Centrum voor Wiskunde en Informatica
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A local limit theorem for *L*-statistics

Department of Operations Research, Statistics, and System Theory   Report BS-R9033   November

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A local limit theorem for L-statistics

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An uniform local central limit theorem is established for the density of a linear combination of order statistics. Higher order approximations are also briefly considered. Our proofs are patterned after VAN ZWET (1977), where closely related arguments can be found. We combine the Fourier inversion theorem with suitable conditioning arguments. The main difficulty is to show that the characteristic function of a linear combination of order statistics is integrable.

(1980) Mathematics Subject Classification: 62E20, 60F05
Key Words & Phrases: local limit theorems, L-statistics, asymptotic expansions, Fourier inversion, convergence in total variation.

1. INTRODUCTION AND MAIN RESULT

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with common distribution function $(df)F$. For $n=1,2,\ldots,X_{1:n} \leqslant \ldots \leqslant X_{n:n}$ denote the ordered $X_1, X_2, \ldots, X_n$. We consider L-statistics (or linear combinations of order statistics) of the form

$$T_n = n^{-1} \sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{i:n} \tag{1.1}$$

where J denotes a fixed weight function on $(0,1)$. Let $F_n^*(x) = P(T_n^* \leqslant x)$, for $-\infty < x < \infty$, where

$$T_n^* = (T_n - ET_n)/\sigma(T_n) \tag{1.2}$$

whenever well-defined.

Many authors have contributed to the problem of the asymptotic normality of L-statistics. For example, in Theorem 2 of STIGLER (1974) it is shown that T_n^* is asymptotically $N(0,1)$ distributed, provided

J is bounded and continuous a.e.. F^{-1}, $EX_1^2 < \infty$ and $\sigma^2(J,F) > 0$, where.

$$\sigma^2(J,F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(F(x))\, J(F(y))\, (F(\min(x,y)) - F(x)F(y))dxdy. \tag{1.3}$$

I.e., it follows from (1.3) that

$$\sup_x |F_n^*(x) - \Phi(x)| \to 0, \quad n \to \infty \tag{1.4}$$

I also refer to the more recent work of MASON (1981) and MASON & SHORACK (1990), which contain the best results so far obtained in this area. On the other hand most L-statistics, which are of practical interest, are covered by condition (1.3). Berry-Esseen bounds and Edgeworth expansions for L-statistics of the form (1.1) are also available (see, e.g. FRIEDRICH (1989), HELMERS (1980), (1982), and VAN ZWET (1977), (1984))

The aim of this paper is to establish an uniform local limit theorem for L-statistics. Sufficient conditions are obtained which guarantee that the density f_n^* of F_n^* exists, and tends uniformly to the standard normal density ϕ, as n get large. Our proof will combine characteristic functions methods with appropriate conditioning arguments in a way closely resembling those used by VAN ZWET (1977). The

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main problem is to show that the characteristic function $\rho_n^*(t) = E\exp(itT_n^*)$ is uniform integrable, for all n sufficiently large, so that the Fourier inversion theorem and the continuity theorem for densities (see, e.g. FELLER (1971), p. 509-510) can be applied.

THEOREM 1.1. *Suppose that assumption* (1.3) *is satisfied. The function J is bounded away from zero on some open subinterval of* (0,1). *The df F is differentiable with bounded derivative f; in addition* $f(F^{-1})$ *satisfies a Lipschitz condition of order* $\delta > 0$ *on (0.1) and is bounded away from zero on closed subintervals of* (0,1). *Then, for all n sufficiently large,* F_n^* *possesses a density* f_n^*, *and, in addition*

$$\lim_{n\to\infty} \sup_x |f_n^*(x) - \phi(x)| = 0. \tag{1.5}$$

In Theorem 1.1 we establish a density analog of a CLT for L-statistics with bounded weights. In addition to Stigler's 1974 conditions (cf. (1.3)), which were already imposed to establish asymptotic normality of F_n^*, we require a weak regularity condition for J and F. Extensions of Theorem 1.1 to L-statistics with unbounded weights are possible. Also weights of a different type - e.g. $c_{in} = n\int_{i-1/n}^{i/n} J(s)ds$ - are easily treated. In stead of employing STIGLER'S (1974) CLT one should use a more general CLT for L- statistics, e.g. the one in MASON (1981).

It follows from (1.5) that

$$\sup_B |P(T_n^* \in B) - \Phi(B)| \to 0, \quad \text{as } n \to \infty \tag{1.6}$$

where the sup is taken over all Borel sets B; i.e. we obtain convergence in total variation. This strong convergence concept is emphasized in the recent monograph by REISS (1989). By combining Theorem 4.5.3 and Lemma 3.1.2 of REISS (1989) results like (1.5) and (1.6) are easily established for L-statistics with non-zero weights on k different order statistic $X_{r_i:n}$, $i = 1,...,k$ and zero weights elsewhere. The number $k = k(n)$ is allowed to increase with n, but at a rate not faster than $o(\sqrt{n})$. The df F must have three derivatives on (shrinking) neighbourhoods of the points $F^{-1}(\frac{r_i}{n+1})$, $i = 1,...,k$. Thus, only L-statistics based on a vanishing fraction of order statistics can be considered by this approach.

In contrast, our results (1.5) and (1.6) apply to linear combinations of order statistics, which allow weights to put on *all* the observations. In addition, our method of proof also yields asymptotic expansions for f_n^*. In section 2 we prove Theorem 1.1. In section 3 we briefly indicate how an asymptotic expansion for f_n^* can be derived. In Theorem 3.1 we give a set of sufficient conditions implying that

$$\sup_x |f_n^*(x) - \tilde{f}_n(x)| = o(n^{-1}), \text{ as } n \to \infty \tag{1.7}$$

holds true, with $\tilde{f}_n(x)$ as in (3.1).

A technical result - a bound for the characteristic function of a smooth function of a single central uniform order statistic - which is needed in the sections 2 and 3, is dealt with in the appendix.

To conclude this section we want to relate our result with recent work of MACHT and WOLF (1989). Clearly, under the assumptions of Theorem 1.1 we easily check with the aid of relation (2.5) with $r = 3$ and the Fourier inversion theorem that $\sup_x |f_n^*(x+y) - f_n^*(x)| \leq K|y|$, for all real y, any $n \geq n_0$ for some natural number n_0, and some constant $K > 0$. In other words the sequence of densities $\{f_n^*\}$ is uniformly Lipschitz of order 1. This smoothness property of the sequence $\{f_n^*\}$ is, in view of Theorem 3 MACHT and WOLF (1989), a sufficient condition for the implication (1.4) $\Rightarrow$ (1.5), or equivalently (1.4) $\Rightarrow$ (1.6), to hold. These authors also obtain a necessary and sufficient condition in terms of $\{f_n^*\}$ for this problem. It is also clear from the Macht-Wolf paper that local limit results like (1.5) may hold even when $\{\rho_n^*\}$ is not uniform integrable and the Fourier-inversion theorem cannot be applied.

2. PROOF OF THEOREM 1.1.

Let, for each $n \geqslant 1$, $(U_{1:n},..., U_{n:n})$ denote the order statistics corresponding to a sample of size n from the uniform distribution on $(0,1)$. Since the joint distribution of $X_{i:n}$, $i=1,..., n$ is the same as that of $F^{-1}(U_{i:n})$, $i=1,...,n$ we may identify T_n with $n^{-1}\Sigma_{i=1}^{n} J(\frac{i}{n+1})F^{-1}(U_{i:n})$.

Let, for each $n \geqslant 1, \rho_n^*(t)=E\exp(itT_n^*)$ denote the characteristic function (ch.f) of T_n^* (cf. (1.2)). By the Fourier inversion theorem and the continuity theorem for densities (cf. FELLER (1971), p. 509-510) we know that to prove Theorem 1.1 it suffices to show that ρ_n^* is integrable, for all n sufficiently large, and in addition that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\rho_n^*(t)-e^{-\frac{1}{2}t^2}|dt = 0 \tag{2.1}$$

To check these requirements we proceed in a number of steps. Theorem 2 of STIGLER (1974) (cf. assumption (1.3)) and Levy's continuity theorem for ch.f's together imply that $\lim_{n\to\infty}\rho_n^*(t)=e^{-\frac{1}{2}t^2}$, uniformly on finite intervals. It easily follows that to prove (2.1) (and the integrability of ρ_n^*) we only have to show that

$$\lim_{M\to\infty} \int_{|t|>M} |\rho_n^*(t)|dt = 0 \tag{2.2}$$

uniformly in n. In other words, we must prove that the sequence $\{|\rho_n^*(t)|\}_{n=1}^{\infty}$ is uniformly integrable.

To proceed we note that, instead of (2.2), we may as well prove that

$$\lim_{M\to\infty} \int_{|t|>M} |\tilde{\rho}_n(t)|dt = 0 \tag{2.3}$$

uniformly in n, where $\tilde{\rho}_n(t)=E\exp\{it\tilde{T}_n\}$, with

$$\tilde{T}_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} J(\frac{i}{n+1})F^{-1}(U_{i:n}) \tag{2.4}$$

This is a simple consequence of Theorem 1 of STIGLER (1974), where it is shown that $\sigma(T_n)\sim n^{-\frac{1}{2}}\sigma(J,F)$, as $n\to\infty$, provided assumption (1.3) is satisfied.

For our purposes in this paper (see also section 3) it will be convenient to prove, in stead of (2.3), a much stronger statement. We shall show that for any positive integer r.

$$|\tilde{\rho}_n(t)| = \mathcal{O}(|t|^{-r}), \text{ as } |t|\to\infty \tag{2.5}$$

uniformly in n. Clearly (2.5) with $r=2$ already gives us (2.3); the assertion (2.5) in its present form will be especially useful in section 3. We note in passing that a bound very similar to (2.5) was established by VAN ZWET (1977). Application of Theorem 4.1 of VAN ZWET (1977) gives the following bound: for any positive integer r there exists a constant $\gamma=\gamma(r)>0$ such that $|\tilde{\rho}_n(t)|=0(|t|^{-r}+e^{-\gamma n})$, as $|t|\to\infty$, uniformly in n. Though van Zwet's bound is very useful when establishing Edgeworth expansions for F_n^*, it obviously fails to imply (2.3), because of the presence of the term $e^{-\gamma n}$ in his bound. In a way all we do in this paper is to modify van Zwet's proof in a suitable manner to get rid of the small term $e^{-\gamma n}$ in his bound and obtain the desired result (2.5).

We shall prove (2.5) with the aid of a few suitable conditioning arguments. To begin with take $m_1=[n\epsilon_1]$ and $m_2=[n\epsilon_2]$, for some $0<\epsilon_1<1-\epsilon_2<1$, and let $V=(V_{1:n-m},..., V_{n-m:n-m})$ denote a vector of order statistics corresponding to a sample of size $n-m$, with $m=m_1+m_2$, from the uniform $(0,1)$ distribution and let V and $(U_{m_1:n}, U_{n-m_2:n})$ be independent. Clearly by conditioning on $(U_{m_1:n}, U_{n-m_2:n})$ and exploiting the conditional independence present, we obtain

$$|\tilde{\rho}_n(t)|\leqslant E_u|E(\exp(it\tilde{T}_n)|U_{m_1:n}, U_{n-m_2:n})| \tag{2.6}$$

$$\leq E_u|E(\exp(itL_n)|U_{m_1:n},\ U_{n-m_2:n})|$$

where

$$L_n = n^{-\frac{1}{2}} \sum_{i=m_1+1}^{n-m_2} J(\frac{i}{n+1})F^{-1}(V_{i-m_1:n-m}(U_{n-m_2:n}-U_{m_1:n}) + U_{m_1:n}) \tag{2.7}$$

Note that the expressions within absolute value in (2.6) denote the conditional ch.f. of T_n (and of L_n), conditionally given $(U_{m_1:n}, U_{n-m_2:n})$. The operator E_u of course refers to the expected value w.r.t. $(U_{m_1:n}, U_{n-m_2:n})$; the r.v. L_n is a trimmed linear combination of a (random) function of uniform order statistics, based on a sample of size $n-m$ from the uniform $(0,1)$ distribution.

Following VAN ZWET (1977), p. 433, in spirit we now employ a second conditioning argument to reduce the conditional ch.f. of L_n, conditionally given $(U_{m_1:n}, U_{n-m_2:n})$, to the ch.f. of a suitable (random) function (depending on $(U_{m_1:n}, U_{n-m_2:n})$) of a single central uniform order statistic. To do this we take $s=[\frac{n-m}{2}]$, let $W^{(1)}=(W^{(1)}_{1:s-1} \ldots, W^{(1)}_{s-1:s-1})$ and $W^{(1)}= (W^{(2)}_{1:n-m-s},\ldots, W^{(2)}_{n-m-s:n-m-s})$ be vectors of order statistics corresponding to samples, of sizes $s-1$ and $n-m-s$ respectively, from the uniform $(0,1)$ distribution, and let $W^{(1)}$, $W^{(2)}$, and $(U_{m_1:n}, U_{n-m_2:n})$ be independent.

Then it is easily checked that the joint distribution of

$$(V_{1:n-m},\ldots, V_{n-m:n-m}) \tag{2.8}$$

is the same as that of

$$\begin{aligned}(V_{s:n-m}\cdot W^{(1)}_{1:s-1},\ldots, V_{s:n-m}W^{(1)}_{s-1:s-1}, V_{s:n-m}, V_{s:n-m}+(1-V_{s:n-m})W^{(2)}_{1:n-m-s},\ldots,\\ V_{s:n-m}+(1-V_{s:n-m})W^{(2)}_{n-m-s:n-m-s})\end{aligned} \tag{2.9}$$

It follows that

$$\begin{aligned}&E(\exp(itL_n)|W^{(1)}, W^{(2)}, U_{m_1:n}, U_{n-m_2:n}) =\\ &= E\ \exp(itn^{-\frac{1}{2}}\tilde{h}(V_{s:n-m}))\end{aligned} \tag{2.10}$$

where the (random) function $\tilde{h}(\cdot)=\tilde{h}(\cdot;W^{(1)}, W^{(2)}, U_{m_1:n},U_{n-m_2:n})$ is given by

$$\begin{aligned}\tilde{h}(v) &= n^{-1}\sum_{i=1}^{s-1}J(\frac{j+m_1}{n+1})F^{-1}(vW^{(1)}_{j:s-1}(U_{n-m_2:n}-U_{m_1:n})+U_{m_1:n})+\\ &+n^{-1}J(\frac{s+m_1}{h+1})F^{-1}(v(U_{n-m_2:n}-U_{m_1:n})+\\ &+U_{m,:n})+n^{-1}\sum_{j=1}^{n-m-s}J(\frac{j+m_1+s}{n+1})\ F^{-1}((W^{(2)}_{j:n-m-s}\\ &+(1-W^{(2)}_{j:n-m-s})v)\ (U_{n-m_2:n}-U_{m_1:n}) + U_{m_1:n})).\end{aligned} \tag{2.11}$$

At this point we need a bound for the ch.f. of $n^{-\frac{1}{2}}\tilde{h}(V_{s:n-m})$ (i.e. (2.10)), for fixed values of $W^{(1)}$, $W^{(2)}$, $U_{m_1:n}$ and $U_{n-m_2:n}$. Application of the Lemma in the Appendix (with h and $U_{j:n}$ replaced by $\tilde{h}$ and $V_{s:n-m}$) implies that, for some $\delta>0$,

$$\begin{aligned}&|E\exp(itn^{-\frac{1}{2}}\tilde{h}(V_{s:n-m}))|W^{(1)}, W^{(2)}, U_{m_1:n}, U_{n-m_2:n})|\\ &= \mathcal{O}(\tilde{c}^{-1}|t|^{-1} + \tilde{c}^{-2-\delta}\tilde{C}^2|t|^{-\delta}n^{-\delta/2})\end{aligned} \tag{2.12}$$

as $|t|\to\infty$, uniformly in n, $W^{(1)}$, $W^{(2)}$, $U_{m_1:n}$ and $U_{n-m_2:n}$.

The quantities $\tilde{c}$ and $\tilde{C}$ appearing in (2.12) can be choosen as follows : $\tilde{c} = \inf_{0<v<1} \tilde{h}'(v)$ and

$$\tilde{C} = \sup_{0<u,v<1} \frac{|\tilde{h}'(u)-\tilde{h}'(v)|}{|u-v|^{\delta}}$$

(cf. the assumptions (A.1) and (A.2) in the appendix). Clearly the definition of $\tilde{h}$ (cf. (2.11)) directly implies that $\tilde{c}$ and $\tilde{C}$ depend on $W^{(1)}$, $W^{(2)}$, $U_{m_1:n}$ and $U_{n-m_2:n}$. With $m=m_1+m_2$ and $s=[\frac{n-m}{2}]$ we have more explicitly

$$\begin{aligned}\tilde{c} &= \tilde{c}(W^{(1)}, W^{(2)}, U_{m_1:n}, U_{n-m_2:n}) = \\ &= \operatorname*{infimum}_{I} F^{-1^{(1)}}(s)\cdot(U_{n-m_2:n}-U_{m_1:n})\cdot \\ &\cdot n^{-1}\left[\sum_{j=1}^{s-1} W^{(1)}_{j:s-1} + \sum_{j=1}^{n-m-s}(1-W^{(2)}_{j:n-m-s})\right]\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\tilde{C} &= \tilde{C}(W^{(1)}, W^{(2)}, U_{m_1:n}, U_{n-m_2:n}) = \\ &= \operatorname*{supremum}_{I} F^{-1^{(1)}}(s)\cdot(U_{n-m_2:n}-U_{m_1:n})\cdot \\ &\cdot n^{-1}\left[\sum_{j=1}^{s-1}(W^{(1)}_{j:s-1})^{\delta} + 1 + \sum_{j=1}^{n-m-s}(1-W^{(2)}_{j:n-m-s})^{\delta}\right]\end{aligned} \tag{2.14}$$

where I denotes the random interval of s-values between $U_{m_1:n}$ and $U_{n-m_2:n}$. To validate (2.12) we must choose $m_1=[n\epsilon_1]$ and $m_2=[n\epsilon_2]$, for suitable values of ϵ_1 and ϵ_2, such that J is bounded away from zero on the interval $(\frac{\epsilon_1}{2}, 1-\frac{\epsilon_2}{2})$. The regularity assumption imposed on J guarantees that this can be done.

Combining (2.10) with (2.12), (2.13) and (2.14) we see that after taking the expected value with respect to $(W^{(1)}, W^{(2)})$ in (2.10), we easily arrive at

$$\begin{aligned}&|E(\exp(itL_n)|U_{m_1:n}, U_{n-m_2:n})| = \\ &= \mathcal{O}(\tilde{d}^{-1}|t|^{-1} + \tilde{d}^{-2-\delta}\tilde{D}^{2}|t|^{-\delta}n^{-\delta/2})\end{aligned} \tag{2.15}$$

as $|t|\to\infty$, uniformly in n and $U_{m_1:n}$ and $U_{n-m_2:n}$. The quantities $\tilde{d}=\tilde{d}(U_{m_1:n}, U_{n-m_2:n})$ and $\tilde{D}=\tilde{D}(U_{m_1:n}, U_{n-m_2:n})$ are given by

$$\tilde{d} = \operatorname*{infimum}_{I} F^{-1^{(1)}}(s)\cdot(U_{n-m_2:n}-U_{m_1:n})$$

and

$$\tilde{D} = \operatorname*{supremum}_{I} F^{-1^{(1)}}(s)\cdot(U_{n-m_2:n}-U_{m_1:n})$$

It now remains to show that $E\tilde{d}^{-p}=\mathcal{O}(1)$, as $n\to\infty$, for $p=1$ and $p=2+\delta$, and also that $E\tilde{D}^{2}=\mathcal{O}(1)$, as $n\to\infty$. To establish these order bounds we employ the smoothness assumption on F and the easily established fact that, $\tilde{d}^{-p} \leqslant M^{p}(U_{n-m_2:n}-U_{m_1:n})^{-p}$, where $M=\sup_x f(x)$. Obviously this yields that $E\tilde{d}^{-p}=\mathcal{O}(1)$, as $n\to\infty$, for any $p>0$. To verify that $E\tilde{D}^{2}=\mathcal{O}(1)$, as $n\to\infty$, is a slightly more involved matter. We will require the full force of the smoothness condition on F. Define $Z=\arg\operatorname{infimum} f(F^{-1}(s))$. Then, clearly,

$$E\tilde{D}^{2} \leqslant E[f(F^{-1}(Z))]^{-2}$$

with $U_{m_1:n} \leq Z \leq U_{n-m_2:n}$. To proceed we write

$$E[f(F^{-1}(Z))]^{-2} = \tag{2.16}$$
$$= E[f(F^{-1}(Z))]^{-2} I(Z) +$$
$$+ E[f(F^{-1}(Z))]^{-2} I^c(Z)$$

where $I(\cdot)$ denotes the indicator function of the set $[\epsilon_1, 1-\epsilon_2]$; $I^c = 1 - I$. Because $f(F^{-1})$ is bounded away from zero on $[\epsilon_1, 1-\epsilon_2]$ one easily infers that $E[f(F^{-1}(Z))]^{-2} I(Z)$ is bounded. To treat $E[f(F^{-1}(Z))]^{-2} I^c(Z)$ we employ the requirement that $f(F^{-1})$ is Lipschitz of order $\delta > 0$ on $(0,1)$. On the set where $I^c(Z) = 1$ we have $U_{m_1:n} \leq Z \leq \epsilon$, or $1 - \epsilon_2 \leq Z \leq U_{n-m_2:n}$. In either case we now use the positivity of $f(F^{-1})$ in ϵ_1 (or in $1-\epsilon_2$) and the Lipschitz condition to find that

$$E[f(F^{-1}(Z))]^{-2} I^c(Z) \leq \tag{2.17}$$
$$\leq E[f(F^{-1}(\epsilon_1)) + \mathcal{O}(|U_{m_1:n} - \epsilon_1|^{\delta})]^{-2} +$$
$$+ E[f(F^{-1}(1-\epsilon_2)) + \mathcal{O}(|U_{n-m_2:n} - (1-\epsilon_2)|^{\delta})]^{-2} =$$
$$= \mathcal{O}(1), \quad \text{as } n \to \infty.$$

The last line involves a Taylor expansion argument and a well-known property of the uniform order statistics $U_{m_1:n}$ and $U_{n-m_2:n}$. This completes the proof of (2.5) with r replaced by δ. To establish the order bound (2.5) in its full strength (i.e. for any positive integer r) we repeat the above argument, and similarly as in VAN ZWET (1977), we condition on $K = [\frac{r}{\delta+1}] + 1$ uniform order statistics $(U_{m_1:n}, U_{m_2:n}, \ldots, U_{m_{K-1}:n}, U_{m_K:n})$ instead of on only two of them. Take m_1 as before, let m_K play the role of m_2, and choose the other m_i 's ($i = 2, \ldots, K$) such that $m_i - m_{i-1} \sim n(1-\epsilon_1-\epsilon_2)K^{-1}$. Clearly, by conditioning on $U = (U_{m_1:n}, \ldots, U_{m_K:n})$ and exploiting the conditional independence present, we obtain (cf. (2.6) and (2.7)).

$$|\tilde{\rho}_n(t)| \leq E_u |E(\exp(it\tilde{T}_n)|U)|$$
$$\leq E_u \prod_{\nu=1}^{K-1} |E(\exp(it\, L_{n\nu})| \tag{2.18}$$

where, for $\nu = 1, 2, \ldots, K-1$,

$$L_{n\nu} = n^{-\frac{1}{2}} \sum_{i=m_\nu+1}^{m_{\nu+1}-1} J(\frac{i}{n+1}) F^{-1}(V_{i-m_\nu:m_{\nu+1}-m_\nu-1}(U_{m_{\nu+1}:n} - U_{m_\nu:n}) + U_{m_\nu:n}) \tag{2.19}$$

Here E_u refers to the expected value w.r.t. U. We can now repeat the argument leading to (2.15) separately for each of the factors appearing in (2.18). Combining all these results and taking the expected value E_u we arrive, after some computations closely resembling (2.16) and (2.17), at the desired result. This completes the proof of (2.5) and the Theorem is proved. □

3. Asymptotic expansions

In this section we indicate very briefly how an valid asymptotic expansion for the density f_n^* of a normalized L-statistic T_n^* can be derived. We combine the order bound (2.5) obtained in the previous section with some results from HELMERS (1980), (1982). Recall that $\rho_n^*(t) = E\{\exp(it\, T_n^*)\}$ and define $\tilde{\tau}_n(t) = \int_{-\infty}^{\infty} \exp\{itx\} \cdot \tilde{f}_n(x),\, dx$, the Fourier-Stieltjes transform of $\tilde{f}_n$, with

$$\tilde{f}_n(x) = \phi(x) \left\{ \frac{\kappa_3}{6n^{1/2}}(x^3 - 3x) + \frac{\kappa_s}{24n}(x^4 - 6x^2 + 3) + \right.$$
$$\left. + \frac{\kappa_3^2}{72n}(x^6 - 15x^4 + 45x^2 - 15) \right\} \tag{3.1}$$

where $\kappa_3=\kappa_3(J,F)$ and $\kappa_4=\kappa_4(J,F)$ are explicitly given in the formula's (2.5) and (2.6) of HELMERS (1980).

Our aim is to show that

$$\sup_x |f_n^*(x)-\tilde{f}_n(x)| = o(n^{-1}) \tag{3.2}$$

as $n\to\infty$, provided the assumptions of Theorem 1.1 of the present paper and those of Theorem 2.1 of HELMERS (1980) are satisfied. The usual argument based on the Fourier inversion theorem implies that we must show that for some sufficiently small $\epsilon>0$

$$\int_{|t|\leq n^\epsilon} |\rho_n^*(t)-\tilde{\tau}_n(t)|dt = o(n^{-1}) \tag{3.3}$$

$$\int_{|t|\geq n^\epsilon} |\rho_n^*(t)|dt = o(n^{-1}) \tag{3.4}$$

and

$$\int_{|t|\geq n^\epsilon} |\tilde{\tau}_n(t)|dt = o(n^{-1}) \tag{3.5}$$

holds, as $n\to\infty$. Together (3.3) - (3.5) will yield the desired result (3.2). Statement (3.4) follows directly from the order bound (2.5) (take, e.g., $r=[\epsilon^{-1}+1]+1$), whereas statement (3.5) is an easy consequence of the explicit formula

$$\tilde{\tau}_n(t) = \exp(\frac{-t^2}{2})\,(1-\frac{it^3\kappa_3}{6n^{1/2}} + \frac{3\kappa_4 t^4-\kappa_3^2 t^6}{72n}) \tag{3.6}$$

It remains to establish (3.3). This statement may be proved by slightly modifying the proof of relation (4.1) of HELMERS (1980). It is easily checked from the proof of Theorem 2.1 of that paper that for any $0<\epsilon<\frac{1}{8}$ assertion (3.3) holds true, under the assumptions of this theorem. Note that this also implies that the choice $r=11$ in (2.5) will be sufficient to give us (3.4). This completes the proof of (3.2) under the assumptions of Theorem 1.1 of the present paper together with those of Theorem 2.1 of HELMERS (1980)

More specifically we obtain the following result.

THEOREM 3.1. *Suppose that the assumptions of Theorem 1.1 are satisfied. In addition suppose that F possesses a finite fourth moment and that J is three times differentiable on* (0,1) *with bounded derivatives. Then* (3.2) *holds.*

APPENDIX

In this appendix we establish a bound for the ch.f of $n^{\frac{1}{2}}h(U_{j:n})$, where $h:(0,1)\to\mathbb{R}$ is sufficiently smooth and $U_{j:n}$ denotes the j^{th} uniform order statistic based on a random sample of size n from the uniform distribution on (0,1).

LEMMA. *Let $h:(0,1)\to\mathbb{R}$ be differentiable with derivative h' and suppose that positive numbers ϵ,c,C,η and δ exists such that $\epsilon n\leq j\leq(1-\epsilon)n$,*

$$h' \geq c \tag{A.1}$$

$$|h'(u)-h'(v)|<C|u-v|^\delta\ ,\ 0<u,v<1 \tag{A.2}$$

Then there exists a positive number M (depending only on η and ϵ) such that for all $|t|\geq\eta$ and all $n\geq 1$

$$|E\exp\{itn^{\frac{1}{2}}h(U_{j:n})\}|\leq \tag{A.3}$$

$$\leqslant M[c^{-1}|t|^{-1}+c^{-2-\delta}C^2 n^{-\frac{\delta}{2}}|t|^{-\delta}].$$

This lemma closely resembles Lemma 4.1 of VAN ZWET (1977). In the special case $\delta=1$ our bound (A.3) is essentially the same as the bound obtained by VAN ZWET (1977) in the proof of his Lemma 4.1. In addition to assumption (A.1) van Zwet requires h to be twice differentiable on $(0,1)$ with a bounded second derivative h''. Assumption (A.2) is somewhat weaker and the resulting bound (A.3) is still sufficient for our purposes. Our method of proof is completely different from the one given in VAN ZWET (1977).

PROOF OF THE LEMMA.
Because of (A.1) the inverse $h^{-1}:\mathbb{R}\to(0,1)$ of h is clearly well-defined. It follows that we may write

$$E\exp\{itn^{-\frac{1}{2}}h(U_{j:n})\} = \tag{A.4}$$

$$= \int_0^1 \exp\{itn^{-\frac{1}{2}}h(u)\}b_{jn}(u)du =$$

$$= \int_{-\infty}^{\infty} \exp\{itx\}g_{jn}(x)dx$$

where b_{jn} denotes the density of $U_{j:n}$ and

$$g_{jn}(x) = n^{-\frac{1}{2}}b_{jn}(h^{-1}(xn^{-\frac{1}{2}}))[h'(h^{-1}(xn^{-\frac{1}{2}}))]^{-1}\cdot I(h(0)\leqslant xn^{-\frac{1}{2}}\leqslant h(1)) \tag{A.5}$$

for $-\infty<x<\infty$. Note that h is bounded on (0,1); $I(A)$ denotes the indicator function of a set A. Since $e^{-i\pi}=-1$ it is easily checked that

$$\int_{-\infty}^{\infty} \exp\{itx\}g_{jn}(x)dx = \tag{A.6}$$

$$= -\int_{-\infty}^{\infty} \exp\{it(x-\frac{\pi}{t})\}g_{jn}(x)dx =$$

$$= -\int_{-\infty}^{\infty} \exp\{itx\}g_{jn}(x+\frac{\pi}{t})dx$$

Similarly as in HEWITT and STROMBERG (1969), p. 401/402 we combine (A.4) and (A.6) to find that

$$2E\exp\{itn^{-\frac{1}{2}}h(U_{j:n})\} = \tag{A.7}$$

$$= \int_{-\infty}^{\infty} \exp\{itx\{(g_{jn}(x)-g_{jn}(x+\frac{\pi}{t})\}\}\}dx$$

In view of (A.5) we see that

$$g_{jn}(x)-g_{jn}(x+\frac{\pi}{t}) = \tag{A.8}$$

$$= a_n(x,t)+b_n(x,t)$$

where

$$a_n(x,t) = n^{-\frac{1}{2}}(b_{jn}(h^{-1}(xn^{-\frac{1}{2}}))- \tag{A.9}$$

$$-b_{jn}(h^{-1}((x+\frac{\pi}{t})n^{-\frac{1}{2}})))[h'(h^{-1}((x+\frac{\pi}{t})n^{-\frac{1}{2}}))]^{-1}$$

and

$$b_n(x,t) = n^{-\frac{1}{2}} b_{jn}(h^{-1}(xn^{-\frac{1}{2}})). \tag{A.10}$$
$$\cdot[(h'(h^{-1}(xn^{-\frac{1}{2}})))^{-1} - (h'(h^{-1}((x+\frac{\pi}{t})n^{-\frac{1}{2}})))^{-1}].$$

Note first that, because of assumption (A.1),

$$\int_{-\infty}^{\infty} |a_n(x,t)|dx \leqslant \tag{A.11}$$
$$\leqslant c^{-1} \int_{h(0)}^{h(1)} |b_{jn}(h^{-1}(z)) - b_{jn}(h^{-1}(z+\pi t^{-1}n^{-\frac{1}{2}}))|dz$$

Again using (A.1) we see that the integral on the r.h.s. of (A.11) can be bounded by

$$c^{-1}\pi|t|^{-1}n^{-\frac{1}{2}} \int_{h(0)}^{h(1)} |b'_{jn}(h^{-1}(z_n))|dz \tag{A.12}$$

where z_n denotes a point between z and $z+\pi t^{-1}n^{-\frac{1}{2}}$. Since $h^{-1}(z_n)=h^{-1}(z)+\underset{\sim}{\mathcal{O}}(|t|^{-1}n^{-\frac{1}{2}}c^{-1})$, uniformly in z, because of assumption (A.1) and an application of the mean value theorem, we find that the integral in (A.12) reduces to

$$\int_0^1 |b'_{jn}(u+\mathcal{O}(|t|^{-1}n^{-\frac{1}{2}}c^{-1}))|h'(u)du \tag{A.13}$$

where the order bound appearing in the argument of b_{jn}' is uniform in u. It is now easily checked hat b''_{jn} has two sign changes on (0,1). It follows that we may split (0,1) into five subintervals, such that on three of these intervals b_{jn}' is monotone, while the remaining two small subintervals contain the points where b''_{jn} change sign and have length $2\eta^{-1}c^{-1}n^{-\frac{1}{2}}$. Using this, the facts that $\int_0^1 |b_{jn}'(u)|du = \mathcal{O}(n^{\frac{1}{2}})$ (cf, e.g., VAN ZWET (1977), p. 432) and $\max_{0<u<1} |b_{jn}'(u)| = \mathcal{O}(n)$, and because $|h'(u) \leqslant C$ for all $u \in (0,1)$ (see (A.16) and the argument following it) we easily arrive at.

$$\int_{-\infty}^{\infty} |a_n(x,t)|dx = \mathcal{O}(\eta^{-1}c^{-1}|t|^{-1}) \tag{A.13}$$

uniformly in n, as $|t| \to \infty$.

Next we deal with $b_n(x,t)$. Because of (A.10) and the assumptions (A.1) and (A.2) we find that

$$|b_n(x,t)| \leqslant c^{-2}C \cdot n^{-\frac{1}{2}}, b_{jn}(h^{-1}(xn^{-\frac{1}{2}})). \tag{A.14}$$
$$\cdot|h^{-1}(xn^{-\frac{1}{2}}) - h^{-1}((x+\frac{\pi}{t})n^{-\frac{1}{2}})|^{\delta} =$$
$$= \mathcal{O}(n^{-\frac{1}{2}-\frac{\delta}{2}}|t|^{-\delta}c^{-2-\delta}C \cdot b_{jn}(h^{-1}(xn^{-\frac{1}{2}})))$$

where we have used the mean value theorem once more. To proceed we note first that

$$\int_{-\infty}^{\infty} b_{jn}(h^{-1}(xn^{-\frac{1}{2}}))dx = \tag{A.15}$$
$$= n^{\frac{1}{2}} \int_{-\infty}^{\infty} b_{jn}(h^{-1}(y))dy =$$

$$= n^{-\frac{1}{2}} \int_0^1 b_{jn}(u) dh(u) =$$

$$= n^{-\frac{1}{2}} \int_0^1 h'(u) b_{jn}(u) du.$$

Let $u_0 \epsilon (0,1)$ such that $h'(u_0) = \inf_{0<u<1} h'(u) = c_0 > 0$. Assumption (A.2) tells us that

$$|h'(u)| \leq |h'(u_0)| + C|u - u_0|^\delta \tag{A.16}$$

for all $0<u<1$. It follows that $|h'(u)| \leq c_0 + C$ for all $0<u<1$ and we obtain

$$\int_0^1 h'(u) b_{jn}(u) du \leq \sup_{0<u<1} |h'(u)| \cdot \int_0^1 b_{jn}(u) du \tag{A.17}$$

$$\leq c_0 + C.$$

In combination with (A.14) and (A.15) this yields

$$\int_{-\infty}^{\infty} |b_n(x,t)| dt = \tag{A.18}$$

$$= \Theta(n^{-\frac{\delta}{2}} |t|^{-\delta} c^{-2-\delta} C(c_0 + C))$$

uniformly in n, as $|t| \to \infty$. Since we may without loss of generality assume that $C \geq c_0$ the order bound in (A.18) can be replaced by $\Theta(n^{-\frac{\delta}{2}} |t|^{-\delta} c^{-2-\delta} C^2)$. This, together with (A.7) and (A.13) completes the proof. □

REFERENCES

[1] W. FELLER (1971). *An Introduction to Probability Theory and Its Applications.* **2**, 2nd.ed. Wiley, New York.

[2] K.O. FRIEDRICH (1989). A Berry-Esseen bound for functions of independent random variables, *Ann. Statist.* **17**, 170-183.

[3] R. HELMERS (1980). Edgeworth expansions for linear combinations of order statistics with smooth weight functions. Ann. Statist. **8**, 1361-1374.

[4] R. HELMERS (1982). *Edgeworth expansions for linear combinations of order statistics,* Mathematical Centre, Tract 105, Amsterdam.

[5] E. HEWITT and K. STROMBERG (1969). *Real and Abstract Analysis.* Springer Verlag.

[6] W. MACHT and W. WOLF (1989). The local limit problem and Hölder-Continuity. Probab.Th.Rel. Fields **82**, 295-305.

[7] D.M. MASON (1981). Asymptotic normality of linear combination of order statistics with a smooth score function. *Ann. Statist.* **9**, 899-904.

[8] D.M. MASON and G.R. SHORACK (1990). Necessary and sufficient conditions for the asymptotic normality of trimmed *L*-statistics, J. Statist. Planning and Inference, **25**, 111-139.

[9] R.-D. REISS (1989). *Approximate Distributions of Order Statistics (with Applications to Non-parametric Statistics),* Springer Series in Statistics.

[10] S.M. STIGLER (1974). Linear functions of order statistics with smooth weight functions. *Ann. Statist.* **2**, 676-693.

[11] W.R. VAN ZWET (1977). Asymptotic expansions for the distribution functions of linear combinations of order statistics. *Proc. Symp. on Statistical Decision Theory and Related Topics* II, 421-437 (ed. D.S. Moore & S.-S. Gupta).

[12] W.R. VAN ZWET (1984). A Berry-Esseen bound for symmetric statistics. *Z. Wahrsch. Verw. Gebiete* **66**, 425-440.