up to 'vertical shifts') each component I of $J \setminus K$ is a straight line segment, or the union of two straight line segments 'above each other', making an angle at a point u in U, as in

(46) 

In the latter case, the straight line segment connecting the end points q and q' of I contains a point $p \in P$, which is an isolated point of J. Moreover, above or under I there is no point in W (i.e., $\pi[I] \cap \pi[W] = \emptyset$). So there is a segment σ of K such that $\pi[I] \subset \pi[\sigma]$ and such that σ is incident with at least one point in P.

The neighbourhood of $\pi^{-1}[v]$ for vertices v of G.

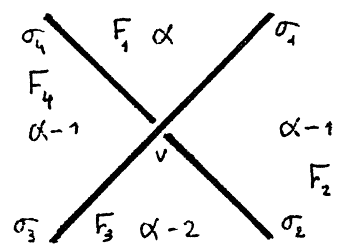
Consider a vertex v of G and its neighbourhood, as in

(47) 

Here F_1, F_2, F_3, F_4 denote the faces of G incident with v. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be the segments incident with $p_v^{\uparrow}$ and $p_v^{\downarrow}$ so that $\pi(\sigma_i)$ is incident with F_i and F_{i+1} ($i = 1, \ldots, 4$, taking indices mod 4).

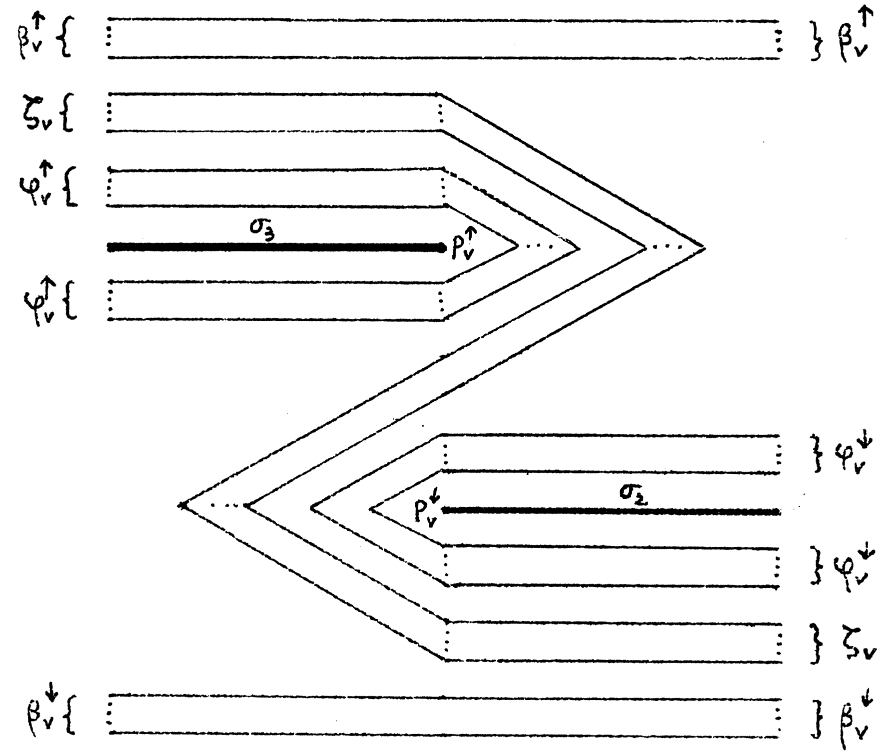
For each $i = 1, \ldots, 4$, choose some subinterval J_i of $\pi[\sigma_i] \cup \{v\} \cup \pi[\sigma_{i-1}]$ containing all points in $\pi[U] \cap (\pi[\sigma_i] \cup \pi[\sigma_{i-1}])$.

First consider the case where $\alpha := \omega(v)$ is odd. Then the diagram is locally as in

(48) 

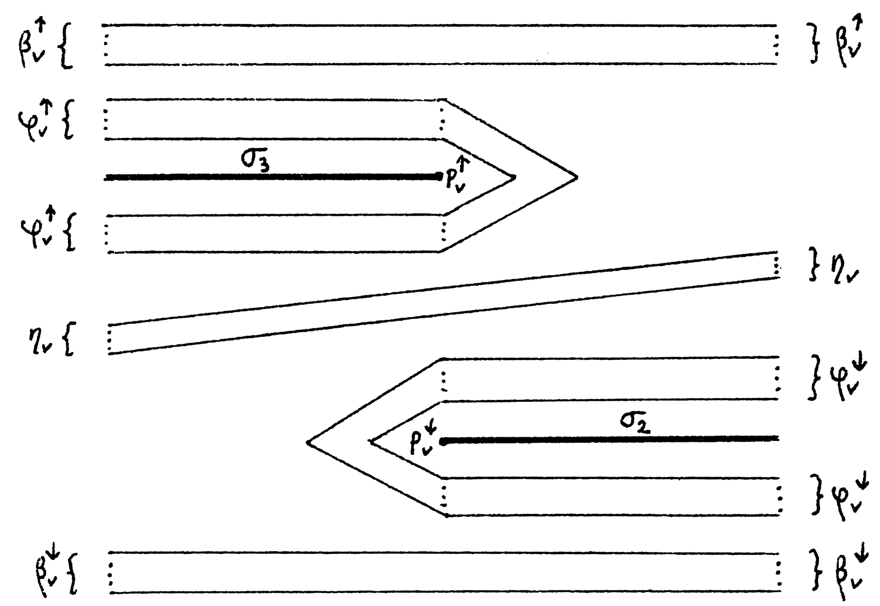
Consider now $\pi^{-1}[J_3]$ as 'seen' from F_3. It is either as in

(49)



or as in

(50)



The numbers $\beta_v^\uparrow, \beta_v^\downarrow, \varphi_v^\uparrow, \varphi_v^\downarrow, \zeta_v, \eta_v$ are the number of occurrences of the given type of curve.

We set $\eta_v := 0$ if (49) applies, and $\zeta_v := 0$ if (50) applies. Define

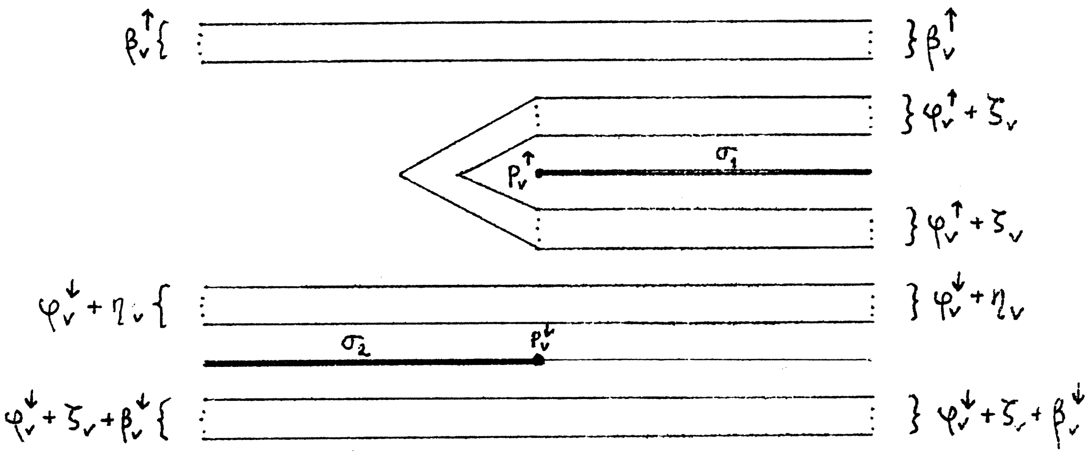
(51) $$\varphi_v := \varphi_v^\uparrow + \varphi_v^\downarrow, \quad \varphi := \sum_{v \in V} \varphi_v, \quad \zeta := \sum_{v \in V} \zeta_v, \quad \eta := \sum_{v \in V} \eta_v.$$

Note that

(52) $$|U| = \varphi + 2\zeta.$$

The plane $\pi^{-1}[J_2]$ seen from F_2 is as in

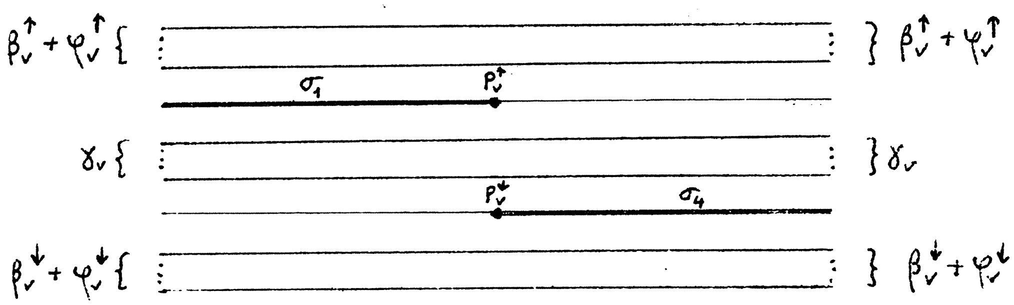
(53)



A symmetric pictures applies to $\pi^{-1}[J_4]$ seen from F_4.

Finally, $\pi^{-1}[J_1]$ seen from F_1 is as in

(54)



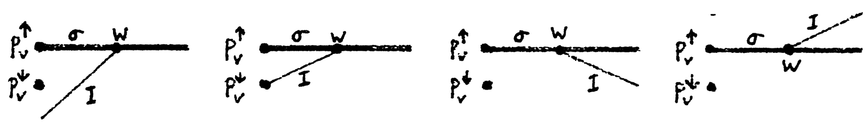
Symmetric pictures and notation hold in case $\omega(v)$ is even.

Segments connecting P and W.

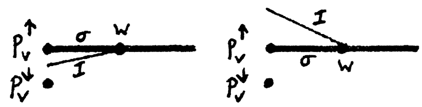
Consider a segment σ incident at one end with a point $p_v^\uparrow$, and at the other end with a point w in W. Let e be the edge of G containing $\pi[\sigma]$. Let I be the component of $(\pi^{-1}[e] \cap \Sigma) \setminus K$ incident with w. Then we have:

Claim 5. *Locally in $\pi^{-1}[e]$, the configuration is like one of the following:*

(55)



Proof. Indeed, the alternative would be that it is one of the configurations in

(56) 

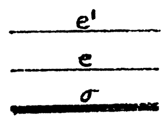
In both of these two cases there is an isotopy shifting along K reducing $\sum_{v\in VG}\omega(v)$, contradicting the minimality assumptions (33). ∎

Similar statements hold for segments connecting $p_v^{\downarrow}$ and a point in W.

The boundaries of components in $\mathcal{C}$.

Consider a component $C \in \mathcal{C}$. Let $\pi[C]$ be contained in face F of G. Then either $\mathrm{bd}(C)$ is a homotopically trivial circuit on $\pi^{-1}[\mathrm{bd}(F)]$, or not. Let $\mathcal{C}_0$ be the collection of components of the first kind, and let $\mathcal{C}_1$ be the collection of components of the second kind. Note that if F is the unbounded face F_0 of G, then all components $C \in \mathcal{C}$ contained in $\pi^{-1}[F]$ belong to $\mathcal{C}_0$ (since $\Sigma \subseteq \mathbb{R}^3$).

In order to study $\mathcal{C}$, consider a segment σ. Let e and e' be two parts of edges of Γ above σ, in such a way that e and e' have the same projection as σ:

(57) 

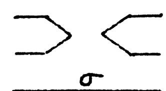
(Here e might be incident with one of the end points of σ). Let e and e' be on the boundaries of components C and C' in $\mathcal{C}$, respectively. Then:

Claim 6. *C and C' are different.*

Proof. Suppose $C = C'$. Then we may assume that there is no other edge part of Γ in between of e and e' with the same projection as σ. Otherwise there would be two such edges e'' and e''' inbetween being part of the boundary of the same C'' in $\mathcal{C}$. (This follows from the fact that if l is a line segment in $\pi^{-1}[\pi[\sigma]]$ connecting e and e', then l is contained in some circuit in $l \cup \mathrm{bd}(C)$ that is homotopically trivial in $\pi^{-1}[\mathrm{bd}(F)]$, where F is the face of G containing $\pi[C]$.)

By replacing e, e' by e'', e''' and repeating the argument, we obtain two 'neighbouring' e, e'.

Now modify Γ by replacing the configuration in figure (57) by that of

(58) 

It is not difficult to check that the new graph Γ' gives a surface Σ' isotopic to Σ, contradicting the minimality of $\sum_{\sigma \text{ segment}} \mu(\sigma)$. ∎

It follows that for any $C \in \mathcal{C}$, with $\pi[C]$ contained in face F of G, and for any $x \in \mathrm{bd}(F)$, $\mathrm{bd}(C)$ has at most three intersections with $\pi^{-1}(x)$.

For any $C \in \mathcal{C}$, let $B(C)$ denote the set of all points in C for which no neighbourhood in C projects one-to-one to $\mathbb{R}^2$.

So

$$\Delta = K \cup \bigcup_{C \in \mathcal{C}} B(C). \tag{59}$$

and hence

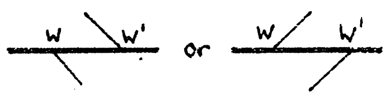
$$H = \pi[\Delta] = G \cup \bigcup_{C \in \mathcal{C}} \pi[B(C)]. \tag{60}$$

If $C \in \mathcal{C}_0$, then Claim 6 implies that $\mathrm{bd}(C)$ has exactly two acute angles, and $\pi[\mathrm{bd}(C)] \neq \mathrm{bd}(F_0)$. We may assume that $|\pi^{-1}(x) \cap C| \leq 2$ for all $x \in F$, and that $B(C)$ is a curve connecting the two acute angles on $\mathrm{bd}(C)$.

Moreover, we may assume that $C = D' \cup B(C) \cup D''$ for two open disks D' and D'' such that both D' and D'' project one-to-one to $\mathbb{R}^2$.

If $C \in \mathcal{C}_1$, then $\mathrm{bd}(C)$ can have some acute angles, but only at the Z-type curves in figure (49) or at segments σ on K that are incident with two points in W, and that are locally as in

(61)

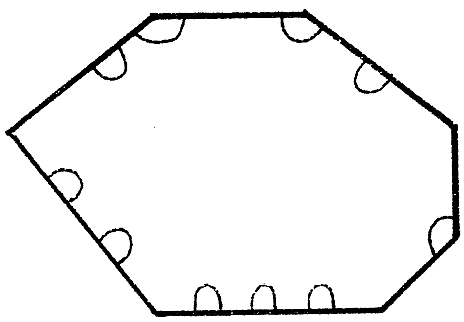


Let us call such a segment a *Z-type* segment.

We may assume that $|\pi^{-1}(x) \cap C| \leq 3$ for all $x \in F$. In fact, we may assume that the set $\{x \in F \mid |\pi^{-1}(x) \cap C| = 3\}$ forms a collection of pairwise disjoint open regions, each corresponding to one Z-type curve or Z-type segment.

The set $B(C)$ forms a disjoint union of curves, each of them connecting two acute angle points on some Z-type curve or segment on the boundary of C. We may assume that $B(C)$ projects one-to-one to $\mathbb{R}^2$, as in

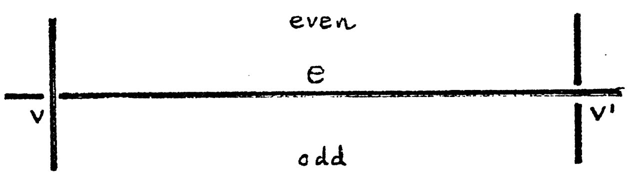
(62)



The graph H along an edge of G.

Consider an edge e of G, let it go from v to v':

(63)

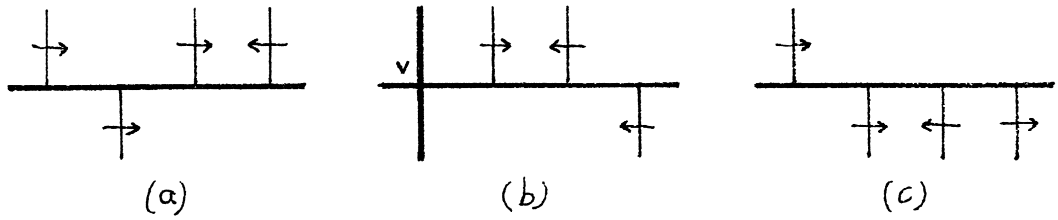


When following e from v to v', we first meet some (or none) points in $\pi[U]$, each having degree 4 in H. Next we meet some (or none) points in $\pi[W]$, each having degree 3 in H. Finally we meet again some (or none) points in $\pi[U]$, each of degree 4 in H.

We first show:

Claim 7. *The following configurations do not occur on any edge e of G:*

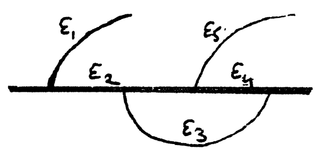
(64)



where v is a vertex of G. Similarly for those configurations arising by exchanging up and down and left and right.

Proof. Configuration (64)(a) gives in the plane $\pi^{-1}[e]$:

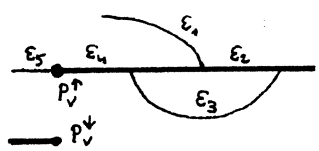
(65)



(up to exchanging up and down). Then the boundary of some component $C \in \mathcal{C}$ contains $\varepsilon_1, \ldots, \varepsilon_5$ (at one side of $\pi^{-1}[e]$ or the other). So bd(C) contains both ε_1 and ε_5. This contradicts Claim 6.

Similarly, configuration (64)(b) gives

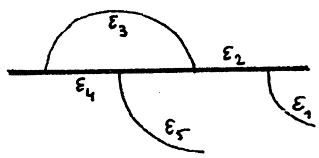
(66)



(up to exchanging up and down), again leading to a contradiction with Claim 6.

Finally, configuration (64)(c) gives

(67)



(up to exchanging up and down), again contradicting Claim 6. ∎

There exists a unique partition of $W \cap \pi^{-1}[e]$ into classes $W_1, W_2, \ldots, W_k$ in such a way that

(69) (i) k is even;

(ii) $\pi[W_1], \ldots, \pi[W_k]$ occur consecutively along e, as in:

(iii) $W_2, \ldots, W_{k-1} \neq \emptyset$;

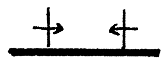
(iv) the arrow crossing any edge ε_w with $w \in W_i$ goes from right to left if i is odd, and from left to right if i is even.

(Again, ε_w denotes the edge of H incident with $\pi(w)$ not being part of G.)

As configuration (64)(a) does not occur, we know that $|W_2, \ldots, |W_{k-1}| \leq 2$. Moreover, the forbidden configurations (64)(b) and (c) imply that $|W_i| = |W_{i+1}|$ for $i = 2, \ldots, k-2$, i even.

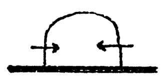
Now if we have two neighbouring edges ε_w and $\varepsilon_{w'}$ with arrows pointing towards each other:

(70)



(up to exchanging up and down in this picture), then they are in fact one and the same edge:

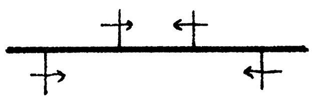
(71)



This follows from the fact that they are projections of some component of $B(C)$ for some $C \in \mathcal{C}_1$, as the segment on K in between is a Z-type segment.

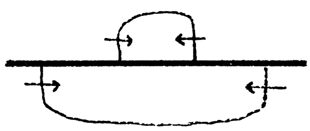
If $|W_i| = |W_{i+1}| = 2$ with $2 \leq i \leq k-2$ and i even, then we have

(72)



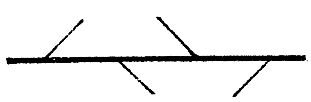
(up to exchanging up and down in this picture). In that case they are part of

(73)

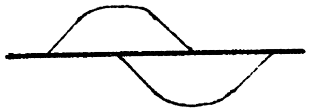


since in $\pi^{-1}[e]$ we have

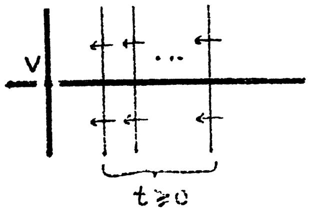
(74)



(up to exchanging up and down), and hence

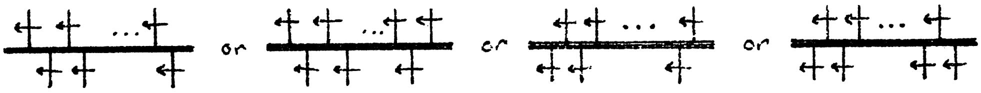
(75) 

Concluding, when following e from v to v', we first meet a number t of points in $\pi[U]$ ($t \geq 0$), each having degree 4 in H:

(76) 

We say that these points of $\pi[U]$ (and their liftings in U) are *near to v*.

Next we meet a series of points in $\pi[W]$ of degree 3. First:

(77) 

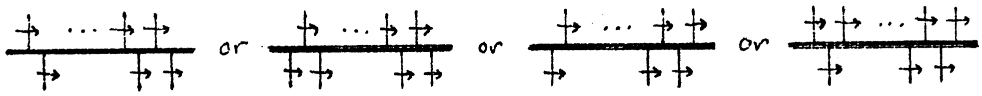
Again we say that these points of $\pi[W]$ (and their liftings in W) are *near to v*.

Next we meet a series of configurations as follows, in some amount and in some order:

(78) 

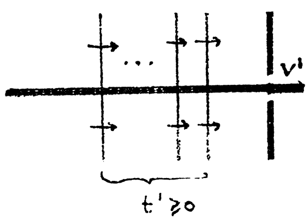
(In fact, Claim 3 gives conditions which configurations can succeed each other.)

After that we have:

(79) 

These points (and their liftings) are called *near to v'*.

Finally, we meet again points of degree 4 in $\pi[U]$

(80) 

These points (and their liftings) are called *near to v'*.

Note:

Claim 8. *All points in U near to a vertex v of G project to at most two edges of G incident with v.*

Proof. This follows directly from (49) and (50). ∎

The graph H in the faces of G.

Let F be a face of G. Let C and C' be two different components in $\mathcal{C}$ contained in $\pi^{-1}[F]$. It follows from the results in the previous part that

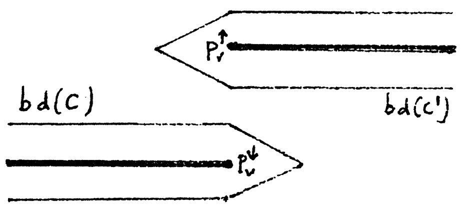
(81) $\pi[B(C)]$ and $\pi[B(C')]$ have a crossing in F, if and only if there are points $u, u' \in U$ such that $u \in \mathrm{bd}(C)$ and $u' \in \mathrm{bd}(C')$, such that u and u' are near to the same vertex v of G, and such that $\pi(u)$ and $\pi(u')$ are on different edges of G (incident with v).

We say that such a crossing is *near to v*. So

(82) each vertex of H of degree 4 in some face of G is a crossing near to some vertex v of G; it can occur in only one of the four faces of G incident with v (viz. the one with smallest μ-value near to v).

If C and C' belong to $\mathcal{C}_0$, then their neighbourhoods near to $\pi^{-1}(v)$ as seen from $\pi^{-1}[F]$ are as in:

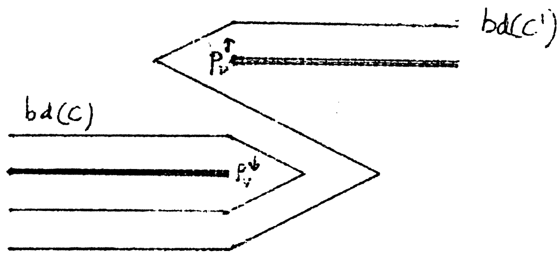
(83)



(up to exchanging up and down).

If $C \in \mathcal{C}_0$ and $C' \in \mathcal{C}_1$, then $\pi^{-1}(v)$ as seen from $\pi^{-1}[F]$ is as in:

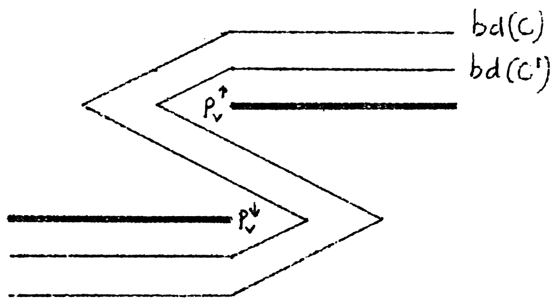
(84)



(up to exchanging up and down and left and right).

If $C, C' \in \mathcal{C}_1$, then then $\pi^{-1}(v)$ as seen from $\pi^{-1}[F]$ is as in:

(85)



(up to exchanging up and down).

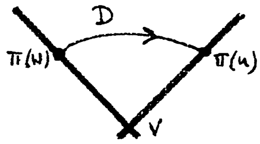
There are no other types of crossings of $B(C)$ and $B(C')$.

We finally note:

Claim 9. *Let v be a vertex of G and let F be a face of G incident with v. Let $w \in W$ and $u \in U$ be near to v. Then there is no directed path in H from $\pi(w)$ to $\pi(u)$ or from $\pi(u)$ to $\pi(w)$ that is contained in F.*

Proof. Let D be a directed path in H from $\pi(w)$ to $\pi(u)$:

(86)

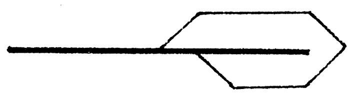


By (81), D does not traverse any other vertex of H. So there exists a component $C \in \mathcal{C}$ contained in $\pi^{-1}[F]$ such that some component of $B(C)$ connects w and u.

If $C \in \mathcal{C}_1$, then each arc in $B(C)$ corresponds to a Z-type curve or Z-type segment, and hence either would connect two points in U or would connect two points in W.

So we know $C \in \mathcal{C}_0$. Let F be the face of G containing $\pi[C]$. Then the boundary of C on $\pi^{-1}[\mathrm{bd}(F)]$ is as

(87)



(up to exchanging up and down and left and right). This contradicts Claim 5. ∎

A lower bound for $\sum_{k=1}^{\infty} \chi(R_{2k})$.

Define for any k:

(88) $$R_k := \text{closure of } \{x \in \mathbb{R}^2 \mid \omega(x) \geq k\}.$$

So $R_k = \emptyset$ if k is large enough. Moreover, let ρ be the number of Z-type segments ((61)).

We show that the Euler characteristics $\chi(R_{2k})$ of the sets R_{2k} satisfy:

Claim 10. $4\sum_{k=1}^{\infty} \chi(R_{2k}) \geq 2|V_{\text{even}}| + |W_{\text{odd}}| + |U| + 2|W^{-}_{\text{odd}}| + 2\eta + 2\rho.$

Proof. We first prove:

Subclaim 10.1. $\sum_{k=1}^{\infty} \chi(R_{2k}) = \frac{1}{2}|\mathcal{C}_1| - \frac{1}{2}b(K) - \sum_{v \in VG} \lfloor \frac{\omega(v)-1}{2} \rfloor + \frac{1}{2}|W_{\text{odd}}| + \frac{1}{2}|U|.$

Here $\lfloor\ \rfloor$ denotes lower integer part.

Proof. We first show that for each face F of G,

(89) $$\sum_{k=1}^{\infty} \chi(R_{2k} \cap F) = \lfloor \frac{1}{2}\kappa_F \rfloor,$$

where κ_F denotes the number of components in $\mathcal{C}_1$ contained in $\pi^{-1}[F]$. Note that κ_F is odd, if and only if F is odd.

For any component C in $\mathcal{C}_0$ one has $\chi(\pi[C]) = 0$ (since C is a union of two disks above each other). Moreover, for any component C in $\mathcal{C}_1$ the set

$$R_C := \{x \in F \mid |\pi^{-1}(x) \cap C| \geq 2\} \tag{90}$$

has $\chi(R_C) = 0$. This implies

$$\begin{aligned} &\sum_{k=1}^{\infty} \chi(R_{2k} \cap F) \\ &= \sum_{C \in \mathcal{C}_0, C \subseteq \pi^{-1}[F]} \chi(\pi[C]) + \sum_{C \in \mathcal{C}_1} \chi(R_C) + \lfloor \frac{1}{2} \sum_{C \in \mathcal{C}_1, C \subseteq \pi^{-1}[F]} \chi(C) \rfloor \\ &= \lfloor \frac{1}{2} \sum_{C \in \mathcal{C}_1, C \subseteq \pi^{-1}[F]} \chi(C) \rfloor \\ &= \lfloor \frac{1}{2} \kappa_F \rfloor. \end{aligned} \tag{91}$$

(The first equality follows from the modularity of the Euler characteristic, i.e., $\chi(A)+\chi(B) = \chi(A \cap B) + \chi(A \cup B)$.)

Adding up (89) over all faces F of G gives:

$$\sum_{k=1}^{\infty} \chi(R_{2k} \setminus G) = \sum_{F \in FG} \lfloor \frac{1}{2} \kappa_F \rfloor = \frac{1}{2} |\mathcal{C}_1| - \frac{1}{2} b(K). \tag{92}$$

Next, let for any vertex v of H, $\tilde{\mu}(v)$ be the maximum value of $\mu(F)$ where F ranges over all faces of H incident with v. So $\tilde{\mu}(v)$ is equal to the largest integer k such that v belongs to R_k. Note that, for each edge e of H with $e \subset G$, $\mu(e)$ is equal to the largest integer k such that e is contained in R_k. Hence

$$\sum_{k=1}^{\infty} \chi(R_{2k} \cap G) = \sum_{v \in VH} \lfloor \frac{\tilde{\mu}(v)}{2} \rfloor - \sum_{e \in EH, e \subset G} \lfloor \frac{\mu(e)}{2} \rfloor. \tag{93}$$

Consider a vertex v of H. If v is also a vertex of G, then $\tilde{\mu}(v) = \omega(v)$. Let e_1, e_2, e_3, e_4 be the edges of H incident with v. We can choose indices so that $\mu(e_1) = \mu(e_2) = \omega(v)$ and $\mu(e_3) = \mu(e_4) = \omega(v) - 1$. Hence

$$\lfloor \frac{\tilde{\mu}(v)}{2} \rfloor - \frac{1}{2} \sum_{i=1}^{4} \lfloor \frac{\mu(e_i)}{2} \rfloor = -\lfloor \frac{\omega(v) - 1}{2} \rfloor. \tag{94}$$

If $v = \pi(u)$ for some $u \in U$, let e_1 and e_2 be the two edges of H incident with v that are contained in G. We can choose indices so that $\mu(e_1) = \tilde{\mu}(v)$ and $\mu(e_2) = \tilde{\mu}(v) - 2$. Hence

$$\lfloor \frac{\tilde{\mu}(v)}{2} \rfloor - \frac{1}{2} \lfloor \frac{\mu(e_1)}{2} \rfloor - \frac{1}{2} \lfloor \frac{\mu(e_2)}{2} \rfloor = \frac{1}{2}. \tag{95}$$

If $v = \pi(w)$ for some $w \in W$, then $\tilde{\mu}(v) = \omega(v) + 1$. Let e_1 and e_2 be the two edges of H incident with v that are contained in G. We can choose indices so that $\mu(e_1) = \tilde{\mu}(v)$ and $\mu(e_2) = \tilde{\mu}(v) - 1$. Hence

$$\lfloor \frac{\tilde{\mu}(v)}{2} \rfloor - \frac{1}{2} \lfloor \frac{\mu(e_1)}{2} \rfloor - \frac{1}{2} \lfloor \frac{\mu(e_2)}{2} \rfloor = \frac{1}{2} \tag{96}$$

if $\tilde{\mu}(v)$ is even, i.e., if $w \in W_{\text{odd}}$. Similarly,

$$(97) \qquad \lfloor\frac{\tilde{\mu}(v)}{2}\rfloor - \frac{1}{2}\lfloor\frac{\mu(e_1)}{2}\rfloor - \frac{1}{2}\lfloor\frac{\mu(e_2)}{2}\rfloor = 0$$

if $\tilde{\mu}(v)$ is odd, i.e., if $w \in W_{\text{even}}$.

Adding up (94) over all $v \in VG$, (95) over all $u \in U$, (96) over all $w \in W_{\text{odd}}$, and (97) over all $w \in W_{\text{even}}$, gives by (93):

$$(98) \qquad \sum_{k=1}^{\infty} \chi(R_{2k} \cap G) = -\sum_{v \in VG} \lfloor\frac{\omega(v)-1}{2}\rfloor + \frac{1}{2}|W_{\text{odd}}| + \frac{1}{2}|U|.$$

Combined with (92), this gives the claimed equality. ∎

Multiplying by 4 gives:

$$(99) \qquad 4\sum_{k=1}^{\infty} \chi(R_{2k}) = 2|\mathcal{C}_1| - 2b(K) - 4\sum_{v \in VG} \lfloor\frac{\omega(v)-1}{2}\rfloor + 2|W_{\text{odd}}| + 2|U|.$$

Rewriting the right hand side gives

$$(100) \qquad 2(|\mathcal{C}| - \frac{1}{2}|W| - \sum_{v \in VG}(\omega(v)-1)) - 2b(K) - 2|\mathcal{C}_0| + 2|V_{\text{even}}| + |W| + 2|W_{\text{odd}}| + 2|U|.$$

The first term here contains the Euler characteristic of Σ as it can be expressed as follows.

Subclaim 10.2. $\chi(\Sigma) = |\mathcal{C}| - \frac{1}{2}|W| - \sum_{v \in VG}(\omega(v)-1)$.

Proof. Since each component in $\mathcal{C}$ is an open disk, one has:

$$(101) \qquad \chi(\Sigma \setminus \pi^{-1}[G]) = |\mathcal{C}|.$$

Moreover,

$$(102) \qquad \chi(\Sigma \cap \pi^{-1}[G]) = \chi(\Gamma) = -\frac{1}{2}|W| - \sum_{v \in VG}(\omega(v)-1).$$

This follows from the fact that all vertices of Γ in $W \cup P$ have degree 3, and all vertices of Γ in $\pi^{-1}[V] \setminus P$ have degree 4. All other vertices of Γ have degree 2. Hence

$$(103) \qquad \begin{aligned} \chi(\Gamma) &= |V\Gamma| - |E\Gamma| \\ &= |W \cup P| - \frac{3}{2}|W \cup P| + \sum_{v \in VG}(\omega(v)-2) - \frac{4}{2}\sum_{v \in VG}(\omega(v)-2) \\ &= -\frac{1}{2}|W| - \sum_{v \in VG}(\omega(v)-1), \end{aligned}$$

since $|P| = 2|VG|$.

Combining (101) and (102) gives the claimed equality. ∎

So (100) is equal to:

$$(104) \qquad 2\chi(\Sigma) - 2b(K) - 2|\mathcal{C}_0| + 2|V_{\text{even}}| + |W| + 2|W_{\text{odd}}| + 2|U|.$$

By assumption (ii) in the Lemma, this is at least

$$(105) \qquad -2|V| - 2|\mathcal{C}_0| + 2|V_{\text{even}}| + |W| + 2|W_{\text{odd}}| + 2|U|.$$

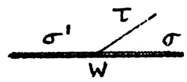
Now $|\mathcal{C}_0|$ satisfies the following equation (recall that ρ is the number of Z-type segments (61) and that ζ is the number of Z-type curves in (49)):

Subclaim 10.3. $|\mathcal{C}_0| = \frac{1}{2}|W| + |U| - \rho - \zeta$.

Proof. For any C in $\mathcal{C}_0$, the boundary bd(C) of C should have exactly two acute angles. Such an acute angle should occur at a point in W or U.

In fact, each point w serves as acute angle for exactly one component in $\mathcal{C}$. For let $w \in W^{\uparrow}$, say

(106)



Then there is one component C, say, in $\mathcal{C}$ that is incident with τ and σ, and one component C', say, in $\mathcal{C}$ that is incident with τ and σ'. C and C' are at different sides of τ.

Now w can serve as acute angle only for C. In fact, w is an acute angle for some C in $\mathcal{C}_0$ if and only if w is not contained in some Z-type segment. So exactly $|W| - 2\rho$ points in W serve as acute angles for components in $\mathcal{C}_0$.

Any point u in U is acute angle for at least one component in $\mathcal{C}_0$ (viz. in the face F_2 or F_4 as in figures (53)). In fact, u is acute angle of two components in $\mathcal{C}_0$, if and only if u is not on a Z-type curve.

Since there are ζ Z-type curves, and each of them contains two points in U, it follows that the points in U make $2|U| - 2\zeta$ acute angles for components in $\mathcal{C}_0$.

So

$$(107) \qquad 2|\mathcal{C}_0| = |W| - 2\rho + 2|U| - 2\zeta,$$

and the claimed equality follows. ∎

Therefore, (105) is equal to

$$(108) \qquad -2|V| + 2\rho + 2\zeta + 2|V_{\text{even}}| + 2|W_{\text{odd}}|.$$

Rewriting gives

$$(109) \qquad \begin{aligned} &-2|V| + |W_{\text{odd}}| + 2|W^-_{\text{odd}}| + \frac{1}{2}(|W^+| - |W^-|) \\ &+ \frac{1}{2}(|W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}|) + 2\rho + 2\zeta + 2|V_{\text{even}}|. \end{aligned}$$

This rewriting being helpful is seen by the following two subclaims.

Subclaim 10.4. $|W^+| - |W^-| = 2v(K)$.

Proof. One directly derives from (27):

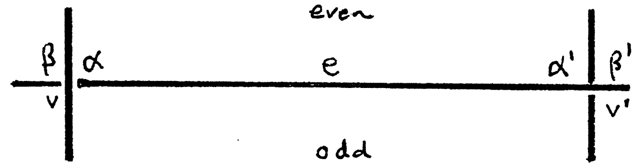
$$(110) \qquad \tau(K,\Sigma) = |W^+| - |W^-| + 2w(K).$$

Since $\tau(K,\Sigma) = 2(v(K)+w(K))$ by assumption, we have the required equality. ∎

Subclaim 10.5. $|W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}| = 2v(K) + 2\varphi + 4\eta$.

Proof. Consider a component e of $K \setminus P$. Let it connect $p_v^\downarrow$ and $p_{v'}^\uparrow$ as in

(111)



In this figure, $\alpha, \beta, \alpha', \beta'$ denote the μ-values in the corresponding faces of H incident with v and v'. Note that α and α' are even.

Define $\xi(e,v) := 1$ if $\beta = \alpha+1$ and $\xi(e,v) := 0$ if $\beta = \alpha - 1$. Similarly, define $\xi(e,v') := 1$ if $\beta' = \alpha'+1$ and $\xi(e,v') := 0$ if $\beta' = \alpha'-1$. (So $\xi(e,v)$ indicates at which side of $p_v^\uparrow$ the surface Σ is attached. Similarly for $\xi(e,v')$.)

Let $v^\uparrow(e)$ denote the number of points in U that are above e, and let $v^\downarrow(e)$ denote the number of points in U that are under e. Let $v(e) := v^\uparrow(e) + v^\downarrow(e)$.

For any $x \in \mathbb{R}^3$, let $\kappa(x)$ denote the number of points in Σ strictly under x, minus the number of points in Σ strictly above x.

We show

$$(112) \qquad \begin{aligned}&\kappa(p_{v'}^\uparrow) - \kappa(p_v^\downarrow) = \\ &\xi(e,v) + \xi(e,v') + 2v(e) + |W^+_{\text{odd}} \cap e| + |W^-_{\text{even}} \cap e| - |W^-_{\text{odd}} \cap e| - |W^+_{\text{even}} \cap e|.\end{aligned}$$

Indeed, when traversing e from $p_v^\downarrow$ to $p_{v'}^\uparrow$, near $p_v^\downarrow$ the number of levels above deleted is $\xi(e,v) + 2v^\uparrow(e)$, while near $p_{v'}^\uparrow$ the number of levels under added is $\xi(e,v') + 2v^\downarrow(e)$.

Moreover, at traversing any point w in W^+_{odd}, if $w \in W^\uparrow$, then one level above is deleted (cf. (23)(a) and (36)(a)), and if $w \in W^\downarrow$, then one level under is added (cf. (23)(c) and (36)(b)).

Similarly, at traversing any point w in W^-_{odd}, if $w \in W^\uparrow$, then one level above is added (cf. (23)(b) and (36)(b)), and if $w \in W^\downarrow$, then one level under is deleted (cf. (23)(d) and (36)(a)).

Symmetric statements hold for $w \in W^-_{\text{even}}$ and $w \in W^+_{\text{even}}$. This shows (112).

Now, for any $v \in VG$, if e and e' are the two components of $K \setminus P$ incident with $p_v^\downarrow$, then $\xi(e,v) + \xi(e',v) = 1$. Similarly for $p_v^\uparrow$.

Hence, adding (112) over all components e of $K \setminus P$ we obtain:

$$(113) \qquad \begin{aligned}&2\Big(\sum_{v\in VG} \kappa(p_v^\uparrow) - \sum_{v \in VG} \kappa(p_v^\downarrow)\Big) \\ &= 2v(K) + 2|U| + |W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}|.\end{aligned}$$

Now from (49) and (50) we see that for any $v \in VG$,

$$(114) \qquad \kappa(p_v^\uparrow) - \kappa(p_v^\downarrow) = 2\varphi_v + 2\zeta_v + 2\eta_v + 2.$$

Hence

$$(115)\qquad \begin{aligned} &|W^+_{\text{odd}}| + |W^-_{\text{even}}| - |W^-_{\text{odd}}| - |W^+_{\text{even}}| \\ &= 4\varphi + 4\zeta + 4\eta + 4v(K) - 2v(K) - 2|U| = 2v(K) + 2\varphi + 4\eta \end{aligned}$$

since $|U| = \varphi + 2\zeta$. ∎

Subclaims 10.4 and 10.5 imply that (109) is equal to

$$(116)\qquad -2|V| + |W_{\text{odd}}| + 2|W^-_{\text{odd}}| + |V| + |V| + \varphi + 2\eta + 2\rho + 2\zeta + 2|V_{\text{even}}|,$$

which equals

$$(117)\qquad |W_{\text{odd}}| + 2|W^-_{\text{odd}}| + \varphi + 2\eta + 2\rho + 2\zeta + 2|V_{\text{even}}|.$$

By (52) this is equal to the right hand side in the Claim. ∎∎

An equality for $\sum_{k=1}^{\infty} |\delta(R_{2k})|$.

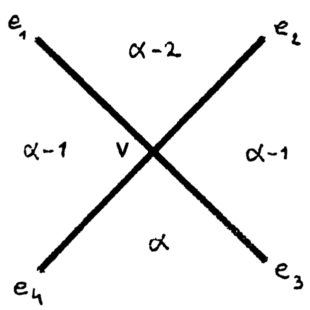
For any closed subset R of $\mathbb{R}^2$, let $\delta(R)$ denote the set of edges of H on G 'sticking out' of R, counting multiplicities. More precisely, it is the set of pairs (e, v) where $v \in VH, e \in EH$ such that e is incident with v, $e \subseteq G$, and $v \in R, e \cap R = \emptyset$.

So if one makes a set of closed curves in $\mathbb{R}^2 \setminus R$ close to the boundary components of R, then these curves will have $|\delta(R)|$ crossings with G.

Claim 11. $\sum_{k=1}^{\infty} |\delta(R_{2k})| = 2|V_{\text{even}}| + |W_{\text{odd}}| + |U|$.

Proof. Consider a vertex v of H. Let $\alpha := \omega(v)$. First let $v \in VG$. Consider a neighbourhood of v:

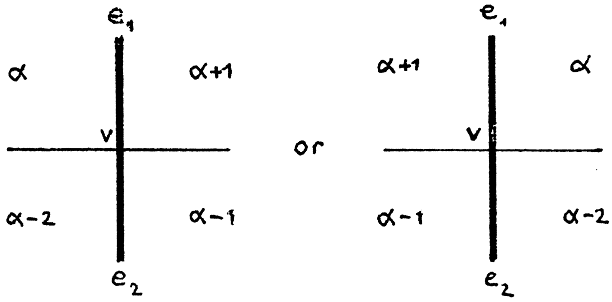
(118)



If α is even, then (e_1, v) and (e_2, v) belong to $\delta(R_\alpha)$, and there are no other pairs (e, v) in any of the sets $\delta(R_{2k})$. If α is odd, then no pair (e, v) belongs to any $\delta(R_{2k})$.

Next let $v \in \pi[U]$. Consider a neighbourhood of v:

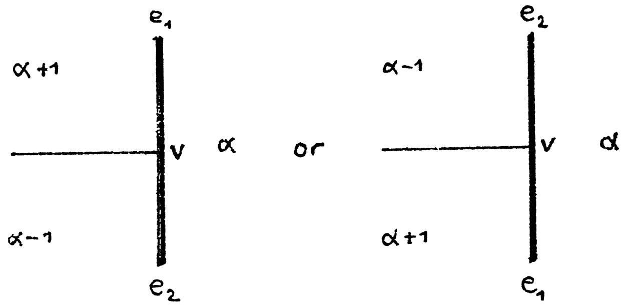
(119)



If α is even then (e_2, v) belongs to $\delta(R_\alpha)$, and if α is odd, then (e_2, v) belongs to $\delta(R_{\alpha+1})$. No other pair (e, v) belongs to any $\delta(R_{2k})$.

Finally, let $v \in \pi[W]$. Consider a neighbourhood of v:

(120)



If α is even, no (e, v) belongs to any $\delta(R_{2k})$. If α is odd, then (e_2, v) belongs to $\delta(R_{\alpha+1})$, and no other pair (e, v) belongs to any $\delta(R_{2k})$.

Adding up over all vertices v of H on G we obtain the Claim. ∎

The remainder of the proof now is to make the following intuitive argument precise. By Claims 10 and 11,

$$(121) \qquad \sum_{k=1}^{\infty} |\delta(R_{2k})| = 2|V_{\text{even}}| + |W_{\text{odd}}| + |U| \le \sum_{k=1}^{\infty} 4\chi(R_{2k}).$$

On the other hand, roughly speaking, since G is well-connected, for each k, $\delta(R_{2k})| \ge 4\chi(R_{2k})$, with equality holding only if each component of R_{2k} covers exactly one vertex of G. Hence it is of type (124)(a) with $\alpha = \omega(v)$ even. Since $|W^+| - |W^-| = 2|V|$, this should hold for each vertex v, while $W^- = \emptyset$ and $W = W^+$. Then it is easy to see that there exists an isotopy bringing Σ to Σ_K.

The graph H'.

Let H' be defined by

$$(122) \qquad H' := \bigcup_{k=1}^{\infty} \text{bd}(R_{2k}).$$

So H' is the subgraph of H consisting of those edges e of H for which $\lfloor \frac{1}{2}\mu(F) \rfloor$ and $\lfloor \frac{1}{2}\mu(F') \rfloor$ differ (by 1), where F and F' are the faces of H incident with e.

So H' contains all of $H \setminus G$, while an edge e of H on G is in H', if and only if $\mu(e)$ is even. H' inherits the orientation from H (cf. (40)).

Note that any edge e of H' that is on the boundary of an odd face of G is oriented counter-clockwise with respect to that face. (So it is oriented clockwise with respect to even faces, except for the unbounded face).

We show:

Claim 12. *Let $D_1, \ldots, D_t$ be a decomposition of the edges of H' into simple directed circuits such that $D_1, \ldots, D_s$ are oriented clockwise and $D_{s+1}, \ldots, D_t$ are oriented counter-clockwise. Then:*

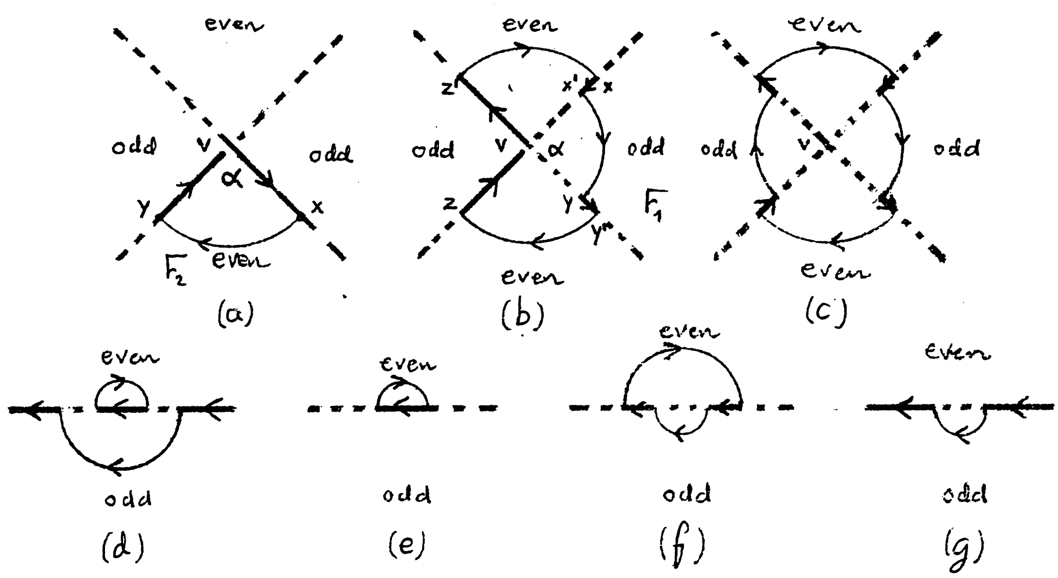
(123)
$$s - (t - s) = \sum_{k=1}^{\infty} \chi(R_{2k}).$$

Proof. We can successively uncross $D_1, \ldots, D_t$. If D_i and D_j cross, we can find two crossings of D_i and D_j such that we can exchange the parts inbetween in such a way that we obtain again two simple directed circuits D_i' and D_j'. Now the number of clockwise oriented circuits among D_i', D_j' is equal to that among D_i, D_j, as one easily checks.

By repeating this, we obtain $D_1, \ldots, D_t$ pairwise noncrossing. Then they should form the boundary components of the sets R_{2k}. For each fixed R_{2k}, $\chi(R_{2k})$ is equal to the number of boundary components that are oriented clockwise, minus the number of boundary components that are oriented counter-clockwise. So (123) follows. ∎

Consider the following configurations:

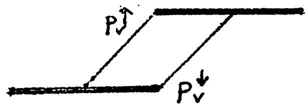
(124)



Here the interrupted line is part of G not in H'. We call the components in (124)(a)–(f) *small components* of H'.

If one of the configurations (124) occurs in H', no other edge of H' is connected to it (by (81) and (82)).

Note that (124)(a) implies that v belongs to V_{even}, and that (124)(b) implies that v belongs to V_{odd}. Moreover, (124)(a) as seen from F_2 is as in

(125) 

Since $\omega(v) = \alpha$ in (124)(a), both x and y belong to $\pi[W]$. Since $\omega(v) = \alpha$ in (124)(b), x, x', y, y' all belong to $\pi[W]$. Hence, by Claim 9, also z and z' belong to $\pi[W]$. Similarly, by Claim 9, all vertices traversed in (124)(c) belong to $\pi[W]$.

By the results following Claim 7, the number ρ is equal to the number of occurrences of configurations (124)(d), (e), (f), and (g).

For any simple closed curve D in $\mathbb{R}^2$ we denote

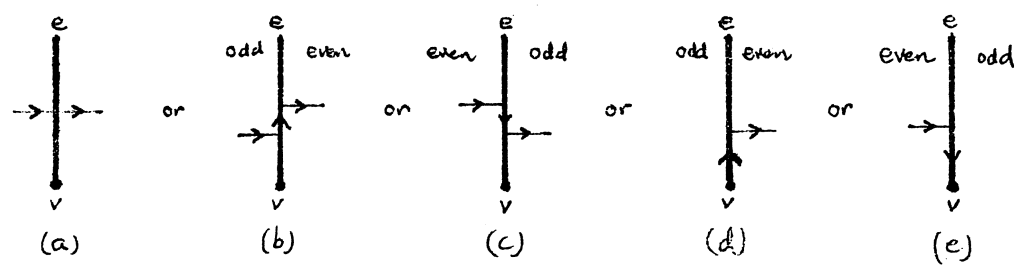
(126) $\qquad R(D) :=$ closed region enclosed by D.

We observe:

Claim 13. *Let D be a simple directed circuit in H', oriented clockwise, such that $R(D) \cap VG = \{v\}$ for some vertex v of G. Then D is of type* (124)(a),(b) *or* (c).

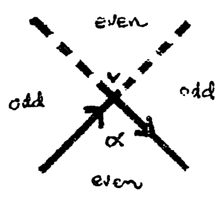
Proof. Note that if D traverses some point in $\pi[U]$ or $\pi[W]$ near to a vertex v' of G, then v' belongs to $R(D)$, implying $v' = v$. Hence D cannot traverse any other points in $\pi[U]$ and $\pi[W]$ than those near to v. (If D would traverse any other point in $\pi[W]$, then it is part of one of the configurations (124)(d)–(g), and hence either $R(D)$ would not contain v, or D would traverse a point in $\pi[W]$ near to some vertex $v' \neq v$.)

So if D intersects one of the edges e incident with v, it intersects e in one of the following ways:

(127) 

First suppose that D does not traverse v. Then by Claim 8, D traverses a point in $\pi[W]$, and hence, by Claim 9, D does not traverse any point in $\pi[U]$. So all vertices of H traversed by D belong to $\pi[W]$, and hence each crossing is of type (127)(b) or (c). Therefore, we have (124)(c).

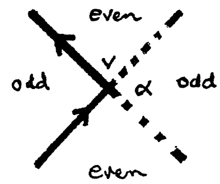
Second suppose D traverses v and v belongs to V_{even}. Then D contains

(128) 

So it contains both (127)(d) and (e). Therefore, D is of type (124)(a).

Finally, suppose D traverses v and v belongs to V_{odd}. Then D contains

(129)



So it contains both (127)(d) and (e). By Claim 9, it follows that D is of type (124)(b). ▮

The space Δ'.

Consider the set

(130) $$\Delta' := (\Delta \cup \bigcup_{v \in VG} e_v) \setminus \{\sigma | \sigma \text{ segment on } K \text{ with } \mu(\sigma) \text{ odd}\}.$$

(As before, e_v denotes the open line segment connecting $p_v^{\downarrow}$ and $p_v^{\uparrow}$.)

Each point in $P \cup W$ is incident with two segments on K, one with even μ-value and one with odd μ-value. Hence Δ' is a 2-regular graph embedded in $\mathbb{R}^3$. So each component of Δ' is a circuit.

Note that

(131) $$H' = \pi[\Delta'].$$

The orientation of H' induces an orientation of Δ'. Each line segment e_v is oriented from $p_v^{\downarrow}$ to $p_v^{\uparrow}$.

Let D be some component of Δ', and consider $\pi[D]$. Each of the components in (124) of H' corresponds to a component of Δ'. We call a component D of Δ' *small* if $\pi[D]$ is small.

The length function l.

For each edge e of H' define the 'length' $l(e)$ of e by:

(132) $$\begin{array}{rcll} l(e) & := & |\bar{e} \cap V_{\text{even}}| & \text{if } e \subseteq G, \\ & := & |\bar{e} \cap G| & \text{if } e \text{ is contained in an even face of } G, \\ & := & 0 & \text{if } e \text{ is contained in an odd face of } G. \end{array}$$

For any $H'' \subseteq H'$ define

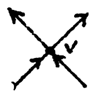
(133) $$l(H'') := \sum_{e \in EH', e \subseteq H''} l(e).$$

Then:

Claim 14. *Let R be a closed region in $\mathbb{R}^2$ such that the boundary* bd(R) *of R is part of H' in such a way that R is at the right hand side of any edge e of H' on* bd(R). *Then*

(134) $$l(\text{bd}(R)) = |\delta(R)|.$$

Proof. Since for any vertex v of H' of degree 4 the edges incident with v are oriented as in

(135) 

$\mathrm{bd}(R)$ consists of pairwise disjoint simple directed circuits.

For any vertex v of H on $G \cap \mathrm{bd}(R)$, define $\alpha(v)$ as follows. If $v \in VG$, let $\alpha(v) := 2$ if $v \in V_{\mathrm{even}}$ and $\alpha(v) := 0$ if $v \in V_{\mathrm{odd}}$. If $v \notin VG$, let $\alpha(v)$ be the number of edges $e \subseteq \mathrm{bd}(R)$ with $v \in \bar{e}$, and e being contained in an even face of G. By definition of l,

$$l(\mathrm{bd}(R)) = \sum_{v \in VH \cap G \cap \mathrm{bd}(R)} \alpha(v). \tag{136}$$

Now, on the other hand, for any $v \in VH \cap G \cap D$, $\alpha(v)$ is equal to the number of edges e of H such that $v \in \bar{e}$ and $e \cap R(D) = \emptyset$. So by definition of $\delta(R)$,

$$|\delta(R)| = \sum_{v \in VH \cap G \cap \mathrm{bd}(R)} \alpha(v). \tag{137}$$

Combining (136) and (137) gives (134). ∎

We next show:

Claim 15. *Each simple directed circuit D in H' is oriented clockwise and has length $l(D) = 4$. Moreover, $W^-_{\mathrm{odd}} = \emptyset$, $\eta = 0$, and configuration* (124)(g) *does not occur.*

Proof. For any oriented curve Q, let x_Q be its beginning point and y_Q be its end point (these points are not part of Q if Q is an open curve).

We first show the following (where we use that the unbounded face F_0 of G is bounded by at least four edges of G):

Subclaim 15.1. *There exist vertices v_1 and v_2 of G on the boundary of the unbounded face F_0 such that v_1 and v_2 are not adjacent in G, and such that for for each $i \in \{1,2\}$ and for each component Q of $\Delta' \cap \pi^{-1}[F_0]$, if the $\pi(x_Q) - \pi(y_Q)$ part of* $\mathrm{bd}(F_0)$ *(in clockwise orientation) contains v_i, then one of x_Q, y_Q is near to v_i.*

Proof. If for each Q the $\pi(x_Q) - \pi(y_Q)$ part of $\mathrm{bd}(F_0)$ contains at most two vertices of G, we can take any two nonadjacent vertices v_1, v_2 of G on $\mathrm{bd}(F_0)$.

If for at least one such component Q the $\pi(x_Q) - \pi(y_Q)$ part of $\mathrm{bd}(F_0)$ contains more than two vertices of G, choose Q maximal in the sense that the $\pi(x_Q) - \pi(y_Q)$ part of $\mathrm{bd}(F_0)$ is as large as possible. Then we choose v_1 and v_2 so that x_Q is near to v_1 and y_Q is near to v_2.

Now v_1 and v_2 have the required properties. For suppose that for some component Q' of $\Delta' \cap \pi^{-1}[F_0]$ the $\pi(x_{Q'}) - \pi(y_{Q'})$ part of $\mathrm{bd}(F_0)$ contains v_1. Since $\pi[Q']$ can cross $\pi[Q]$ only near to v_1 or v_2, and since we have chosen Q maximal, it follows that $x_{Q'}$ or $y_{Q'}$ is near to v_1.

Moreover, v_1 and v_2 are nonadjacent, since otherwise we could replace the component $C \in \mathcal{C}_0$ with $B(C) = Q$ through an isotopy of S^3 by a component $C' \in \mathcal{C}_0$ so that $Q' := B(C')$ is a curve with the property that the $\pi(x_{Q'}) - \pi(y_{Q'})$ part of $\mathrm{bd}(F_0)$ contains no other vertices of G than (possibly) v_1 and v_2. As this isotopy reduces $\sum_{v \in VG \cap \mathrm{bd}(F_0)} \omega(v)$, we obtain a contradiction to the minimality assumption. ∎

Let $\mathcal{D}$ denote the collection of all boundary components of all R_{2k} that are oriented clockwise and that are not in (124)(d), (e), or (f). Let ρ' denote the number of small components of type (124)(d), (e), or (f). So

$$\sum_{k=1}^{\infty} \chi(R_{2k}) \leq |\mathcal{D}| + \rho'. \tag{138}$$

Let e_1', e_1'' be the two edges incident with v_1 on $\mathrm{bd}(F_0)$, and let e_2', e_2'' be the two edges incident with v_2 on $\mathrm{bd}(F_0)$.

For any simple directed circuit D let again $R(D)$ denote the closed region enclosed by D. Moreover, let $r_1(D)$ be equal to the number of sets among $e_1', \{v_1\}, e_1''$ that are contained in $R(D)$. So $r_1(D) \in \{0,1,2,3\}$. Similarly, let $r_2(D)$ be equal to the number of sets among $e_2', \{v_2\}, e_2''$ that are contained in $R(D)$.

This is used in showing:

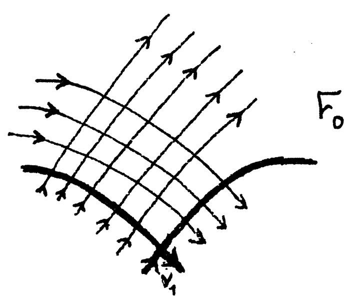
Subclaim 15.2. *There is no crossing (i.e., vertex of H of degree 4) in F_0 near to v_1 or near to v_2.*

Proof. Suppose the Subclaim is not true, and suppose without loss of generality that there exists a crossing in F_0 near to v_1. This implies that

$$(\#D \in \mathcal{D} \mid r_1(D) = 3) < (\#D \in \mathcal{D} \mid r_1(D) = 1). \tag{139}$$

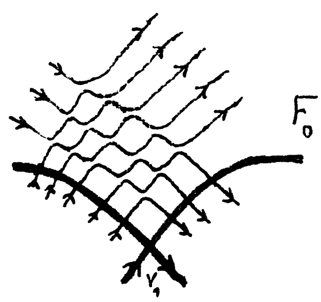
The reason is that the crossings in F_0 near to v_1 are locally as in:

(140)



It implies that v belongs to V_{even} and that the curves in $\mathcal{D}$ are locally as in:

(141)



So the number of $D \in \mathcal{D}$ with $r_1(D) = 3$ is strictly less than those with $r_1(D) = 1$. (Note that by the conditions in Subclaim 15.1, all D in $\mathcal{D}$ with $r_1(D) = 3$ occur (partly) in (141).)

It implies that for the curve $D \in \mathcal{D}$ traversing v_1 one has that $R(D) \cap VG \neq \{v_1\}$, since otherwise it would be of type (124)(a), implying that there are no crossings near to v_1 (cf. (125)).

Now we distinguish two cases.

Case 1. There is no crossing in F_0 near to v_2. This similarly implies:

(142) $$(\#D \in \mathcal{D} \mid r_2(D) = 3) < (\#D \in \mathcal{D} \mid r_2(D) = 1)$$

and v_2 belongs to V_{even}.

Now for each $D \in \mathcal{D}$ we have:

(143) $$l(D) \geq 8 - 2|r_1(D) - 1| - |r_2(D) - 1|.$$

To see this, let $\tilde{D}$ be a closed curve encircling D and very close to D, in a such a way that $\tilde{D}$ has exactly $l(D) = |\delta(R(D))|$ crossings with G. Then showing (143) is simple case-checking, using the facts that $R(D)$ should contain at least one vertex of G (as it is oriented clockwise, and as it not a small component of type (124)(d), (e), or (f)), and that hence, by the well-connectedness of K, $\tilde{D}$ should cross G often enough.

[Without loss of generality, $r_2(D) \leq r_1(D)$. First assume $r_2(D) = 0$. If $r_1(D) = 0$, then $l(D) \geq 4$, since $\emptyset \neq R(D) \cap VG \neq VG$, implying $l(D) = |\delta(R(D))| \geq 4$. If $r_1(D) = 1$, then $l(D) \geq 6$, since $\tilde{D}$ crosses e_1' and e_1'' and since $v_2 \notin R(D)$, while $R(D) \cap VG \neq \{v_1\}$. If $r_1(D) \geq 2$, then $l(D) \geq 4$, since $v_2 \notin R(D), v_1 \in R(D)$.

Second assume $r_2(D) = 1$. If $r_1(D) = 1$, then $l(D) \geq 8$, since $\tilde{D}$ crosses each of e_1', e_1'', e_2', e_2'', implying that $\tilde{D}$ can be decomposed into two curves $\tilde{D}_1$ and $\tilde{D}_2$ with end points all in F_0, where $\tilde{D}_1$ crosses e_1' and e_2' (say), and $\tilde{D}_2$ crosses e_1'' and e_2''. Then each of $\tilde{D}_1$ and $\tilde{D}_2$ crosses G at least four times. So $\tilde{D}$ crosses G at least eight times. If $r_1(D) = 2$, then $l(D) \geq 6$, since $\tilde{D}$ crosses one of e_1', e_1'' and each of e_2', e_2''. Again by decomposing $\tilde{D}$ into $\tilde{D}_1$ and $\tilde{D}_2$ one sees that $l(D) \geq 6$. If $r_1(D) = 3$, then $l(D) \geq 4$, since $\tilde{D}$ crosses both e_2' and e_2''.

Third assume $r_2(D) = 2$. If $r_1(D) = 2$, then $l(D) \geq 4$, since $\tilde{D}$ crosses at least two of the edges e_1', e_1'', e_2', e_2''. If $r_1(D) = 3$, then $l(D) \geq 2$, since $\tilde{D}$ crosses at least one of the edges e_2', e_2''.

Finally, if $r_1(D) = r_2(D) = 3$, then $l(D) \geq 0$.]

Claim 10, (138), (139), (142), (143), and Claims 14 and 11 imply:

(144) $$\begin{aligned}
&2|V_{\text{even}}| + |W_{\text{odd}}| + |U| + 2|W_{\text{odd}}^-| + 2\eta + 2\rho - 4\rho' \\
&\leq 4(\textstyle\sum_{k=1}^{\infty} \chi(R_{2k})) - 4\rho' \leq 4|\mathcal{D}| \\
&< (2|\mathcal{D}| - 2(\#D \in \mathcal{D}|r_1(D) = 3) + 2(\#D \in \mathcal{D}|r_1(D) = 1)) \\
&+ (2|\mathcal{D}| - 2(\#D \in \mathcal{D}|r_2(D) = 3) + 2(\#D \in \mathcal{D}|r_2(D) = 1)) \\
&= \textstyle\sum_{D\in\mathcal{D}}(4 - 2|r_1(D) - 1|) + \sum_{D\in\mathcal{D}}(4 - 2|r_2(D) - 1|) \\
&= \textstyle\sum_{D\in\mathcal{D}}(8 - 2|r_1(D) - 1| - |r_2(D) - 1|) \\
&\leq \textstyle\sum_{D\in\mathcal{D}} l(D) \leq (\sum_{k=1}^{\infty} l(\text{bd}(R_{2k}))) - 2\rho' \\
&= (\textstyle\sum_{k=1}^{\infty} |\delta(R_{2k})|) - 2\rho' = 2|V_{\text{even}}| + |W_{\text{odd}}| + |U| - 2\rho'.
\end{aligned}$$

Since $\rho' \leq \rho$, this gives a contradiction.

Case 2. There is no crossing in F_0 near to v_2. So

$$(145) \qquad (\#D \in \mathcal{D} \mid r_2(D) = 3) = 0.$$

Now for each $D \in \mathcal{D}$ one has

$$(146) \qquad l(D) \geq 6 - 2|r_1(D) - 1|.$$

To see this, again let $\tilde{D}$ be a closed curve encircling D and very close to D, in a such a way that $\tilde{D}$ has exactly $l(D) = |\delta(R(D))|$ crossings with G. Then showing (146) is again simple case-checking, using the fact that $R(D)$ contains at least one vertex of G and using the well-connectedness of K.

[If $r_1(D) = 0$ then $l(D) \geq 4$, since $v_1 \notin R(D)$, and hence $\tilde{D}$ has at least four crossings with G. If $r_1(D) = 1$, then $l(D) \geq 6$, since $\tilde{D}$ crosses both e_1' and e_1'' and since $r_2(D) \leq 2$. If $r_1(D) = 2$, then $l(D) \geq 4$, since $\tilde{D}$ crosses at least one of the edges e_1', e_1'' and since $r_2(D) \leq 2$. If $r_1(D) = 3$, then $l(D) \geq 2$, since $r_2(D) \leq 2$.]

Now by Claim 10, (138), (139), (146), and Claims 14 and 11:

$$(147) \qquad \begin{aligned} &2|V_{\mathrm{even}}| + |W_{\mathrm{odd}}| + |U| + 2|W^-_{\mathrm{odd}}| + 2\eta + 2\rho - 4\rho' \\ &\leq 4(\textstyle\sum_{k=1}^{\infty} \chi(R_{2k})) - 4\rho' \leq 4|\mathcal{D}| \\ &< 4|\mathcal{D}| - 2(\#D \in \mathcal{D}|r_1(D) = 3) + 2(\#D \in \mathcal{D}|r_1(D) = 1) \\ &= \textstyle\sum_{D\in\mathcal{D}}(6 - 2|r_1(D) - 1|) \leq \sum_{D\in\mathcal{D}} l(D) = (\sum_{k=1}^{\infty} l(\mathrm{bd}(R_{2k}))) - 2\rho' \\ &= (\textstyle\sum_{k=1}^{\infty} |\delta(R_{2k})|) - 2\rho' = 2|V_{\mathrm{even}}| + |W_{\mathrm{odd}}| + |U| - 2\rho'. \end{aligned}$$

Since $\rho' \leq \rho$, this is a contradiction. This proves Subclaim 15.2. ∎

This gives:

Subclaim 15.3. *Let D be a simple directed circuit in H', oriented clockwise, and not being a small component of type* (124)(d), (e) *or* (f). *Then* $l(D) \geq 4$.

Proof. By Subclaim 15.2, $r_1(D) \leq 2$ and $r_2(D) \leq 2$. If $VG \not\subseteq R(D)$, then, as $R(D)$ contains at least one vertex, $l(D) = |\delta(R(D))| \geq 4$, by the well-connectedness of K.

If $VG \subseteq R(D)$, then $1 \leq r_1(D) \leq 2$ and $1 \leq r_2(D) \leq 2$. So any curve $\tilde{D}$ encircling D and close to D crosses at least one of e_1', e_1'' and at least one of e_2', e_2''. So $l(D) \geq 4$. ∎

Now by Claim 10, (138), Subclaim 15.3, and Claims 14 and 11:

$$(148) \qquad \begin{aligned} &2|V_{\mathrm{even}}| + |W_{\mathrm{odd}}| + |U| + 2|W^-_{\mathrm{odd}}| + 2\eta + 2\rho - 2\rho' \\ &\leq 4(\textstyle\sum_{k=1}^{\infty} \chi(R_{2k})) - 2\rho' \leq 4|\mathcal{D}| + 2\rho' \leq (\sum_{D\in\mathcal{D}} l(D)) + 2\rho' \\ &\leq \textstyle\sum_{k=1}^{\infty} l(\mathrm{bd}(R_{2k})) = \sum_{k=1}^{\infty} |\delta(R_{2k})| = 2|V_{\mathrm{even}}| + |W_{\mathrm{odd}}| + |U|. \end{aligned}$$

Since $\rho' \leq \rho$, it follows that we have equality throughout in (148). Hence $W^-_{\mathrm{odd}} = \emptyset$ and $\eta = 0$ and $\rho' = \rho$. So configuration (124)(g) does not occur.

Moreover, H' has no simple directed circuit D that is oriented counter-clockwise. Otherwise we could decompose H' into simple directed circuits $D_1, \ldots, D_t$ where $D_t = D$, and where for some $s < t$, $D_1, \ldots D_s$ are oriented clockwise, and $D_{t+1}, \ldots, D_t$ are oriented counter-clockwise. This implies by Subclaim 15.3 and Claim 12,

$$
\begin{aligned}
(149) \qquad \sum_{k=1}^{\infty} l(\mathrm{bd}(R_{2k})) = l(H') = \sum_{i=1}^{t} l(D_i) \\
\geq 4s - 2\rho' > 4(s - (t - s)) - 2\rho' = 4\sum_{k=1}^{\infty} \chi(R_{2k}) - 2\rho',
\end{aligned}
$$

contradicting equality in (148).

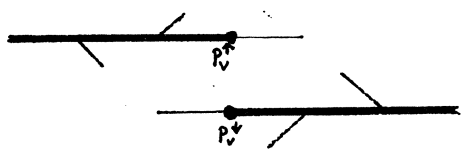
It similarly follows that for each simple directed circuit D one has $l(D) = 4$. ∎

This implies:

Claim 16. *Configuration* (124)(b) *does not occur.*

Proof. Consider $\pi^{-1}[v]$ from face F_1. We see, since $W^-_{\mathrm{odd}} = \emptyset$:

(150)



This contradicts Claim 5. ∎

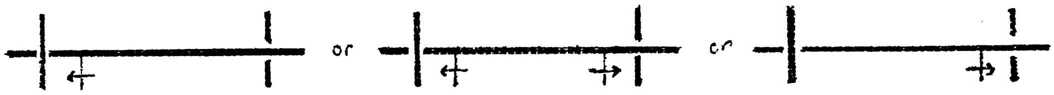
Moreover:

Claim 17. $W = W^+_{\mathrm{odd}}$ *and* $\varphi = 0$. *Configurations* (124)(d) *and* (f) *do not occur.*

Proof. By Claim 15, $W^-_{\mathrm{odd}} = \emptyset$. We next show $W^-_{\mathrm{even}} = \emptyset$. Suppose $W^-_{\mathrm{even}} \neq \emptyset$. Let $w \in W^-_{\mathrm{even}}$, and let e be the edge of G containing $\pi(w)$. So, by Claim 4, all points in W that project to e, belong to W^-_{even}.

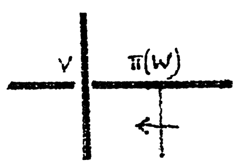
Since by Claim 15 configuration (124)(g) does not occur, it implies that e is as in

(151)



and there are no other points in $\pi[W]$ on e. Without loss of generality, we may assume that

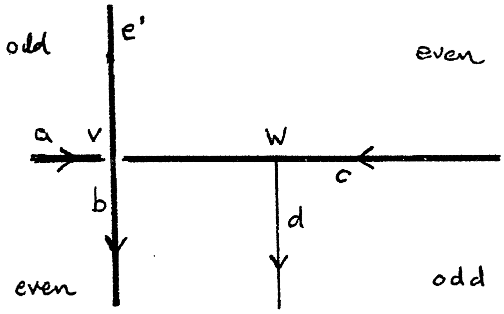
(152)



occurs. Then on e, left to $\pi(w)$ all edges of H' are entering e from above, and leaving e from below. Moreover, right to $\pi(w)$ all edges of H' are entering e from below, and leaving e from above:

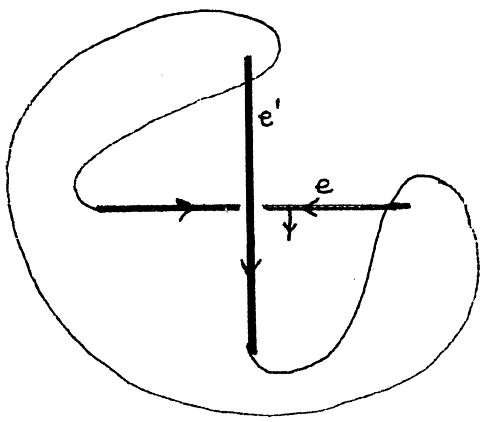
(153) 

Now first assume that v belongs to V_{even}. Let a, b, c, d be as in

(154) 

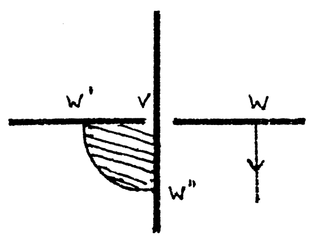
Then H' contains a simple directed circuit D containing a and b, and a simple directed circuit D' containing c and d. We may assume that D' encloses D (except for touchings).

Again, let $R(D)$ denote the closed region enclosed by D. Suppose $R(D) \cap VG \neq \{v\}$. Then $VG \subseteq R(D)$, and the boundary of D has crossings with the edges e and e' as in

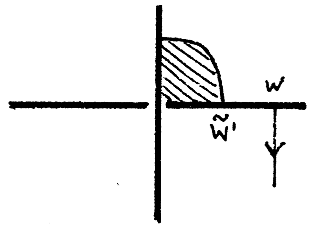
(155) 

But in that case D' cannot enclose D.

So we know $R(D) \cap VG = \{v\}$. Hence by Claim 13, $R(D)$ is the shaded region in

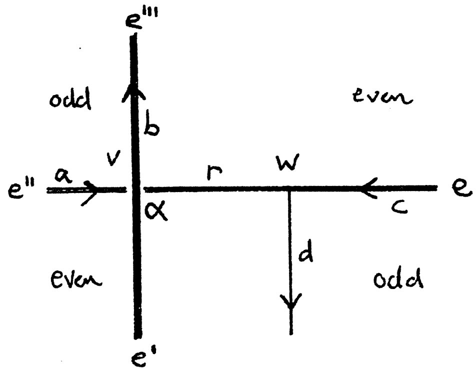
(156) 

with $w', w'' \in W^+$. We can switch the component C in $\mathcal{C}$ with $B(C)$ having w and w' as angle points, to the other side of v. That is, (156) becomes

(157) 

However, now $\tilde{w}'$ belongs to W^+ while $w \in W^-$, contradicting the minimality of Σ (cf. Claim 4).

Next assume that v belongs to V_{odd}. Let a, b, c, d be as in

(158) 

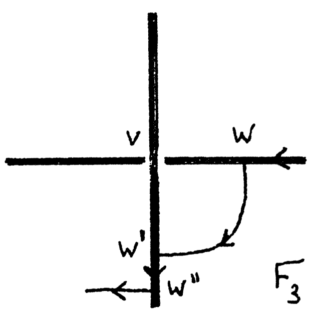
Now there are no points in $\pi[U]$ on part r (since $v \in V_{\text{odd}}$ and hence $\omega(v) = \alpha$). It follows that H' contains a simple directed circuit D containing a, b, c and d. Since e is fully contained in $R(D)$, we know $v' \in R(D)$, and hence $R(D) \cap VG \neq \{v\}$.

Let e', e'', e''' be the edges of G as indicated in (158). Let E' be the set of edges of G not contained in $R(D)$ but with at least one end point in $R(D)$. Since $l(D) = 4$, $|E'| \leq 4$.

If $|E'| = 4$, then two edges from E' form a cut either with e and e''' or with e' and e''. Hence, by the well-connectedness of K, $E' = \{e, e', e'', e'''\}$, contradicting the fact that $e \subseteq R(D)$.

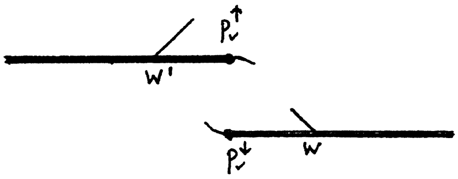
Suppose next that $|E'| = 3$. Then there are exactly two edges in G that intersect $R(D)$ in exactly one of their end points. This contradicts the well-connectedness of K.

So $|E'| = 2$. Then either $E' \cup \{e, e''\}$ or $E' \cup \{e', e''\}$ is a cut. Hence, by the well-connectedness of K, $E' = \{e, e'''\}$ or $E' = \{e', e''\}$. Since e is fully contained in $R(D)$, it follows that $E' = \{e', e''\}$. So e' is not fully contained in $R(D)$. This implies that D should have a crossing with edge e'. Since $\omega(v) = \alpha$, this cannot be in a point in $\pi[U]$ near to v. So near to v we have

(159) 

However, since $W_{\mathrm{odd}}^{-} = \emptyset$, we have that $w', w'' \in W^{+}$. In particular, $w' \in W_{\mathrm{even}}^{+}$. Let $C \in \mathcal{C}$ be the component in $\pi^{-1}[F_3]$ with $B(C) = Q$. The angles of $\mathrm{bd}(C)$ seen from F_3 are as in

(160)



This is not possible.

So $W^{-} = \emptyset$. Now by Subclaims 10.5 and 10.4

(161) $$\begin{aligned} |W_{\mathrm{odd}}^{+}| - |W_{\mathrm{even}}^{+}| &= 2|V| + 2\varphi + 2\eta, \\ |W_{\mathrm{odd}}^{+}| + |W_{\mathrm{even}}^{+}| &= 2|V|. \end{aligned}$$

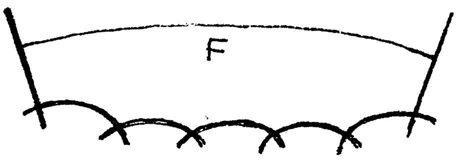
Hence $W_{\mathrm{even}}^{+} = \emptyset$ and $\varphi = \eta = 0$.

It follows that configurations (124)(d) and (f) do not occur. ∎

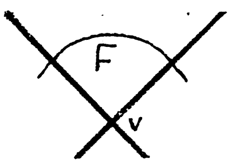
Consider any component Q of $\Delta' \setminus \pi^{-1}[G]$. So Q connects two points in $W \cup U$. Let $\pi[Q]$ be contained in face F of G.

Then Q can be of one of the following three types:

(162) (i) $Q = B(C)$ for some $C \in \mathcal{C}_0$:



(ii) $Q \subseteq B(C)$ for some $C \in \mathcal{C}_1$, and Q connects two points in U on a Z-type curve near to a vertex v of G:



(iii) $Q \subseteq B(C)$ for some $C \in \mathcal{C}_1$, and Q connects two points in W on a Z-type segment on an edge e of G:

Since $W_{\mathrm{even}} = \emptyset$, if F is odd then (162)(iii) does not apply. If F is odd and (162)(i) applies then Q must connect two points in U. Moreover, since $\varphi = 0$, if (162)(i) applies, then any end point of Q must be connected to a component Q' of type (162)(ii).

It follows that each component D of Δ' is of one of the following three types:

(163) (i) for any face F of G, any component Q of $D \cap \pi^{-1}[F]$ is of type (162)(i) if F is odd, and of type (162)(ii) if F is even;

(ii) for any face F of G, any component Q of $D \cap \pi^{-1}[G]$ is of type (162)(i) if F is even, and of type (162)(ii) if F is odd;

(iii) D is of type (124)(e).

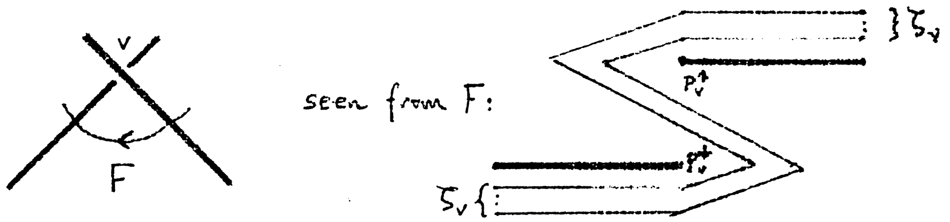
For any $x \in \mathbb{R}^3$ let $\omega^{\uparrow}(x)$ be the number of points in Σ strictly above x, and let $\omega^{\downarrow}(x)$ denote the number of points in Σ strictly under x.

Claim 18. *For each $v \in V_{\text{even}}$, $\zeta_v = 0$.*

Proof. Consider a component D of type (163)(i). So $D \setminus \pi^{-1}[G]$ consists, alternatingly, of components of type (162)(i) in odd faces and (162)(ii) in even faces, each connecting two points in U.

So when following x along D we see that $\omega^{\uparrow}(x) - \omega^{\downarrow}(x)$ decreases by $2\zeta_v + 2$ when traversing any component Q of type (162)(ii) near vertex v of G (so $v \in V_{\text{even}}$):

(164)



Now adding up all changes of $\omega^{\uparrow}(x) - \omega^{\downarrow}(x)$ over D we should obtain 0. Hence, adding up over all such D, gives

(165)
$$\sum_{v \in V_{\text{even}}} \zeta_v(2\zeta_v + 2) = 0.$$

This implies that $\zeta_v = 0$ for each $v \in V_{\text{even}}$. ∎

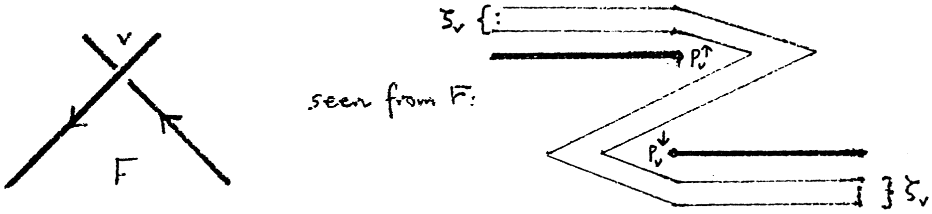
From this we derive:

Claim 19. $V_{\text{odd}} = \emptyset, U = \emptyset$.

Proof. Consider a component D of type (163)(ii) that is not of type (124)(e). So D consists of parts of K and of components Q of type (162)(i) (in even faces) and (ii) (in odd faces).

Again we will check how $\omega^{\uparrow}(x) - \omega^{\downarrow}(x)$ changes when x traverses D. Since $\zeta_v = \eta_v = \varphi_v = 0$ if $v \in V_{\text{even}}$, $\omega^{\uparrow}(x) - \omega^{\downarrow}(x)$ does not change near to v when $v \in V_{\text{even}}$. As in Claim 18, at any point $v \in V_{\text{odd}}$, when following one of the components (162)(ii), $\omega^{\uparrow}(x) - \omega^{\downarrow}(x)$ increases by $2\zeta_v + 2$.

Near any point $v \in V_{\text{odd}}$, when traversing the component through e_v, the value of $\omega^{\uparrow}(x) - \omega^{\downarrow}(x)$ increases by $2\zeta_v$.

(166) 

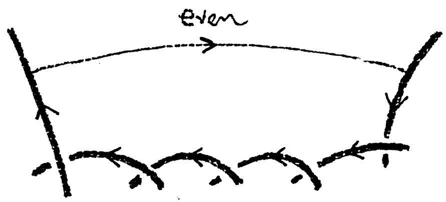
Near any point $v \in V_{\text{even}}$, when traversing the component through e_v, the value of $\omega^{\uparrow}(x) - \omega^{\downarrow}(x)$ decreases by 2. Finally, at any point $w \in W$, the value of $\omega^{\uparrow}(x) - \omega^{\downarrow}(x)$ increases by 1 (since $W = W^{+}_{\text{odd}}$). This last also holds for components of type (162)(iii).

Adding up over all changes and over all D of types (163)(ii) and (iii), we obtain

$$\sum_{v \in V_{\text{odd}}} 2\zeta_v(\zeta_v + 1) + \sum_{v \in V_{\text{odd}}} 2\zeta_v + |W| - 2|V_{\text{even}}| = 0. \tag{167}$$

Hence, since $|W| = 2|V|$, we see that $V_{\text{odd}} = \emptyset$. Since moreover by Claims 17 and 18, $\zeta_v = \eta_v = \varphi_v = 0$ for each $v \in V_{\text{even}}$, we have $U = \emptyset$. ▮

It follows that each component D of Δ' not of type (124)(e) has projection like:

(168) 

However, since $|\delta(R(D))| = 4$ (where $R(D)$ is the closed region enclosed by D), it should be a component of type (124)(a). Since $|W| = 2|V|$, it follows that all components of Δ' are of this type. This implies that there exists an isotopy of $\mathbb{R}^3$ bringing Σ to Σ_K. ▮▮▮

5. Theorem B

We finally show:

Theorem B. *Let K and K' be well-connected alternating links such that the unbounded faces of $\pi[K]$ and $\pi[K']$ are even. If there is an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$, then the diagrams of K and K' are equivalent.*

Proof. Let Φ be an isotopy of S^3 bringing Σ_K to $\Sigma_{K'}$. Let $\psi(x) := \Phi(1, x)$ for all $x \in S^3$. So $\psi[\Sigma_K] = \Sigma_{K'}$.

Again, let H_K be the planar graph obtained by putting a vertex in each odd face of $\pi[K]$, joining any two such vertices by an edge if the corrresponding odd faces have a crossing in common. So for each vertex v of $\pi[K]$ there is an edge, denoted by ε_v, of H_K (and an edge denoted by e_v in Σ_K).

The graph $H_{K'}$ is derived similarly from K'. Now ε'_v denotes the edge of $H_{K'}$ corresponding to vertex v of $\pi[K']$. Let e'_v denote the edge in $\Sigma_{K'}$ corresponding to vertex v of $\pi[K']$.

For each even face F of $\pi[K]$, we fix a simple closed curve C_F on Σ_K as follows. Let $F_1, \ldots, F_t$ be the odd faces incident with F, and let $v_1, \ldots, v_t$ be the vertices of $\pi[K]$ incident with F. Then C_F is a closed curve on Σ_K traversing the faces $D_{F_1}, \ldots, D_{F_t}$ of Σ_K and crossing each of the edges $e_{v_1}, \ldots, e_{v_t}$ exactly once, and not traversing any other face of Σ_K or crossing any other edge of Σ_K.

(Recall that $D_F = \pi^{-1}[F] \cap \Sigma_K$ for each face F of $\pi[K]$, and that e_v is the edge of Σ_K connecting $p_v^{\downarrow}$ and $p_v^{\uparrow}$.)

By the well-connectedness of K we can take the curves C_F in such a way that, for any two faces F_1, F_2 of $\pi[K]$, C_{F_1} and C_{F_2} have at most one crossing. In fact C_{F_1} and C_{F_2} have exactly one crossing, if and only if $\overline{F_1}$ and $\overline{F_2}$ intersect, viz. in a vertex v of $\pi[K]$. (That is, if and only if F_1 and F_2 are contained in adjacent faces of H_K.) We may assume that this crossing occurs on e_v.

For any even face F of $\pi[K]$, let B_F denote the circuit in H_K bounding the face of H_K containing F.

Now for each even face F of $\pi[K]$, $\psi[C_F]$ is a closed curve on $\Sigma_{K'}$. We may assume that each edge e'_v in $\Sigma_{K'}$ is crossed only a finite number of times by $\psi[C_F]$. For each even face F of $\pi[K]$ and each edge $e = \varepsilon'_v$ of $H_{K'}$, let

$$(169) \qquad x(F, e) := \text{number of times } \psi[C_F] \text{ crosses } e'_v.$$

Define for each even face F of $\pi[K]$:

$$(170) \qquad B'_F := \{e \in EH_{K'} \mid x(F, e) \text{ is odd}\}.$$

Since $\psi[C_F]$ is a closed curve, we know that B'_F is a cycle (= edge-disjoint union of circuits) in $H_{K'}$.

We show:

Claim 20. *For each edge e of $H_{K'}$ there exist even faces $F_1 \neq F_2$ of $\pi[K]$ such that $e \in B'_{F_1} \cap B'_{F_2}$.*

Proof. Choose an edge e of $H_{K'}$, say $e = \varepsilon'_v$, where v is a vertex of $\pi[K']$. First note that

$$(171) \qquad \sum_F x(F, e) \text{ is even}$$

(where F ranges over all even faces F of $\pi[K]$). This follows from the fact that $\bigcup_F C_F$ is null-homologous on Σ_K, implying that $\bigcup_F \psi[C_F]$ is null-homologous on $\psi[\Sigma_K]{=}\Sigma_{K'}$.

(171) implies that we only have to show that there exists one face F such that $e \in B'_F$. Make a closed curve D in $\mathbb{R}^3 \setminus \Sigma_{K'}$ close to e'_v and encircling e'_v. This can be done in such a way that $\mathrm{lk}(D, C'_{F'}) = 1$, where F' is one of the two even faces of $\pi[K']$ incident with vertex v of $\pi[K']$.

Then

$$(172) \qquad \mathrm{lk}(\psi^{-1}[D], \psi^{-1}[C'_{F'}]) = \mathrm{lk}(D, C'_{F'}) = 1.$$

So there exists a closed curve C (viz. $\psi^{-1}[C'_{F'}]$), on Σ_K such that $\mathrm{lk}(\psi^{-1}[D], C) = 1$. Since C is homologous to a combination of curves C_F in Σ_K (for even faces F of $\pi[K]$), there exists at least one even face F of $\pi[K]$ such that $\mathrm{lk}(\psi^{-1}[D], C_F)$ is odd. Hence $\mathrm{lk}(D, \psi[C_F])$

is odd. This implies that $\psi[C_F]$ crosses e'_v an odd number of times. So e belongs to B'_F. ∎

Next:

Claim 21. *For each even face F of $\pi[K]$ one has: $|B_F| = |B'_F|$. Moreover, each edge of $H_{K'}$ is contained in exactly two of the cycles B'_F.*

Proof. For any simple closed curve C' on $\Sigma_{K'}$ and any $e = e'_v$ on $\Sigma_{K'}$ define:

$$(173)\qquad \begin{aligned}&\gamma(C',e) := \\ &[(\text{number of times } C' \text{ crosses } e \text{ in one direction}) \\ &-(\text{number of times } C' \text{ crosses } e \text{ in the other direction})]^2.\end{aligned}$$

(So here we choose, temporarily, a 'left hand side' and a 'right hand side' of e. Clearly, the definition is independent of this choice.)

Then it is easy to check that

$$(174)\qquad \tau(C',\Sigma_{K'}) = \sum_v \gamma(C',e'_v),$$

where v ranges over all vertices of $\pi[K']$.

Moreover, for each even face F of $\pi[K]$ and each vertex v of $\pi[K']$, $x(F,\varepsilon'_v)$ is odd, if and only if $\gamma(\psi[C_F],e'_v)$ is odd.

Hence for each even face F of $\pi[K]$:

$$(175)\qquad |B'_F| \le \sum_v \gamma(\psi[C_F],e'_v) = \tau(\psi[C_F],\Sigma_{K'}) = \tau(C_F,\Sigma_K) = |B_F|$$

(where again v ranges over vertices of $\pi[K']$). Moreover, since by Claim 20 each edge e of $H_{K'}$ is contained in at least two edges of B'_F:

$$(176)\qquad \sum_F |B'_F| \ge 2v(K') = 2v(K) = \sum_F |B_F|,$$

where F ranges over all even faces of $\pi[K]$.

Combining (175) and (176) gives the Claim. ∎

Next we show:

Claim 22. *Let F_1 and F_2 be two even faces of $\pi[K]$. Then $|B'_{F_1} \cap B'_{F_2}|$ is odd, if and only if F_1 and F_2 are in adjacent faces of H_K.*

Proof. First assume that F_1 and F_2 are not in adjacent faces of H_K. So by assumption, C_{F_1} and C_{F_2} are disjoint. Then also $\psi[C_{F_1}]$ and $\psi[C_{F_2}]$ are disjoint We may assume that the projections $\pi[\psi[C_{F_1}]]$ and $\pi[\psi[C_{F_2}]]$ are closed curves in $\mathbb{R}^2$such that, they only cross at vertices of $\pi[K']$, in such a way that in a vertex v of $\pi[K']$ there are

$$(177)\qquad x(F_1,e'_v)\cdot x(F_2,e'_v)$$

crossings of $\pi[\psi[C_{F_1}]]$ with $\pi[\psi[C_{F_2}]]$.

Since the total number of crossings of $\pi[\psi[C_{F_1}]]$ with $\pi[\psi[C_{F_2}]]$ is even, we know that

$$(178) \qquad \sum_v x(F_1, e'_v) \cdot x(F_2, e'_v)$$

is even. Since (178) has the same parity as $|B'_{F_1} \cap B'_{F_2}|$, we know that $|B'_{F_1} \cap B'_{F_2}|$ is even.
If F_1 and F_2 are in adjacent faces, one similarly shows that $|B'_{F_1} \cap B'_{F_2}|$ is odd. ∎

In fact we have:

Claim 23. *For any two even faces F_1 and F_2 of $\pi[K]$, $|B'_{F_1} \cap B'_{F_2}| = 1$ if F_1 and F_2 are contained in adjacent faces of H_K, and $|B'_{F_1} \cap B'_{F_2}| = 0$ otherwise.*

Proof. By Claims 21 and 22 and by the well-connectedness of K,

$$(179) \qquad \begin{aligned} &2v(K) \\ &= \text{number of pairs } (F_1, F_2) \text{ of two adjacent faces of } H_K \\ &\leq \sum_{(F_1,F_2), F_1 \neq F_2} |B'_{F_1} \cap B'_{F_2}| = \sum_{F_1} \Big(\sum_{F_2 \neq F_1} |B'_{F_1} \cap B'_{F_2}| \Big) \\ &= \sum_{F_1} |B'_{F_1}| = \sum_{F_1} |B_{F_1}| = 2v(K). \end{aligned}$$

So the inequality is attained with equality, and the claim follows. ∎

We can now define a function

$$(180) \qquad \theta : EH_K \longrightarrow EH_{K'}$$

as follows. For $e \in EH_K$, let F_1 and F_2 be the two even faces of $\pi[K]$ contained in the faces of H_K incident with e. Let

$$(181) \qquad B'_{F_1} \cap B'_{F_2} = \{e'\}.$$

Then define $\theta(e) := e'$. By Claim 21, this function is one-to-one, and hence onto (since $|EH_K| = |EH_{K'}|$).

Moreover, for each even face F of $\pi[K]$, $\theta[B_F] = B'_F$, since

$$(182) \qquad \theta[B_F] = \bigcup_{F' \neq F} \theta[B_F \cap B_{F'}] = \bigcup_{F' \neq F} (B'_F \cap B'_{F'}) = B'_F.$$

So for each cycle B in H_K the set $\theta[B]$ is a cycle in $H_{K'}$ (since B is a binary sum of circuits B_F, and hence $\theta[B]$ is a binary sum of cycles B'_F).

Now both H_K and $H_{K'}$ are 3-vertex-connected planar graphs (by the well-connectedness of K and K'), with $|VH_K| = b(K) = b(K') = |VH_{K'}|$ and $|EH_K| = v(K) = v(K') = |EH_{K'}|$. Hence, by Whitney's theorem [10] H_K and $H_{K'}$ are the same plane graph, up to rerouting edges through the unbounded face, and up to turning the graph upside down. This implies that the diagrams of K and K' can be obtained from each other by the operations (3). That is, K and K' have equivalent diagrams. ∎∎∎

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