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The Two-Dimensional Random Walk, its Hitting Process and its Classification

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Summary

The classification of the states of a two-dimensional random walk on the lattice $\mathbb{N} \times \mathbb{N}$ in $\mathbb{R}_2$ has been investigated by several authors mainly under some conditions concerning the supports of the components of the one-step displacement vectors and the existence of their first moments. In the present study an analysis is developed by which a detailed classification can be obtained. The technique is illustrated for the two-dimensional analogon of the imbedded queue length process of the M/G/1 queue. The hitting point process with the coordinate axes plays an essential role in the analysis. Next to this process the hitting point process of a similar type of random walk on the lattice in the upper half plane is studied. With this information the classification is readily obtained for the case with non zero-drift in the interior of the state space. The developed technique can be also applied by using results from Fluctuation Theory to random walks for which the first moments of the various components of the one-step displacement vectors do not exist. The last section discusses the zero drift case, a complete classification is here obtained, and for the positive recurrent case several state probabilities are evaluated.

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1. Introduction

The present study is mainly concerned with the classification of the states, in terms of nonrecurrent, null and positive recurrent, of a two-dimensional random walk $\{\mathbf{z}_n, n=0,...\}$ with state space S the set of lattice points with nonnegative integer valued coordinates in $\mathbb{R}_2$. The random walk is semi-homogeneous, i.e. the one-step displacement vectors from interior points of S are identically distributed, also those out from points on the positive x-axis, and similarly for points on the positive y-axis, see (2.2).

The problem of the classification of the states has been discussed by several authors of which we mention Malyshev [11], Fayolle [12], Nauta [13] and Vaninski and Lazareva [14]. In all these studies it is assumed that all components of the various one-step displacement vectors possess finite first moments.

Malyshev's study is restricted to the case with all these components having a finite support; his analysis is based on the use of martingales. Fayolle assumes the existence of finite second moments and his investigation starts from the construction of an appropriate Lyapounov function. Vaninski and Lazareva's approach is quite similar to that of Malyshev but they can do without the assumption of finite supports. In all these studies only classification in terms of nonrecurrent and recurrent is discussed. Nauta gives a more detailed classification, actually only for a special class of two-dimensional random walks, although it is a very general class. In his analysis he starts from the functional equation of the random walk (of a type as in (2.8) below), and by transforming it into a boundary value problem he derives his results.

In the present study the problem is approached by a rather detailed analysis of the hitting point process $\{\mathbf{k}_m, m=1,2,...\}$ of the $\mathbf{z}_n$-process with the boundary B of S, i.e. the set of lattice points on the coordinate axes.

In section 2 the $\mathbf{z}_n$-process is defined. It is assumed that it is aperiodic, that its state space is

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irreducible and that the supports of the components of the various one-step displacement vectors are all contained in $\{-1,0,...\}$, see (2.2) and (2.3); for a weakening of this assumption see the end of this section and the remarks 5.3 and 5.4 of section 5. Section 2 also contains a derivation of the functional equation for the $\mathbf{z}_n$-process, see (2.8), a discussion of zero tuples of the kernel in relation to entrance times and the formulation of the so-called hitting point identity; for details about the derivation of this identity, see [8].

In section 3 the hitting point process $\{\mathbf{k}_m, m=1,2,...\}$ is defined and its stochastic structure is exposed; this process is a nonhomogeneous Markov chain, see (3.9). Theorem 3.1 formulates the functional equation for the hitting point process, it is of a similar type as the hitting point identity.

The hitting point process is still a rather complicated process due to the fact that hitting points may occur on both coordinate axes. Therefore, we first study in section 4 the semi-bounded random walk $\{\mathbf{w}_n, n=0,1,2,...\}$ on the set of lattice points in the upper half plane of $\mathbb{R}_2$. The hitting point process $\{\mathbf{h}_m, m=1,2,...\}$ of the $\mathbf{w}_n$-process with the x-axis is actually a one dimensional, homogeneous random walk; it is easily characterized in particular by using results from Fluctuation Theory. With the observation that the sample functions of the $\mathbf{k}_m$-process contain cycles of which the asymptotic properties can be derived from the $\mathbf{h}_m$-process, the classification of the states of the $\mathbf{k}_m$-process and consequently also of those of the $\mathbf{z}_n$- process can be obtained, except for the case in which the $\mathbf{z}_n$-process has zero drift in the interior $S \setminus B$ of its state space. With the exclusion of this case the conditions for the classification of the states of the $\mathbf{k}_m$- and of the $\mathbf{z}_n$-process are formulated in theorem 5.1. These conditions here been obtained before, cf. [11], [12]. The presented analysis is, however, quite different and new, it leads to a good insight in the probabilistic meaning of the various conditions and provides starting points for a further analysis of these processes.

In section 6 the case with zero drift is analyzed, it requires another approach. The analysis is based on a theorem concerning the first moment of the entrance time out from an interior point of S into B (see [17]; and theorem 2.3 below), this moment may be finite. With the aid of the hitting point identity a complete classification of the $\mathbf{k}_m$- and the $\mathbf{z}_n$-process is obtained, see theorems 6.1 and 6.2. It is shown that the conditions for the $\mathbf{k}_m$-process do not require the finiteness of the second moments of the various one-step displacement vectors (see corollary 6.1). For the positive recurrent $\mathbf{z}_n$-process with zero drift detailed information is obtained concerning several of its state probabilities and its first moments. The results are mainly all new*.

The $\mathbf{z}_n$-process studied is in fact the two-dimensional analogon of the imbedded queue length process of the M/G/1 queueing model. This is one of the main reasons for assuming that the supports of the various variables are restricted to be contained in $\{-1,0,1,2,...\}$, since the two-dimensional analogon of the M/G/1 system is of importance in Queueing Theory; moreover under this assumption the analysis is fairly simple. In remark 5.3 it is pointed out that the method used in this study can be easily generalised when no restrictions are made on the supports of the various variables supposing that their first moments do exist. In remark 5.4 it is pointed out that our analysis can also rather easily be extended if the assumption of the existence of the first moments is not introduced; such an extension then needs fully the terminology and results of Fluctuation Theory.

2. Definitions, assumptions and notation

In this section we describe the two-dimensional stochastic process

$$\mathbf{z}_n, \quad n=0,1,2,...,$$

with

$$\mathbf{z}_n = (\mathbf{x}_n, \mathbf{y}_n) \in S := \{0,1,2,...\} \times \{0,1,2,...\}, \tag{2.1}$$

so that S, the set of lattice points with integer valued coordinates in the first quadrant of $\mathbb{R}_2$, is the

* Recently Malyshev and Fayolle seem to have obtained results for the zero-drift case, they are, however, not yet known in detail to the present author when preparing this preprint.

state space of the $\mathbf{z}_n$-process.

To define the structure of the $\mathbf{z}_n$-process we introduce several sequences of stochastic variables, viz.:

i. for every fixed $k = 0,1,2,3,$ (2.2)

$(\xi_n^{(k)},\eta_n^{(k)})$, $n=0,1,...,$ is a sequence of i.i.d. stochastic vectors with

$$(\xi_n^{(k)},\eta_n^{(k)}) \in S;$$

ii. the four families $\{(\xi_n^{(k)},\eta_n^{(k)}),\ n=0,1,...\}$, $k=0,1,2,3$, are independent families.

The structure of the $\mathbf{z}_n$-process is defined by the following recursive relations.
For $n=0,1,2,...,$

i. $z_0 \equiv (x_0,y_0) \in S$ is the starting point of the $\mathbf{z}_n$-process, i.e. (2.3)

$$\mathbf{z}_0 = z_0 = (x_0,y_0)$$

ii.
$$\mathbf{x}_{n+1} = [\mathbf{x}_n - 1]^+ + \xi_n^{(k)},$$
$$\mathbf{y}_{n+1} = [\mathbf{y}_n - 1]^+ + \eta_n^{(k)},$$

here

$$\begin{aligned} k &= 3 \text{ for } \mathbf{x}_n>0,\ \mathbf{y}_n>0,\\ &= 2 \text{ for } \quad =0, \quad >0,\\ &= 1 \text{ for } \quad >0, \quad =0,\\ &= 0 \text{ for } \quad =0, \quad =0; \end{aligned}$$

and

$$[x]^+ := \max(0,x), \quad x \text{ real}.$$

REMARK 2.1 From (2.2) and (2.3) it is readily seen that the $\mathbf{z}_n$-process is a discrete time parameter Markov chain with state space S. □

NOTATION 2.1. For $k=0,1,2,3$:
i. (ξ_k,η_k) shall stand for a stochastic vector with the same state space and distribution as $(\xi_n^{(k)},\eta_n^{(k)})$, i.e. (2.4)

$$(\xi_k,\eta_k) \sim (\xi_n^{(k)},\eta_n^{(k)}),$$

ii.
$$\phi_k \equiv \phi_k(p_1,p_2) := \mathrm{E}\{p_1^{\xi_k} p_2^{\eta_k}\}, \quad |p_1| \leq 1,\ |p_2| \leq 1,$$
$$\mu_k := \mathrm{E}\{\xi_k\}, \quad \nu_k := \mathrm{E}\{\eta_k\}.$$

Concerning the bivariate generating functions ϕ_k we introduce the

ASSUMPTION 2.1.
i. For $k = 0,1,2,3,$

$$|\phi_k(p_1,p_2)| = 1 \text{ for } |p_1| = 1, |p_2| = 1 \Rightarrow p_1 = 1, p_2 = 1; \tag{2.5}$$

ii. for every $(i,j) \in S$ the coefficient of $p_1^i p_2^j$ in the series expansion of $[\phi_k(p_1,p_2)/p_1 p_2]^n$ with $|p_1| = 1, |p_2| = 1$, n being a positive integer, is positive for n sufficiently large.

iii. $\mu_k < \infty, \quad \nu_k < \infty, \quad k=0,1,2.$

□

REMARK 2.2. The assumption 2.1.i implies that the $\mathbf{z}_n$-process is aperiodic, cf. [16], and 2.1.ii guarantees that its state space S is irreducible. The assumption 2.1.iii is a rather natural one; however, in sections 4 and 6 it will be shown that for some of the results to be obtained we can do without assumption (2.5)iii, note that assumption (2.5)ii implies

$$\mu_k > 0, \quad \nu_k > 0, \quad k = 0,1,2,3. \tag{2.6}$$

□

Define for $|r|<1$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$;

$$\mathrm{E}_{z_0}\{p_1^{\mathbf{x}_n} p_2^{\mathbf{y}_n}\} := \mathrm{E}\{p_1^{\mathbf{x}_n} p_2^{\mathbf{y}_n} | \mathbf{x}_0 = x_0, \mathbf{y}_0 = y_0\}, \quad n = 0,1,2,..., \tag{2.7}$$

$$\Phi_{z_0}(r,p_1,p_2) := \sum_{n=0}^{\infty} r^n \, \mathrm{E}_{z_0}\{p_1^{\mathbf{x}_n} p_2^{\mathbf{y}_n}\}.$$

REMARK 2.3. From (2.3) it is readily verified that

$$\mathrm{E}_{z_0}(p_1^{\mathbf{x}_n} p_2^{\mathbf{y}_n}), \quad n = 0,1,2,...,$$

is recursively uniquely defined and that $\Phi_{z_0}(r,p_1,p_2)$ satisfies for $|r|<1$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$:

i. $$[p_1p_2 - r\phi_3]\,\Phi_{z_0}(r,p_1,p_2) = p_1^{x_0+1} p_2^{y_0+1} + r[p_1p_2\phi_0 - \phi_3]\Phi_{z_0}(r,0,0) + \tag{2.8}$$
$$r[p_1\phi_2 - \phi_3]\{\Phi_{z_0}(r,0,p_2) - \Phi_{z_0}(r,0,0)\} + r[p_2\phi_1 - \phi_3]\{\Phi_{z_0}(r,p_1,0) - \Phi_{z_0}(r,0,0)\},$$

ii. $\Phi_{z_0}(r,p_1,p_2)$, $|r|<1$ is for every fixed p_2 with $|p_2|\leqslant 1$ regular in p_1 for $|p_1|<1$, continuous in p_1 for $|p_1|\leqslant 1$, and similarly with p_1 and p_2 interchanged. □

DEFINITION 2.1. For the $\mathbf{z}_n$-process we define:

i. $$B_{00} := \{0,0\}, \quad B_{10} := \{1,2,...\}\times\{0\}, \quad B_{01} = \{0\}\times\{1,2,...\}, \tag{2.9}$$
$$B = B_{00} \cup B_{10} \cup B_{01};$$

ii. $$\mathbf{n}(z_0) := \inf_{n=0,1,...} \{n : \mathbf{z}_n \in B | \mathbf{z}_0 = z_0 \in S\},$$
$$:= \infty \text{ if } \mathbf{z}_n \notin B \text{ for all } n = 1,2,...;$$

iii. $$\mathbf{k}(z_0) \equiv (\mathbf{k}_1(z_0), \mathbf{k}_2(z_0)) := \mathbf{z}_{\mathbf{n}(z_0)} \equiv (\mathbf{x}_{\mathbf{n}(z_0)}, \mathbf{y}_{\mathbf{n}(z_0)}) \text{ if } \mathbf{n}(z_0) < \infty,$$
$$:= (\infty, \infty) \qquad \text{if} \qquad = \infty.$$

Obviously B is the boundary of the state space S and $\mathbf{n}(z_0)$ the first entrance time from $z_0 \in S$ into B; further it is seen that $\mathbf{k}(z_0)$ is the hitting point of B. It follows from definition 2.1 that

$$\mathbf{k}_1(z_0)\mathbf{k}_2(z_0) = 0. \tag{2.10}$$

DEFINITION 2.2. The function

$$Z(r,p_1,p_2) := p_1p_2 - r\phi_3(p_1,p_2), \quad |r|\leqslant 1, \; |p_1|\leqslant 1, \; |p_2|\leqslant 1, \tag{2.11}$$

is called the *kernel* of the functional relation (2.8); a pair $(\hat{p}_1,\hat{p}_2)$ with $|\hat{p}_1|\leqslant 1$, $|\hat{p}_2|\leqslant 1$ is called a *zero tuple* of the kernel if

$$Z(r,\hat{p}_1,\hat{p}_2) = 0. \tag{2.12}$$

For a discussion concerning the existence, construction and the properties of zero tuples the reader is referred to the studies [1],...,[8]. In the present study we are mainly concerned with zero tuples of a special type to be described below.

For fixed r with $|r|\leqslant 1$, $r\neq 1$ and fixed p_2 with $|p_2|=1$, it is readily verified by applying Rouche's theorem and using (2.5) that the function

$$\frac{1}{p_2}Z(r,p_1,p_2)\equiv p_1-\frac{z}{p_2}\phi_3(p_1,p_2)\equiv p_1-r\mathrm{E}\{p_1^{\xi_3}p_2^{\eta_3-1}\},\quad |p_1|\leqslant 1, \tag{2.13}$$

has exactly one zero $p_1=P_1(r,p_2)$; consequently,

$$(\hat{p}_1,\hat{p}_2):=(P_1(r,\hat{p}_2),\hat{p}_2),\quad |\hat{p}_2|=1, \tag{2.14}$$

is a zero tuple of the kernel; analogously the zero tuple

$$(\hat{p}_1,\hat{p}_2):=(\hat{p}_1,P_2(r,\hat{p}_1)),\quad |\hat{p}_1|=1, \tag{2.15}$$

is defined.

REMARK 2.4. For the definition of $P_1(r,p_2)$ and $P_2(r,p_1)$ for $r\rightarrow 1$, $|r|<1$, see remark 2.5. □

DEFINITION 2.3. For the component random walk $\{\mathbf{x}_n, n=0,1,...\}$ on $\{0,1,2,...\}$ of the $\mathbf{z}_n$-process:

$$\mathbf{x}_{n+1}=[\mathbf{x}_n-1]^+ +\boldsymbol{\xi}_n^{(3)},\quad n=0,1,...,\qquad \mathbf{x}_0=x_0, \tag{2.16}$$

denote by $\mathbf{M}_{x_0}^{(31)}$ the first entrance time into $\{0\}$ when starting in $x_0\in\{0,1,2,...\}$; analogously $\mathbf{M}_{y_0}^{(32)}$ is defined for the random walk $\{\mathbf{y}_n,\ n=0,1,...\}$ with

$$\mathbf{y}_{n+1}=[\mathbf{y}_n-1]^+ +\boldsymbol{\eta}_n^{(3)},\quad n=0,1,...,\qquad \mathbf{y}_0=y_0.$$

Obviously*

$$\mathbf{M}_0^{(31)}=0,\ \ \mathbf{M}_{x_0}^{(31)}\in\{x_0,x_0+1,...\},\ \ x_0\in\{1,2,...\},$$
$$\mathbf{M}_{y_0}^{(32)}=0,\ \ \mathbf{M}_{y_0}^{(32)}\in\{y_0,y_0+1,...\},\ \ y_0\in\{1,2,...\}. \tag{2.17}$$

Put

$$\mathbf{m}_{31}:=\mathbf{M}_1^{(31)},\quad \mathbf{m}_{32}:=\mathbf{M}_1^{(32)}, \tag{2.18}$$

and let

$$\{\mathbf{m}_{31,n},\ n=1,2,...\}\text{ and }\{\mathbf{m}_{32,n},\ n=1,2,...\}, \tag{2.19}$$

be two independent families each consisting of i.i.d. stochastic variables with

$$\mathbf{m}_{31,n}\sim\mathbf{m}_{31},\quad \mathbf{m}_{32,n}\sim\mathbf{m}_{32}; \tag{2.20}$$

further put

$$\mathbf{X}_n^{(31)}:=\sum_{m=1}^n\boldsymbol{\xi}_m^{(3)},\quad \mathbf{Y}_n^{(32)}:=\sum_{m=1}^n\boldsymbol{\eta}_m^{(3)},\qquad n=1,2,3,...,$$
$$:=0,\qquad\qquad :=0,\qquad\qquad n=0. \tag{2.21}$$

The following theorem formulates some wellknown results concerning one-dimensional random walks of the type (2.16).

*) For typographical reasons we shall sometimes use the notation $\mathbf{M}^{(31)}(x_0)$ in stead of $\mathbf{M}_{x_0}^{(31)}$.

THEOREM 2.1.

i. For $x_0, y_0 \in \{1,2,\dots,\}$, (2.22)

$$\mathbf{M}^{(31)}_{x_0} \sim \sum_{n=1}^{x_0} \mathbf{m}_{31,n}, \quad \mathbf{M}^{(32)}_{y_0} \sim \sum_{n=1}^{y_0} \mathbf{m}_{32,n};$$

ii. for $|r| \leqslant 1$,

$$P_1(r,p_2) = \mathrm{E}\{r^{\mathbf{m}_{31}} p_2^{\mathbf{Y}^{(32)}_{\mathbf{m}_{31}} - \mathbf{m}_{31}}\} = \mathrm{E}\{r^{\mathbf{m}_{31}} p_2^{\sum_{n=1}^{\mathbf{m}_{31}} (\boldsymbol{\eta}^{(3)}_n - 1)}\}, \quad |p_2| = 1,$$

$$P_2(r,p_1) = \mathrm{E}\{r^{\mathbf{m}_{32}} p_1^{\mathbf{X}^{(31)}_{\mathbf{m}_{32}} - \mathbf{m}_{32}}\} = \mathrm{E}\{r^{\mathbf{m}_{32}} p_1^{\sum_{n=1}^{\mathbf{m}_{32}} (\boldsymbol{\xi}^{(3)}_n - 1)}\}, \quad |p_1| = 1,$$

with

$$P_1(1,p_2) := \lim_{r \to 1} P_1(r,p_2) = \mathrm{E}\{p_2^{\mathbf{Y}_{\mathbf{m}_{31}} - \mathbf{m}_{31}} (\mathbf{m}_{31} < \infty)\}, \qquad |p_2| = 1,$$

$$P_1(1,p_1) := \lim_{r \to 1} P_2(r,p_1) = E\{p_1^{\mathbf{X}_{\mathbf{m}_{32}} - \mathbf{m}_{32}} (\mathbf{m}_{32} < \infty)\}, \qquad |p_1| = 1;$$

iii.

$$P_1(1,1) := \lim_{r \uparrow 1} P_1(r,1) = Pr\{\mathbf{m}_{31} < \infty\} = 1 \quad \text{if } \mu_3 \leqslant 1,$$
$$< 1 \quad \text{if } \quad > 1,$$

$$P_2(1,1) := \lim_{r \uparrow 1} P_2(r,1) = Pr\{\mathbf{m}_{32} < \infty\} = 1 \quad \text{if } \nu_3 \leqslant 1,$$
$$< 1 \quad \text{if } \quad > 1,$$

iv.

$$\mathrm{E}\{\mathbf{m}_{31} | \mathbf{m}_{31} < \infty\} = \frac{P_1(1,1)}{\mathrm{E}\{(1-\xi_3) P_1^{\xi_3}(1,1)\}},$$
$$= \frac{1}{1-\mu_3} \qquad \text{if } \mu_3 < 1,$$

$$\mathrm{E}\{\mathbf{m}_{32} | \mathbf{m}_{32} < \infty\} = \frac{P_2(1,1)}{\mathrm{E}\{(1-\eta_3) P_2^{\eta_3}(1,1)\}}$$
$$= \frac{1}{1-\nu_3} \qquad \text{if } \nu_3 < 1.$$

PROOF. Per transition $\mathbf{x}_n$, cf. (2.16), can decrease by one at most, so it follows that the first entrance time $\mathbf{M}^{(31)}_{x_0}$ is the sum of the entrance times from $\{k\}$ into $\{k-1\}$, $k = 1,2,\dots, x_0$, which are obviously independent and identically distributed; hence (2.22)i follows. To prove (2.22)ii take $x_0 = 1$ then by the same argument on which (2.22)i is based it is seen that for $|r| \leqslant 1$, $r \neq 1$, $|p_2| = 1$,

$$\mathrm{E}\{r^{\mathbf{m}_{31}} p_2^{\sum_{n=1}^{\mathbf{m}_{31}} (\boldsymbol{\eta}^{(3)}_n - 1)}\} = r \sum_{k=0}^{\infty} \mathrm{E}\{(\boldsymbol{\xi}^{(3)}_1 = k) p_2^{\boldsymbol{\eta}_1 - 1} \mathrm{E}^k \{r^{\mathbf{m}_{31}} p_2^{\sum_{n=1}^{\mathbf{m}_{31}} (\boldsymbol{\eta}^{(3)}_n - 1)}\}\}$$

$$= \frac{r}{p_2} \phi_3(\mathrm{E}\{r^{\mathbf{m}_{31}} p_2^{\sum_{n=1}^{\mathbf{m}_{31}} (\boldsymbol{\eta}^{(3)}_n - 1)}\}, p_2^{\mathbf{y}_1}\}.$$

By noting that for

$$|r| \leqslant 1, \quad r \neq 1, \quad |p_2| = 1,$$

$$z - \frac{r}{p_2} \phi_3(z, p_2)$$

has a unique zero in $|z| \leqslant 1$, cf. (2.13), it follows from the definition of $P_1(r,p_2)$ that $P_1(r,p_2)$ is given by (2.22)ii for $|r|\leqslant 1$, $r\neq 1$, $|p_2|=1$. Obviously, $P_1(r,p_2)$, $|r|\leqslant 1$, $r\neq 1$, $|p_2|=1$ has a limit for $r\to 1$, and hence the definition in (2.22)ii is justified. The third statement is a wellknown result concerning $\mathbf{m}_{31}$. The fourth statement follows by differentiating the following relation, with respect to r,

$$p_2 P_1(r,p_2)= r\phi_3(P_1(r,p_2),p_2), \quad |r|<1,$$

and by taking in the resulting expression with $p_2=1$ the limit for $r\to 1$. □

REMARK 2.5. It is not difficult to show by using (2.5) that $P_1(1,p_2)$, $|p_2|=1$, $p_2\neq 1$ is the unique zero of $Z(1,p_1,p_2)$ in $|p_1|\leqslant 1$, and further that $P_1(1,1)$ is a zero of $Z(1,p_1,1)$ in $|p_1|\leqslant 1$. However, the latter may be not the unique one in $|p_1|\leqslant 1$; it is unique if $\mu_3\leqslant 1$; if $\mu_3>1$ then $p_1=1$ is a second zero in $|p_1|\leqslant 1$. We define for $|p_2|=1$,

$$P_1(1,p_2):=\lim_{\substack{r\to 1\\ |r|<1}} P_1(r,p_2),$$

and it is seen that $P_1(1,p_2)$ is a zero in $|p_1|\leqslant 1$ of $Z(1,p_1,p_2)$, $|p_2|=1$. □

REMARK 2.6. If ξ_3 and η_3 are independent, so that

$$\phi_3(p_1,p_2)=\phi_3(p_1,1)\phi_3(1,p_2), \quad |p_1|\leqslant 1, |p_2|\leqslant 1,$$

then $(\hat{p}_1,\hat{p}_2)$ as given by: for $|z|=1$, $|r|\leqslant 1$,

$$\hat{p}_1=\mathrm{E}\{(z\sqrt{r})^{\mathbf{m}_{31}}\}, \quad \hat{p}_2=\mathrm{E}\{(z^{-1}\sqrt{r})^{\mathbf{m}_{32}}\}, \tag{2.23}$$

is a zero tuple of $Z(r,p_1,p_2)$. Note that $\hat{p}_1$ is the unique zero in $|p_1|\leqslant 1$ of

$$p_1-z\sqrt{r}\,\phi_3(p_1,1), \quad |r|<1,$$

and $\hat{p}_2$ that in $|p_2|\leqslant 1$ of

$$p_2-z^{-1}\sqrt{r}\,\phi_3(1,p_2), \quad |r|<1.$$

□

REMARK 2.7. The so called *hitting point identity* has been derived and extensively discussed in the author's studies [3],...,[8]. In our present notation this identity reads:
for $(\hat{p}_1,\hat{p}_2)$ a zero tuple of $Z(r,p_1,p_2)$, $|r|\leqslant 1$, $r\neq 1$, and with $z_0=(x_0,y_0)\in S$,

$$\hat{p}_1^{x_0}\hat{p}_2^{y_0}=\mathrm{E}\{r^{\mathbf{n}(z_0)}\hat{p}_1^{\mathbf{k}_1(z_0)}\hat{p}_2^{\mathbf{k}_2(z_0)}\}, \tag{2.24}$$

which may be also written as:

$$\hat{p}_1^{x_0}\hat{p}_2^{y_0}=\mathrm{E}\{r^{\mathbf{n}(z_0)}\hat{p}_1^{\mathbf{k}_1(z_0)}(\mathbf{k}(z_0)\in B_{10})\}+\mathrm{E}\{r^{\mathbf{n}(z_0)}(\mathbf{k}(z_0)\in B_{00})\}+ \mathrm{E}\{r^{\mathbf{n}(z_0)}\hat{p}_2^{\mathbf{k}_2(z_0)}(\mathbf{k}(z_0)\in B_{01})\}. \tag{2.25}$$

These relations also hold for $r=1$, i.e.

$$\hat{p}_1^{x_0}\hat{p}_2^{y_0}=\mathrm{E}\{\hat{p}_1^{\mathbf{k}_1(z_0)}\hat{p}_2^{\mathbf{k}_2(z_0)}(\mathbf{n}(z_0)<\infty)\}, \tag{2.26}$$

for $(\hat{p}_1,\hat{p}_2)$ a zero tuple of $Z(1,p_1,p_2)$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, under the condition that such a zero tuple is the limit for $r\to 1$, $|r|<1$ of a zero tuple of $Z(r,p_1,p_2)$; it is readily proved that this is always the case if $\mu_3\leqslant 1$, $\nu_3\leqslant 1$, cf. remark 2.5. □

REMARK 2.8. From (2.8)i and ii it follows that for every zero tuple $(\hat{p}_1,\hat{p}_2)$ of $Z(r,p_1,p_2)$ with $|r|\leqslant 1$, $r\neq 1$ holds:

$$\hat{p}_1^{x_0}\hat{p}_2^{y_0}=(1-r\hat{\phi}_0)\,\Phi_{z_0}(r,0,0)+$$

$$\{1-r\frac{\hat{\phi}_1}{\hat{p}_1}\}\{\Phi_{z_0}(r,\hat{p}_1,0)-\Phi_{z_0}(r,0,0)\}+\{1-r\frac{\hat{\phi}_2}{\hat{p}_2}\}\{\Phi_{z_0}(r,0,\hat{p}_2)-\Phi_{z_0}(r,0,0)\},$$

with

$$\hat{\phi}_j \equiv \phi_j(\hat{p}_1,\hat{p}_2), \quad j=0,1,2. \qquad \square$$

THEOREM 2.2. *For* $z_0 \in S \setminus B$,

$$i.\ \Pr\{\mathbf{M}^{(31)}_{x_0}<\infty\}=1 \text{ or } \Pr\{\mathbf{M}^{(32)}_{y_0}<\infty\}=1 \Rightarrow \Pr\{\mathbf{n}(z_0)<\infty\}=1; \tag{2.27}$$

$$ii.\ \mu_3<1 \text{ or } \nu_3<1 \Rightarrow \mathrm{E}\{\mathbf{n}(z_0)\}<\infty;$$

$$iii.\ \mu_3<1 \Rightarrow \frac{x_0-\mathrm{E}\{\mathbf{k}_1(z_0)\}}{1-\mu_3}=\mathrm{E}\{\mathbf{n}(z_0)\},$$

$$\nu_3<1 \Rightarrow \frac{y_0-\mathrm{E}\{\mathbf{k}_2(z_0)\}}{1-\nu_3}=\mathrm{E}\{\mathbf{n}(z_0)\},$$

$$\mu_3<1,\ \nu_3<1 \Rightarrow \frac{x_0-\mathrm{E}\{\mathbf{k}_1(z_0)\}}{1-\mu_3}=\frac{y_0-\mathrm{E}\{\mathbf{k}_2(z_0)\}}{1-\nu_3};$$

$$iv.\ \mu_3=1,\ \nu_3<1 \Rightarrow \mathrm{E}\{\mathbf{k}_1(z_0)\}=x_0,$$

$$\mu_3<1,\ \nu_3=1 \Rightarrow \mathrm{E}\{\mathbf{k}_2(z_0)\}=y_0.$$

PROOF. From (2.9) ii we have

$$\mathbf{n}(z_0)=\min(\mathbf{M}^{(31)}_{x_0},\ \mathbf{M}^{(32)}_{y_0}), \tag{2.28}$$

from which (2.27) i directly follows. From (2.28) it follows easily that

$$\mathrm{E}\{\mathbf{n}(z_0)\}\leqslant \min(\{\mathrm{E}\{\mathbf{M}^{(31)}_{x_0}\},\ \mathrm{E}\{\mathbf{M}^{(32)}_{y_0}\}). \tag{2.29}$$

From (2.22) i and iv we have

$$\begin{aligned} \mathrm{E}\{\mathbf{M}^{(31)}_{x_0}\} &= \frac{x_0}{1-\mu_3} \text{ if } \mu_3<1; & \mathrm{E}\{\mathbf{M}^{(32)}_{y_0}\} &= \frac{y_0}{1-\nu_3} \text{ if } \nu_3<1,\\ &=\infty \quad ,, \quad =1; & &=\infty \quad ,, \quad =1, \end{aligned} \tag{2.30}$$

and so (2.27) ii results from (2.29) and 2.30).

To prove (2.27) iii we start from the hitting point identity (2.25) with, cf. (2.14),

$$\hat{p}_1=P_1(r,\hat{p}_2), \quad |\hat{p}_2|=1,$$

i.e., cf. (2.25), for $|r|\leqslant 1$, $|p_2|=1$,

$$\begin{aligned} P_1^{x_0}(r,p_2)p_2^{y_0} = {} & \mathrm{E}\{r^{\mathbf{n}(z_0)}\, P_1^{\mathbf{k}_1(z_0)}(r,p_2)(\mathbf{k}_1(z_0)\in B_{00}\cup B_{10})\}+\\ & \mathrm{E}\{r^{\mathbf{n}(z_0)}p_2^{\mathbf{k}_2(z_0)}(\mathbf{k}_2(z_0)\in B_{01})\}. \end{aligned} \tag{2.31}$$

Take $p_2=1$ in (2.31) and differentiate with respect to r, it then follows: for $|r|\leqslant 1$,

$$\begin{aligned} &[x_0P_1^{x_0}(r,1)-\mathrm{E}\{\mathbf{k}_1(z_0)r^{\mathbf{n}(z_0)}P_1^{\mathbf{k}_1(z_0)}\}]\,\frac{\frac{\mathrm{d}}{\mathrm{d}r}P_1(r,1)}{P_1(r,1)}=\\ &r^{-1}\,\mathrm{E}\{\mathbf{n}(z_0)r^{\mathbf{n}(z_0)}[P_1^{\mathbf{k}_1(z_0)}(r,1)(\mathbf{k}_1(z_0)\in B_{00}\cup B_{10})+(\mathbf{k}_2(z_0)\in B_{01})]\}. \end{aligned} \tag{2.32}$$

Since cf. (2.22) ii and (2.27) ii,

$$\mu_3<1 \Rightarrow P_1(1,1)=1,\ \frac{\mathrm{d}}{\mathrm{d}r}P_1(r,1)=\mathrm{E}\{\mathbf{m}_{31}\}=\frac{1}{1-\mu_3},\ \mathrm{E}\{\mathbf{n}(z_0)\}<\infty,$$

the first relation of (2.27) iii follows from (2.32) by letting $r \to 1$ and by using dominated convergence; the second one follows by symmetry and from these the third one. The first relation of (2.27) iv also follows from (3.32) by letting $r \to 1$, since $\nu_3 < 1$ implies $\mathrm{E}\{\mathbf{n}(z_0)\} < \infty$ and $P_1(r,1) \to 1$, $\frac{\mathrm{d}}{\mathrm{d}r} P_1(r,1) \to \infty$ for $r \to 1$ if $\mu_3 = 1$. □

THEOREM 2.3. *For* $\mu_3 = \nu_3 = 1$, $\mathrm{E}\{\xi_3^2\} < \infty$, $\mathrm{E}\{\eta_3^2\} < \infty$, $z_0 \in S \setminus B$,

i. $\Pr\{\mathbf{n}(z_0) < \infty\} = 1$; (2.33)

ii. $\mathrm{E}\{\mathbf{k}_1(z_0)\} = x_0, \quad \mathrm{E}\{\mathbf{k}_2(z_0)\} = y_0$;

iii. if $\mathrm{E}\{(\xi_3 - 1)(\eta_3 - 1)\} < 0$ *then* $\mathrm{E}\{\mathbf{n}(z_0)\} = \dfrac{x_0 y_0}{-\mathrm{E}\{(\xi_3-1)(\eta_3-1)\}}$ *and*

$$\mathrm{E}\{(\mathbf{k}_1(z_0) - x_0)^2\} = \frac{x_0 y_0\, \mathrm{E}\{(\xi_3-1)^2\}}{-\mathrm{E}\{(\xi_3-1)(\eta_3-1)\}}, \quad \mathrm{E}\{(\mathbf{k}_2(z_0) - y_0)^2\} = \frac{x_0 y_0\, \mathrm{E}\{(\eta_3-1)^2\}}{-\mathrm{E}\{(\xi_3-1)(\eta_3-1)\}};$$

iv. if $\mathrm{E}\{(\xi_3 - 1)(\eta_3 - 1)\} \geqslant 0$ *then* $\mathrm{E}\{\mathbf{n}(z_0)\} = \infty$ *and*

$\mathrm{E}\{\mathbf{k}_1^2(z_0)\} = \infty, \quad \mathrm{E}\{\mathbf{k}_2^2(z_0)\} = \infty.$

PROOF. Since

$$\mu_3 = 1 \Rightarrow \Pr\{\mathbf{M}_{x_0}^{(31)} < \infty\} = 1; \quad \nu_3 = 1 \Rightarrow \Pr\{\mathbf{M}_{y_0}^{(32)} < \infty\} = 1,$$

it is seen that (2.33) i follows from (2.28). The proof of the other statements is fairly complicated, and lengthy, for it we refer the reader to the study [17]. □

3. DESCRIPTION OF THE HITTING POINT PROCESS

For the $\mathbf{z}_n$-process starting at $z_0 = (x_0, y_0) \in S$ (see section 2) denote by

$$\mathbf{t}_m, \quad m = 1,2,..., \tag{3.1}$$

the successive epochs at which this process hits the boundary B after leaving z_0; we put

$$\mathbf{t}_1 := 0, \quad \mathbf{k}_1 := z_0 \quad \text{if} \quad z_0 \in B. \tag{3.2}$$

Put

$$\mathbf{k}_m \equiv (\mathbf{k}_m^{(1)}, \mathbf{k}_m^{(2)}) := \mathbf{z}_{\mathbf{t}_m} \equiv (\mathbf{x}_{\mathbf{t}_m}, \mathbf{y}_{\mathbf{t}_m}), \quad m = 1,2,..., \tag{3.3}$$

so

$$\{\mathbf{k}_m, \quad m = 1,2,...\}$$

is the *hitting point process.*

To describe the relations between the successive hitting points it is firstly noted that (2.3) implies: for $m = 1,2,...$,

$$\begin{aligned} \mathbf{z}_{\mathbf{t}_m+1} = &\ (\mathbf{k}_m^{(1)} + \xi_{\mathbf{t}_m}^{(1)} - 1, \eta_{\mathbf{t}_m}^{(1)}) \quad \text{for} \quad \mathbf{k}_m^{(1)} \geqslant 1, \ \mathbf{k}_m^{(2)} = 0, \\ = &\ (\xi_{\mathbf{t}_m}^{(2)}, \mathbf{k}_m^{(2)} + \eta_{\mathbf{t}_m}^{(2)} - 1) \quad \text{for} \quad\quad = 0, \quad\quad \geqslant 1, \\ = &\ (\xi_{\mathbf{t}_m}^{(0)}, \eta_{\mathbf{t}_m}^{(0)}) \quad\quad \text{for} \quad\quad = 0, \quad\quad = 0. \end{aligned} \tag{3.4}$$

From definition (2.3) it follows that for $\mathbf{k}_{m+1}$, i.e. the next hitting point after $\mathbf{t}_m$, holds:

$$\begin{aligned} &\text{i.} \quad \mathbf{k}_{m+1} \in B_{10} \Leftrightarrow \mathbf{M}_{\mathbf{y}_{\mathbf{t}_m+1}}^{(32)} < \mathbf{M}_{\mathbf{k}_{\mathbf{t}_m+1}}^{(31)}, \\ &\text{ii.} \quad \ldots \in B_{00} \Leftrightarrow \quad \ldots \quad = \quad \ldots, \\ &\text{iii.} \quad \ldots \in B_{01} \Leftrightarrow \quad \ldots \quad > \quad \ldots, \end{aligned} \tag{3.5}$$

since, cf. (2.9)ii and definition 2.3,

$$\mathbf{n}_1 := \mathbf{n}(z_0) = \min(\mathbf{M}^{(31)}_{x_0}, \mathbf{M}^{(32)}_{y_0}),$$
$$\mathbf{n}_{m+1} := \mathbf{n}(\mathbf{z}_{\mathbf{t}_m+1}) = \min(\mathbf{M}^{(31)}_{\mathbf{x}_{\mathbf{t}_m+1}}, \mathbf{M}^{(32)}_{\mathbf{y}_{\mathbf{t}_m+1}}), \quad m=1,2,\ldots. \tag{3.6}$$

Obviously we have

$$\mathbf{t}_1 = \mathbf{n}(z_0),$$
$$\mathbf{t}_{m+1} - \mathbf{t}_m = 1 + \mathbf{n}_{m+1}, \quad m = 1,2,\ldots. \tag{3.7}$$

By noting that $\mathbf{n}_{m+1}+1$ is the number of steps of the random walk $\mathbf{z}_n$ after which the next hitting of B after $\mathbf{t}_m$ occurs, it is seen that

$$\Big(\sum_{h=1}^{\mathbf{n}_{m+1}} (\boldsymbol{\xi}^{(3)}_{\mathbf{t}_m+h} - 1), \ \sum_{h=1}^{\mathbf{n}_{m+1}} (\boldsymbol{\eta}^{(3)}_{\mathbf{t}_m+h} - 1)\Big), \tag{3.8}$$

is the displacement vector of the $\mathbf{z}_n$-process during the time interval $\mathbf{t}_{m+1} - \mathbf{t}_m - 1$. Consequently, from (3.4) for $m=1,2,\ldots,$

$$\mathbf{k}^{(1)}_{m+1} = \mathbf{k}^{(1)}_m + \boldsymbol{\xi}^{(h)}_{\mathbf{t}_m} - \boldsymbol{\delta}_h + \sum\nolimits_{j=1}^{\mathbf{n}_{m+1}} (\boldsymbol{\xi}^{(3)}_{\mathbf{t}_m+j} - 1)), \tag{3.9}$$
$$\mathbf{k}^{(2)}_{m+1} = \mathbf{k}^{(2)}_m + \boldsymbol{\eta}^{(h)}_{\mathbf{t}_m} - \boldsymbol{\epsilon}_h + \sum\nolimits_{j=1}^{\mathbf{n}_{m+1}} (\boldsymbol{\eta}^{(3)}_{\mathbf{t}_m+j} - 1),$$

with for $h = 0,1,2,$ respectively,

$$\begin{aligned}
\boldsymbol{\xi}^{(h)}_{\mathbf{t}_m} &= \boldsymbol{\xi}^{(0)}_{\mathbf{t}_m}, & \boldsymbol{\eta}^{(h)}_{\mathbf{t}_m} &= \boldsymbol{\eta}^{(0)}_{\mathbf{t}_m}, & \boldsymbol{\delta}_h &= 0, & \boldsymbol{\epsilon}_h &= 0 \ \text{ for } \ \mathbf{k}^{(1)}_m = 0, & \mathbf{k}^{(2)}_m &= 0,\\
&= \boldsymbol{\xi}^{(1)}_{\mathbf{t}_m}, & &= \boldsymbol{\eta}^{(1)}_{t_m}, & &= 1, & &= 0 \ \text{ for } \quad \geqslant 1, & &= 0,\\
&= \boldsymbol{\xi}^{(2)}_{\mathbf{t}_m}, & &= \boldsymbol{\eta}^{(2)}_{\mathbf{t}_m}, & &= 0, & &= 1 \ \text{ for } \quad = 0, & &\geqslant 1;
\end{aligned} \tag{3.10}$$

note that in (3.9) at least one of the righthand sides is zero.

It remains to specify $\mathbf{k}_1$. By a similar argument as above it is seen that (3.2) implies

$$\mathbf{k}^{(1)}_1 = x_0 + \sum_{h=1}^{\mathbf{n}_1} (\boldsymbol{\xi}^{(3)}_h - 1), \tag{3.11}$$
$$\mathbf{k}^{(2)}_1 = y_0 + \sum_{h=1}^{\mathbf{n}_1} (\boldsymbol{\eta}^{(3)}_h - 1),$$

where at least one of the righthand sides is zero.

The relations (3.9) and (3.11) describe the structure of the hitting point process $\{\mathbf{k}_m, m=1,2,\ldots\}$. It is readily seen that this process is a discrete time parameter Markov process with state space B and with stationary but nonhomogeneous transition probabilities.

To investigate the $\mathbf{k}_m$-process we introduce the following generating functions. For $|t|<1$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$K_1(t,p_1) := \sum_{m=1}^{\infty} t^m \mathrm{E}\{p_1^{\mathbf{k}^{(1)}_m} (\mathbf{k}^{(1)}_m \geqslant 1, \mathbf{k}^{(2)}_m = 0)(\mathbf{n}_m < \infty) \,|\, \mathbf{z}_0 = z_0\}, \tag{3.12}$$
$$K_0(t) := \sum_{m=1}^{\infty} t^m \mathrm{E}\{(\mathbf{k}^{(1)}_m = 0, \mathbf{k}^{(2)}_m = 0)(\mathbf{n}_m < \infty) \,|\, \mathbf{z}_0 = z_0\},$$
$$K_2(t,p_2) := \sum_{m=1}^{\infty} t^m \mathrm{E}\{p_2^{\mathbf{k}^{(2)}_m} (\mathbf{k}^{(1)}_m \geqslant 0, \mathbf{k}^{(2)}_m \geqslant 1)(\mathbf{n}_m < \infty) \,|\, \mathbf{z}_0 = z_0\},$$

From the structure of the $\mathbf{k}_m$-process, cf. (3.9) and (3.11), a relation between the generating functions defined in (3.12) may be derived. However, that relation can be more easily obtained by using the hitting point identity (2.26). The idea in using this identity is that we consider $\mathbf{z}_{\mathbf{t}_m+1}$ as the starting point of the $\mathbf{z}_n$-processes and then apply the hitting point identity for the hitting point $\mathbf{k}_{m+1}$.

DEFINITION 3.1. A zero tuple $(\hat{p}_1,\hat{p}_2)$ of $Z(1-,p_1,p_2)$, cf. (2.11), is a zero tuple of $Z(1,p_1,p_2)$ if it is the limit for $r\to 1$, $|r|\leqslant 1$ of a zero tuple of $Z(r,p_1,p_2)$.

THEOREM 3.1. *For* $|t|<1$, $z_0\in S$ *and* $(\hat{p}_1,\hat{p}_2)$ *a zero tuple of* $Z(1-,p_1,p_2)$:

$$t\hat{p}_1^{x_0}\hat{p}_2^{y_0} = \{1-t\frac{\hat{\phi}_1}{\hat{p}_1}\}K_1(t,\hat{p}_1) + \{1-t\hat{\phi}_0\}K_0(t)+\{1-t\frac{\hat{\phi}_2}{\hat{p}_2}\}K_2(t,\hat{p}_2). \tag{3.13}$$

PROOF. Let $(\hat{p}_1,\hat{p}_2)$ be a zero tuple of $Z(1-,p_1,p_2)$ then we have from (2.26),

$$\begin{aligned}\hat{p}_1^{x_0}\hat{p}_2^{y_0} = {} & \mathrm{E}_{z_0}\{\hat{p}_1^{\mathbf{k}_1}(\mathbf{k}_1^{(1)}\geqslant 1,\mathbf{k}_1^{(2)}=0)(\mathbf{n}_1<\infty)\} + \mathrm{E}_{z_0}\{(\mathbf{k}_1^{(1)}=0,\mathbf{k}_1^{(2)}=0)(\mathbf{n}_1<\infty)\}+\\ & \mathrm{E}_{z_0}\{\hat{p}_2^{\mathbf{k}_2}(\mathbf{k}_1^{(1)}=0,\,\mathbf{k}_1^{(2)}\geqslant 1)(\mathbf{n}_1<\infty)\},\end{aligned} \tag{3.14}$$

and with probability one

$$\begin{aligned}&\hat{p}_1^{\mathbf{k}_m^{(1)}+\xi_{\mathbf{t}_m}^{(1)}-1}\hat{p}_2^{\eta_{\mathbf{t}_m}^{(1)}}(\mathbf{k}_m^{(1)}\geqslant 1,\mathbf{k}_m^{(2)}=0)(\mathbf{n}_m<\infty)+\\ &\hat{p}_1^{\xi_{\mathbf{t}_m}^{(0)}}\hat{p}_2^{\eta_{\mathbf{t}_m}^{(0)}}(\mathbf{k}_m^{(1)}=0,\mathbf{k}_m^{(2)}=0)(\mathbf{n}_m<\infty)+\\ &\hat{p}_1^{\xi_{\mathbf{t}_m}^{(2)}}\hat{p}_2^{\mathbf{k}_m^{(2)}+\eta_{\mathbf{t}_m}^{(2)}-1}(\mathbf{k}_m^{(1)}\geqslant 0,\mathbf{k}_m^{(2)}\geqslant 1)(\mathbf{n}_m<\infty)=\\ &\mathrm{E}\{\hat{p}_1^{\mathbf{k}_{m+1}^{(1)}}\hat{p}_2^{\mathbf{k}_{m+1}^{(2)}}(\mathbf{n}_{m+1}<\infty)|\mathbf{z}_{\mathbf{t}_m+1},\mathbf{t}_m\}.\end{aligned} \tag{3.15}$$

By noting that

$$(\mathbf{k}_m^{(1)},\mathbf{k}_m^{(2)}) \text{ and } (\xi_{\mathbf{t}_m}^{(h)},\,\eta_{\mathbf{t}_m}^{(h)}),\ h=0,1,2,$$

are independent it follows from (3.15) by taking expectations that

$$\begin{aligned}&\frac{\hat{\phi}_1}{\hat{p}_1}\mathrm{E}\{\hat{p}_1^{\mathbf{k}_m^{(1)}}(\mathbf{k}_m^{(1)}\geqslant 1,\,\mathbf{k}_m^{(2)}=0)(\mathbf{n}_m<\infty)|\mathbf{z}_0=z_0\}+\\ &\hat{\phi}_0\mathrm{E}\{(\mathbf{k}_m^{(1)}=0,\,\mathbf{k}_m^{(2)}=0)(\mathbf{n}_m<\infty)|\mathbf{z}_0=z_0\}+\\ &\frac{\hat{\phi}_2}{\hat{p}_2}\mathrm{E}\{p_2^{\mathbf{k}_m^{(2)}}(\mathbf{k}_m^{(1)}=0,\,\mathbf{k}_m^{(2)}\geqslant 1)(\mathbf{n}_m<\infty)|\mathbf{z}_0=z_0\}=\\ &\mathrm{E}\{\hat{p}_1^{\mathbf{k}_{m+1}^{(1)}}\hat{p}_2^{\mathbf{k}_{m+1}^{(2)}}(\mathbf{n}_{m+1}<\infty)|\mathbf{z}_0=z_0\}.\end{aligned} \tag{3.16}$$

Multiplying (3.14) by t and (3.16) by t^{m+1} and summing the result over $m=1,2,\ldots$ yields (3.13). □

REMARK 3.1. Note that $K_1(t,p)$ and $K_2(t,p)$ are both regular for $|p|<1$, continuous for $|p|\leqslant 1$, for every fixed t with $|t|<1$; comparison of (2.27) and (3.13) shows that these relations have a similar analytic structure. For the analysis of relations with such a structure see the author's studies, e.g. [1], [3], [7], [8]. □

As already mentioned above $B_{10}\cup B_{00}\cup B_{01}$ is the state space of the $\mathbf{k}_m$-process and a sample function of the $\mathbf{k}_m$-process may be split up into cycles of hitting points which all belong to the same component of B, i.e. to B_{10},B_{00} or B_{01}.

DEFINITION 3.2. A sequence of hitting points $\mathbf{k}_{m+h}$, $h=1,2,\ldots,N$, with

$$\mathbf{k}_{m+1}\in B_{10},\quad h=1,\ldots,N;\quad 1\leqslant N\leqslant\infty,$$

$$\mathbf{k}_m \notin B_{10}, \quad \mathbf{k}_{m+N+1} \notin B_{10}, \quad \mathbf{k}_0 := (0,0),$$

will be called a b_{10} - cycle, analogously b_{00}- and b_{01} - cycles are defined; $\mathbf{b}_{10}$ shall stand for the number of hitting points of which a b_{10}- cycle consists.

Consider two successive hitting points both belonging to a same b_{10}- cycle, say $\mathbf{k}_m$ and $\mathbf{k}_{m+1}$, so that:

$$\mathbf{k}_m^{(1)} \geqslant 1, \mathbf{k}_m^{(2)} = 0; \mathbf{k}_{m+1}^{(1)} \geqslant 1, \mathbf{k}_{m+1}^{(2)} = 0. \tag{3.17}$$

Then, cf. (3.7),

$$\mathbf{n}_{m+1} = \mathbf{M}^{(32)}_{\mathbf{y}_{\mathbf{t}_{m+1}}} = \mathbf{M}^{(32)}(\boldsymbol{\eta}^{(1)}_{\mathbf{t}_m}), \tag{3.18}$$

and, cf. (3.9),

$$\mathbf{k}_{m+1}^{(1)} = \mathbf{k}_m^{(1)} + \xi^{(1)}_{\mathbf{t}_m} - 1 + \sum_{h=1}^{\mathbf{n}_{m+1}} (\xi^{(3)}_{\mathbf{t}_m+h} - 1); \quad \mathbf{n}_{m+1} = \mathbf{M}^{(32)}(\boldsymbol{\eta}^{(1)}_{\mathbf{t}_m}), \tag{3.19}$$

analogously, if $\mathbf{k}_m$ and $\mathbf{k}_{m+1}$ both belong to a same b_{01}-cycle then

$$\mathbf{k}_{m+1}^{(2)} = \mathbf{k}_m^{(2)} + \boldsymbol{\eta}^{(2)}_{\mathbf{t}_m} - 1 + \sum_{h=1}^{\mathbf{n}_{m+1}} (\boldsymbol{\eta}^{(3)}_{\mathbf{t}_m+h} - 1); \quad \mathbf{n}_{m+1} = \mathbf{M}_{(31)}(\xi^{(2)}_{\mathbf{t}_m}); \tag{3.20}$$

and if they both belong to a same b_{00}-cycle then

$$\begin{aligned} &\mathbf{k}_{m+1}^{(1)} = \xi^{(0)}_{\mathbf{t}_m} + \sum_{h=1}^{\mathbf{n}_{m+1}} (\xi^{(3)}_{\mathbf{t}_m+h} - 1) = 0, \\ &\qquad\qquad \mathbf{n}_{m+1} = \mathbf{M}^{(32)}(\boldsymbol{\eta}^{(0)}_{\mathbf{t}_m}) = \mathbf{M}^{(31)}(\xi^{(0)}_{\mathbf{t}_m}). \\ &\mathbf{k}_{m+1}^{(2)} = \boldsymbol{\eta}^{(0)}_{\mathbf{t}_m} + \sum_{h=1}^{\mathbf{n}_{m+1}} (\boldsymbol{\eta}^{(3)}_{\mathbf{t}_m+h} - 1) = 0, \end{aligned} \tag{3.21}$$

Put

$$\begin{aligned} \boldsymbol{\sigma}_{00}^{(1)} &:= \xi_1^{(0)} + \sum_{h=1}^{\mathbf{M}^{(32)}(\boldsymbol{\eta}_1^{(0)})} (\xi_h^{(3)} - 1), \\ \boldsymbol{\sigma}_{00}^{(2)} &:= \boldsymbol{\eta}_1^{(0)} + \sum_{h=1}^{\mathbf{M}^{(31)}(\xi_1^{(0)})} (\boldsymbol{\eta}_h^{(3)} - 1), \\ \boldsymbol{\sigma}_{10} &:= \xi_1^{(1)} - 1 + \sum_{h=1}^{\mathbf{M}^{(32)}(\boldsymbol{\eta}_1^{(1)})} (\xi_h^{(3)} - 1), \\ \boldsymbol{\sigma}_{01} &:= \boldsymbol{\eta}_1^{(2)} - 1 + \sum_{h=1}^{\mathbf{M}^{(31)}(\xi_1^{(2)})} (\boldsymbol{\eta}_h^{(3)} - 1). \end{aligned} \tag{3.22}$$

REMARK 3.2. The hitting points belonging to a same b_{10}-cycle may be considered, cf. (3.19) and (3.22), to form a realisation of a one-dimensional random walk of which the one-step displacements have the same distribution as $\boldsymbol{\sigma}_{10}$; analogously for those hitting points belonging to a same b_{01}-cycle. □

For the further analysis of the $\mathbf{k}_m$-process we need the following

LEMMA 3.1. (cf. assumption 2.1iii)

$$\begin{aligned} \text{i.}\quad & \mathrm{E}\{p_1^{\boldsymbol{\sigma}_{10}}\} = \frac{1}{p_1}\phi_1(p_1, P_2(1,p_1)), && |p_1| = 1, \\ & \mathrm{E}\{p_2^{\boldsymbol{\sigma}_{01}}\} = \frac{1}{p_2}\phi_2(P_1(1,p_2), p_2), && |p_2| = 1, \\ & \mathrm{E}\{p_1^{\boldsymbol{\sigma}_{00}^{(1)}}\} = \phi_0(p_1, P_2(1,p_1)), && |p_1| = 1, \\ & \mathrm{E}\{p_2^{\boldsymbol{\sigma}_{00}^{(2)}}\} = \phi_0(P_1(1,p_2), p_2)), && |p_2| = 1; \end{aligned} \tag{3.23}$$

$$\text{ii. } r_1 := \mathrm{E}\{\sigma_{10}\} = \mu_1 - 1 + \frac{\nu_1}{1-\nu_3}(\mu_3 - 1) \quad \text{for } \nu_3 < 1,$$
$$= \infty \qquad \text{for } \quad = 1, \ \mu_3 > 1,$$
$$= -\infty \qquad \text{for } \quad = 1, \ \mu_3 < 1;$$

$$r_2 := \mathrm{E}\{\boldsymbol{\sigma}_{01}\} = \nu_2 - 1 + \frac{\mu_2}{1-\mu_3}(\nu_3 - 1) \quad \text{for } \mu_3 < 1,$$
$$= \infty \qquad \text{for } \quad = 1, \ \nu_3 > 1,$$
$$= -\infty \qquad \text{for } \quad = 1, \quad < 1;$$

iii. for $\nu_3 = 1$, $\mu_3 = 1$ and $\mathrm{E}\{\xi_3^2\} < \infty$, $\mathrm{E}\{\boldsymbol{\eta}_3^2\} < \infty$,

$$r_1 = \mathrm{E}\{\boldsymbol{\sigma}_{10}\} = \mu_1 - 1 - \nu_1 \frac{\mathrm{E}\{(\xi_3 - 1)(\boldsymbol{\eta}_3 - 1)\}}{E\{(\boldsymbol{\eta}_3 - 1)^2\}},$$

$$r_2 = \mathrm{E}\{\boldsymbol{\sigma}_{01}\} = \nu_2 - 1 - \mu_2 \frac{\mathrm{E}\{(\xi_3 - 1)(\boldsymbol{\eta}_3 - 1)\}}{E\{(\xi_3 - 1)^2\}}.$$

PROOF. From (2.2)i and ii, and from theorem 2.1 with $r = 1$ it follows readily that for $|p_2| = 1$,

$$\mathrm{E}\{p_1^{\sigma_{10}}\} = \mathrm{E}\{p_1^{\xi_1 - 1}\, \mathrm{E}^{\eta_1}\{p_1^{\sum_{h=1}^{m_{31}}(\xi_h^{(3)} - 1)}\}\}$$
$$= \mathrm{E}\{p_1^{\xi_1 - 1}\, P_2^{\eta_1}(1, p_1)\} = \frac{1}{p_1}\phi_1(p_1, P_2(1, p_1)).$$

This proves the first relation of (3.23)i, the others are proved analogously.

Note that $P_2(1,p_1)$, $|p_1| = 1$ is the unique zero of $Z(1-, p_1, p_2)$ in $|p_2| \leqslant 1$, cf. (2.13), theorem 2.1 and definition 3.1, so that

$$p_1 P_2(1, p_1) - \phi_3(p_1, P_2(1, p_1)) = 0.$$

From the latter relation $\dfrac{\mathrm{d}}{\mathrm{d}p_1} P_2(1, p_1)$ is readily calculated for $p_1 = 1$; by using

$$\mathrm{E}\{\boldsymbol{\sigma}_{10}\} = \frac{\mathrm{d}}{\mathrm{d}p_1} \mathrm{E}\{p_1^{\sigma_{10}}\}|_{p_1 = 1},$$

the expression for r_1 is readily derived if $\nu_3 \leqslant 1$. Similarly for r_2. Note that the expression for r_1 may be also derived directly from (3.22) by using Wald's theorem for a random sum of i.i.d. stochastic variables if $\nu_3 < 1$, since $\mathbf{M}^{(32)}(\boldsymbol{\eta}_1^{(1)})$ is a stopping time for the sequence $\xi_n^{(3)} - 1$, $n = 1, 2, \ldots$.

The proof of the third statement requires a more refined analysis. This is due to the fact that the function

$$p_2 - \phi_3(1, p_2) \quad \text{with } \nu_3 = 1,$$

has in $|p_2| \leqslant 1$ a unique zero $p_2 = P_2(1,1) = 1$ which has multiplicity two. This implies that $P_2(r, 1)$, $|r| \leqslant 1$ has a branch point at $r = 1$. From

$$P_2(r, p_1) - \frac{r}{p_1}\phi_3(p_1, P_2(r, p_1)) = 0, \ |p_1| = 1, \ |r| < 1,$$

it follows readily that for $|r| < 1$,

$$\lim_{\substack{p_1 \to 1 \\ |p_1| = 1}} \frac{\partial P_2(r, p_1)}{\partial p_1} = -\frac{\mathrm{E}\{P_2^{\eta_3}(r, 1)\} - \mathrm{E}\{\xi_3 P_2^{\eta_3}(r, 1)\}}{1 - r\mathrm{E}\{\boldsymbol{\eta}_3 P_2^{\eta_3 - 1}(r, 1)\}}. \tag{3.24}$$

In [17] (cf. lemma 3.2.ii) it has been shown for the case with $\mu_3=\nu_3=1$, $E\{\xi_3^2\}<\infty$, $E\{\eta_3^2\}<\infty$ that for $|r|<1$, $r\to 1$,

$$\frac{1-P_2(r,1)}{\sqrt{1-r}}=\frac{1}{rE\{(\eta_3-1)^2\}}\{-\sqrt{1-r}+\sqrt{1-r+2rE\{(\eta_3-1)^2\}}\}+O(\sqrt{1-r}). \tag{3.25}$$

We next consider the three terms in the righthand side of (3.24), and write for $|r|<1$, $r\to 1$,

$$\begin{aligned}
&\text{i.} \quad E\{P_2^{\eta_3}(r,1)\}=r\phi_3(1,P_2(r,1))=1-(1-P_2(r,1)),\\
&\text{ii.} \quad E\{\xi_3P_2^{\eta_3}(r,1)\}=E\{\xi_3[1-(1-P_2(r,1))]^{\eta_3}\}=\\
&\qquad E\{\xi_3[1-\eta_3(1-P_2(r,1))+o(1-P_2(r,1))]=\\
&\qquad 1-E\{\xi_3\eta_3\}(1-P_2(r,1))+o(\sqrt{1-r}),\\
&\text{iii.} \quad E\{\eta_3P_2^{\eta_3-1}(r,1)\}=E\{\eta_3[1-(1-P_2(r,1))]^{\eta_3-1}\}=\\
&\qquad E\{\eta_3-\eta_3(\eta_3-1)(1-P_2(r,1))+o(1-P_2(r,1))\}=\\
&\qquad 1-E\{(\eta_3-1)^2\}(1-P_2(r,1))+o(1-P(r)).
\end{aligned} \tag{3.26}$$

Inserting the expressions (3.26) in (3.24) leads to, cf. (2.22)ii, for $|r|<1$, $r\to 1$,

$$\lim_{\substack{p_1\to 1\\|p_1|=1}}\frac{\partial P_2(r,p_1)}{\partial p_1}=E\{r^{m_{32}}\textstyle\sum_{h=1}^{m_{32}}(\xi_h^{(3)}-1)\}= \tag{3.27}$$

$$\frac{-r+r[1-P_2(r,1)]+r-rE\{\xi_3\eta_3\}[1-P_2(r,1)]}{1-r+rE\{(\eta_3-1)^2\}[1-P_2(r,1)]}+O(\sqrt{1-r}).$$

From (3.25) we obtain (see also [17], Lemma 3.2),

$$\lim_{\substack{|r|<1\\r\to 1}}\frac{1-P_2(r,1)}{\sqrt{1-r}}=\sqrt{2}\,E^{-1/2}\{(\eta_3-1)^2\}, \tag{3.28}$$

Hence the lefthand side of (3.27) has a limit for $|r|<1$, $r\to 1$ and so by monotone convergence we obtain from (3.27),

$$\begin{aligned}
E\{\textstyle\sum_{h=1}^{m_{32}}(\xi_h^{(3)}-1)\}&=\frac{1-E\{\xi_3\eta_3\}}{E\{(\eta_3-1)^2\}}\\
&=-\frac{E\{(\xi_3-1)(\eta_3-1)\}}{E\{(\eta_3-1)^2\}},
\end{aligned} \tag{3.29}$$

since $\mu_3=E\{\xi_3\}=\nu_3=E\{\eta_3\}=1$.

From (2.22)i and (3.28) it follows readily that

$$E\{\textstyle\sum_{h=1}^{M^{(32)}(\eta_1^{(1)})}(\xi_h^{(3)}-1)\}=-\nu_1\frac{E\{(\xi_3-1)(\eta_3-1)\}}{E\{(\eta_3-1)^2\}},$$

which leads directly to the first relation of (3.23)iii, the second one follows analogously. □

REMARK 3.3. Note that the expectations in (3.23)ii do exist without any conditions on higher moments of ξ_3 and η_3. The conditions $E\{\xi_3^2\}<\infty$, $E\{\eta_3^2\}<\infty$ are sufficient for the existence of a limit for $r\to 1$ of the right hand side of (3.24); these conditions can be weakened, however, we shall not investigate this here. Obviously, by the same technique as used above $E\{\sigma_{00}^{(1)}\}$ and $E\{\sigma_{00}^{(2)}\}$ can be determined. □

REMARK 3.4. In remark 3.2 it has been argumented that the hitting points belonging to a same, say, b_{01}-cycle may be considered as a realisation of a random walk, however, the initial point of such a cycle is a stochastic point. Here we shall show that this point has a finite first moment.

Let $(\mathbf{k}_m^{(1)},0)$ be a point of a b_{10}-cycle, then it is the last point of this cycle if

$$\mathbf{M}^{(31)}(\mathbf{k}_m^{(1)}+\boldsymbol{\xi}_{\mathbf{t}_m}^{(1)}-1)<\mathbf{M}^{(32)}(\boldsymbol{\eta}_{\mathbf{t}_m}^{(1)}),$$

and conditionally on this event $(0,\mathbf{k}_{m+1}^{(2)})$ is then the initial point of the next b_{01}-cycle. From theorem 2.2 ii and iii it then follows, cf. (3.6), that for $\nu_3<1$,

$$\mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\}=\mathrm{E}\{\boldsymbol{\eta}_{t_m}^{(1)}\}-(1-\nu_3)\mathrm{E}\{\mathbf{n}_{m+1}\}<\mathrm{E}\{\boldsymbol{\eta}_{t_m}^{(1)}\}=\nu_1<\infty,$$

and from theorem 2.3 ii for $\mu_3=1$, $\nu_3=1$, $\mathrm{E}\{\boldsymbol{\xi}_3^2\}<\infty$, $\mathrm{E}\{\boldsymbol{\eta}_3^2\}<\infty$,

$$\mathrm{E}\{\mathbf{k}_{m+1}^{(2)}\}=\mathrm{E}\{\boldsymbol{\eta}_{t_m}^{(1)}\}=\nu_1<\infty.$$

Consequently, if $\nu_3\leqslant 1$ then the initial point of a b_{01}-cycle has a finite first moment, it is readily seen that this also hold if the b_{01}-cycle is preceded by a b_{00}-cycle. Similar results apply for the b_{10}-cycles. □

The results obtained in the present section will be used in section 5 to formulate the necessary and sufficient conditions for the states of the $\mathbf{k}_m$-process to be nonrecurrent, null recurrent or positive recurrent.

4. ON THE SEMI-BOUNDED TWO DIMENSIONAL RANDOM WALK

The hitting point process $\{\mathbf{k}_m, m=1,2,...\}$, defined in the previous section, is a rather complicated process, this being due to the alternation of sequences of hitting points with the x-axis and those with the y-axis. Therefore, we first study the semi-bounded random walk

$$\mathbf{w}_n\equiv(\mathbf{u}_n,\mathbf{v}_n),\quad n=0,1,2,..., \tag{4.1}$$

with state space

$$T:=\{...,-1,0,1,...\}\times\{0,1,2,...\},$$

and structure: for $n=0,1,2,...;$

if $\mathbf{v}_n>0$, (4.2)

$$\mathbf{u}_{n+1}=\mathbf{u}_n+\boldsymbol{\xi}_n^{(3)}-1,$$
$$\mathbf{v}_{n+1}=\mathbf{v}_n+\boldsymbol{\eta}_n^{(3)}-1;$$

if $\mathbf{v}_n=0$,

$$\mathbf{u}_{n+1}=\mathbf{u}_n+\boldsymbol{\xi}_n^{(1)}-1,$$
$$\mathbf{v}_{n+1}=\boldsymbol{\eta}_n^{(1)};$$
$$\mathbf{w}_0\equiv w_0=(u_0,v_0)\in T.$$

Put for $|r|<1$, $|p_1|=1$, $|p_2|\leqslant 1$,

$$\Omega_{w_0}(r,p_1,p_2):=\sum_{n=0}^{\infty}r^n\,\mathrm{E}\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}|\mathbf{w}_0=w_0\}. \tag{4.3}$$

It follows readily from (4.2) and (4.3) that for $|r|<1$, $|p_1|=1$, $|p_2|\leqslant 1$,

$$\left[1-r\,\frac{\phi_3(p_1,p_2)}{p_1p_2}\right]\Omega_{w_0}(r,p_1,p_2)=p_1^{u_0}p_2^{v_0}+r\left[\frac{\phi_1(p_1,p_2)}{p_1}-\frac{\phi_3(p_1,p_2)}{p_1p_2}\right]\Omega_{w_0}(r,p_1,0). \tag{4.4}$$

Consequently, since $\Omega_{w_0}(r,p_1,p_2)$ is for fixed $|r|<1$, fixed p_1 with $|p_1|=1$, regular in p_2 for $|p_2|<1$, continuous for $|p_2|\leqslant 1$, we have for:

$$|r|<1,\ |\hat{p}_1|=1,\ \hat{p}_2=P_2(r,\hat{p}_1):$$

$$\hat{p}_1^{u_0}\hat{p}_2^{v_0}=\{1-r\frac{\phi_1(\hat{p}_1,\hat{p}_2)}{\hat{p}_1}\}\,\Omega_{w_0}(r,\hat{p}_1,0). \tag{4.5}$$

Next we denote by

$$\mathbf{t}_m,\quad m=1,2,..., \tag{4.6}$$

the successive hitting epochs of the $\mathbf{w}_n$-process with the boundary B_0 of T,

$$B_0:=\{...,-1,0,1,...\}\times\{0\}, \tag{4.7}$$

after leaving the initial state w_0. The hitting point process

$$\{\mathbf{h}_m,\quad m=1,2,...\}$$

is defined by

$$\mathbf{h}_m:=\mathbf{w}_{\mathbf{t}_m}=(\mathbf{u}_{\mathbf{t}_m},0),\quad m=1,2,.... \tag{4.8}$$

Put

$$\mathbf{N}_1:=\inf_{n=0,1,...}\{n:\mathbf{w}_n\in B_0|\mathbf{w}_0=w_0\}, \tag{4.9}$$

$$:=\infty\quad \text{if } \mathbf{w}_n\in T\setminus B_0 \text{ for all } n=1,2,...;$$

$$\mathbf{N}_{m+1}:=\inf_{n=1,2,...}\{n:\mathbf{w}_{\mathbf{t}_m+n}\in B_0\}$$

$$:=\infty\quad \text{if } \mathbf{w}_{\mathbf{t}_m+n}\in T\setminus B_0 \text{ for all } n=1,2,...;$$

so $\mathbf{N}_m$, $m=1,2,...$, are successive entrance times into B_0.

From the structure of the $\mathbf{w}_n$-process, cf. (4.2), it is seen that, cf. definition 2.3,

$$\mathbf{N}_1=\mathbf{M}^{(32)}_{v_0}, \tag{4.10}$$

$$\mathbf{N}_{m+1}=\mathbf{M}^{(32)}_{\mathbf{v}_{\mathbf{t}_m+1}},\quad m=1,2,....$$

Obviously the structure of the $\mathbf{h}_m$-process is described by

$$\mathbf{h}_1=w_0+\textstyle\sum_{h=1}^{\mathbf{N}_1}(\boldsymbol{\xi}_h^{(3)}-1), \tag{4.11}$$

$$\mathbf{h}_{m+1}=\mathbf{h}_m+\boldsymbol{\xi}^{(1)}_{\mathbf{t}_m}-1+\textstyle\sum_{h=1}^{\mathbf{N}_{m+1}}(\boldsymbol{\xi}^{(3)}_{\mathbf{t}_m+h}-1),\quad m=1,2,....$$

Put for $|t|<1$, $|p|=1$,

$$H(t,p):=\sum_{m=1}^{\infty}t^m\mathrm{E}_{w_0}\{p^{\mathbf{h}_m}\}, \tag{4.12}$$

it then follows readily from (4.11) that for $|t|<1$, $|\hat{p}_1|=1$, $\hat{p}_2=P_2(1,\hat{p}_1)$,

$$t\hat{p}_1^{u_0}\hat{p}_2^{v_0}=\{1-t\frac{\phi_1(\hat{p}_1,\hat{p}_2)}{\hat{p}_1}\}H(t,\hat{p}_1). \tag{4.13}$$

Put

$$\boldsymbol{\sigma}^{(10)}_m:=\boldsymbol{\xi}^{(1)}_{\mathbf{t}_m}-1+\textstyle\sum_{h=1}^{\mathbf{N}_{m+1}}(\boldsymbol{\xi}^{(3)}_{\mathbf{t}_m+h}-1),\quad m=1,2,...., \tag{4.14}$$

so that $\boldsymbol{\sigma}_m^{(10)}, m=1,2,...,$ are i.i.d. stochastic variables with distribution that of $\boldsymbol{\sigma}_{10}$, cf. (3.22).
Since, cf. (4.10) and (2.22),

$$\begin{aligned} \Pr\{\mathbf{N}_m<\infty\} &= P_2(1,1) \;\; =1 \;\text{ for } \nu_3\leqslant 1, \\ & \qquad\qquad\;\; <1 \;\text{ for } \;\; >1, \\ \mathrm{E}\{\mathbf{N}_m\} &= \frac{\nu_1}{1-\nu_3} \qquad\quad \text{for } \nu_3<1, \\ &= \infty \qquad\qquad\quad\; \text{for } \;\; =1, \end{aligned} \tag{4.15}$$

it follows that for $m=1,2,...,$

$$\begin{aligned} \Pr\{|\mathbf{h}_m|<\infty\} = P_2(1,1)\, \phi_1^{m-1}(1,P_2(1,1)) &= 1 \;\text{ for } \nu_3\leqslant 1, \\ &< 1 \;\text{ for } \;>1. \end{aligned} \tag{4.16}$$

Hence

$$\begin{aligned} \Pr\{\mathbf{w}_n\in B_0, \text{i.o.}\} = P_2(1,1) &= 1 \;\text{ for } \nu_3\leqslant 1, \\ &= 0 \;\text{ for } \nu_3>1. \end{aligned} \tag{4.17}$$

Obviously, the hitting point process $\{\mathbf{h}_m, m=1,2,...\}$ is a one-dimensional random walk with one-step displacement distribution that of $\boldsymbol{\sigma}_{10}$, cf. (3.22), with $\mathbf{h}_1$ given by (4.11).
From Fluctuation Theory, cf. [9], it now follows that, cf. Lemma 3.1, with probability one for $\nu_3\leqslant 1$;

$$\begin{aligned} r_1 = \mathrm{E}\{\boldsymbol{\sigma}_{10}\}<0 &\Rightarrow \lim_{m\to\infty} \mathbf{h}_m = -\infty, \sup_{m=1,2,...} \mathbf{h}_m<\infty, \\ =0 &\Rightarrow \liminf \mathbf{h}_m = -\infty, \limsup \mathbf{h}_m = \infty, \\ >0 &\Rightarrow \inf_{m=1,2,...} \mathbf{h}_m > -\infty, \lim_{m\to\infty} \mathbf{h}_m = \infty, \end{aligned} \tag{4.18}$$

with r_1 as defined in Lemma 3.1, supposing that $\mathrm{E}\{\boldsymbol{\xi}_3^2\}<\infty$, $\mathrm{E}\{\boldsymbol{\eta}_3^2\}<\infty$ if $\mu_3=\nu_3=1$.

REMARK 4.1. Actually for the conclusion in (4.18) the existence of $\mathrm{E}\{\boldsymbol{\sigma}_{10}\}$ is not necessary, as it is seen from Fluctuation Theory; viz. by putting

$$\begin{aligned} P_1 &:= \textstyle\sum_{m=1}^{\infty} \frac{1}{m} \Pr\{\boldsymbol{\sigma}_1^{(10)}+...+\boldsymbol{\sigma}_m^{(10)}<0\}, \\ R_1 &:= \textstyle\sum_{m=1}^{\infty} \frac{1}{m} \Pr\{\boldsymbol{\sigma}_1^{(10)}+...+\boldsymbol{\sigma}_m^{(10)}>0\}, \end{aligned} \tag{4.19}$$

a basic result in Fluctuation Theory states that (4.18) holds if in (4.18):

$$\begin{aligned} r_1<0 \;\text{ is replaced by }\; & P_1=\infty, \; R_1<\infty, \\ =0 \qquad\;\; \text{,,} \qquad\;\; & \;\;\;\;\, =\infty, \;\;\;\; =\infty, \\ >0 \qquad\;\; \text{,,} \qquad\;\; & \;\;\;\;\, <\infty, \;\;\;\; =\infty. \end{aligned} \tag{4.20}$$

REMARK 4.2. From Fluctuation Theory it is wellknown that

$$Q_1 := \textstyle\sum_{m=1}^{\infty} \frac{1}{m} \Pr\{\boldsymbol{\sigma}_1^{(10)}+...+\boldsymbol{\sigma}_m^{(10)}=0\}<\infty,$$

if

$$\Pr\{\boldsymbol{\sigma}_1^{(10)}\neq 0\}>0,$$

which is a very weak condition. □

We conclude this section with some observations concerning the $\mathbf{w}_n$-process and applications of some results from Fluctuation Theory.

The $\mathbf{w}_n$-process is said to be *continuous to the west, to the south-west* and *to the south,* because per transition $\mathbf{u}_n$ and also $\mathbf{v}_n$ can decrease by at most one.

If $\nu_3>1$ then, cf. (4.17), the $\mathbf{h}_m$-process consists of only a finite number of hitting points with probability one. Consequently all states of the $\mathbf{w}_n$-process are for $\nu_3>1$ nonrecurrent, note that assumption 2.1ii implies that the state space T of the $\mathbf{w}_n$-process is irreducible.

Put for $w_0 \in S \setminus B_{00} \cup B_{10}$,

$$\mathrm{N}(w_0) := \inf_{n=1,2,\ldots} \{n : \mathbf{w}_n \in B_{00} \cup B_{10}\}, \tag{4.21}$$
$$= \infty \text{ if } \mathbf{w}_n \in S \setminus B_{00} \cup B_{10} \text{ for all } n=1,2,\ldots.$$

Because the $\mathbf{w}_n$-process is continuous to the west it follows from (4.18) that if $\nu_3 \leqslant 1$ then:

$$\Pr\{\mathrm{N}(w_0)<\infty\}=1 \quad \text{for } r_1 \leqslant 0, \tag{4.22}$$
$$<1 \quad \text{for } r_1>0.$$

From definition 3.2 it is evident that the sequence of successive hitting points $\mathbf{h}_m$ of the $\mathbf{w}_n$-process with B_{10}, with n restricted to $n=1,2,\ldots,\ \mathrm{N}(w_0)-1$, forms a b_{10}-cycle. Hence if $\nu_3 \leqslant 1$ then

$$\Pr\{\mathbf{b}_{10}<\infty\}=1 \quad \text{for } r_1 \leqslant 0. \tag{4.23}$$

It is also readily seen that if $\nu_3 \leqslant 1$, $r_1>0$, then the hitting point process $\{\mathbf{h}_m, m=1,2,\ldots\}$ contains with probability one a b_{10}-cycle consisting of an infinite number of hitting points, i.e.

$$\nu_3 \leqslant 1, r_1>0 \Rightarrow \Pr\{\exists \text{ a } b_{10}-\text{cycle with } \mathbf{b}_{10}=\infty\}=1. \tag{4.24}$$

As mentioned below (4.17), the $\mathbf{h}$-process is a discrete time parameter Markov chain and Fluctuation Theory may be applied for the analysis of the sequence of i.i.d. stochastic variables defined in (4.14), in particular the properties of the descending ladder variable of this sequence are important here. From Fluctuation Theory it readily follows that if $\nu_3 \leqslant 1$ then $r_1 \leqslant 0$ iff the descending ladder variable is finite with probability one, actually this statement is equivalent with (4.23). It further follows that the descending ladder variable has a finite first moment if $\nu_3 \leqslant 1$, $r_1<0$, whereas for $\nu_3 \leqslant 1$, $r_1=0$ this moment is infinite. Using remark 3.4, this observation leads directly to:

for $\nu_3 \leqslant 1$,

$$\mathrm{E}\{\mathbf{b}_{10}\}<\infty \text{ if } r_1<0, \tag{4.25}$$
$$=\infty \text{ if } r_1=0.$$

5. Nonrecurrence, null recurrence and positive recurrence of the $\mathbf{k}_m$- and the $\mathbf{z}_n$-process.

In this section we derive the necessary and sufficient conditions for the characterisation of the states of the state space B of the $\mathbf{k}_m$-process and similarly for the state space S of the $\mathbf{z}_n$-process with *exclusion* of the case $\mu_3=\nu_3=1$, which will be discussed in section 6.

The $\mathbf{z}_n$-process has been defined in section 2, and the $\mathbf{k}_m$-process, i.e. the process of the successive hitting points of $\mathbf{z}_n$ with the boundary B of S, has been described in section 3.

We start with a rather trivial result.

Lemma 5.1. *For $\mu_3>1$, $\nu_3>1$ both the $\mathbf{z}_n$ and the $\mathbf{k}_m$-process are nonrecurrent.*

PROOF. From one-dimensional random walk theory it follows directly, cf. (2.9)ii, that

$$\mu_3 > 1, \nu_3 > 1 \Rightarrow \Pr\{\mathbf{n}(z_0) < \infty\} < 1 \quad \text{for } z_0 \in S \setminus B, \tag{5.1}$$

$\mathbf{n}(z_0)$ being the first entrance time out from $z_0 \in S \setminus B$ into B in the $\mathbf{z}_n$-process. Since the state space S of this process is irreducible it follows that all states of S are nonrecurrent in the $\mathbf{z}_n$-process.

From assumption 2.1 it follows that

$$\Pr\{\mathbf{z}_n \in B, \text{i.o.}\} = 0,$$

so that (5.1) implies that the hitting point process $\mathbf{k}_m$, $m = 1,2,\ldots$, consists of only a finite number of points with probability one. Since the state space B of the $\mathbf{k}_m$-process is obviously irreducible it follows that the $\mathbf{k}_m$-process is nonrecurrent if $\mu_3 > 1$, $\nu_3 > 1$. □

REMARK 5.1. Lemma 5.1 shows that in our further analysis it suffices to consider only the cases with

$$\mu_3 \leqslant 1 \quad \textit{or} \quad \nu_3 \leqslant 1. \tag{5.2}$$

The concepts of positive- and of null recurrence are based on the first moments of the return time of a state. It is for this reason that we have to make a distinction between the *clocks* for the $\mathbf{z}_n$- and for the $\mathbf{k}_m$-process. In both these processes time is measured in units of their transitions since the start of the $\mathbf{z}_n$-process, which implies that at the hitting epoch $\mathbf{t}_j$, cf. (3.1), the $\mathbf{z}_n$-*clock* stands at $\mathbf{t}_j$, whereas the $\mathbf{k}_m$-*clock* stands then at j. □

REMARK 5.2. Note that entrance- and return times for the $\mathbf{z}_n$-process are measured on the $\mathbf{z}_n$-clock, those of the $\mathbf{k}_m$-process on the $\mathbf{k}_m$-clock. □

DEFINITION 5.1. A recurrent state of S in the $\mathbf{z}_n$-process is positive recurrent if its average return time on the $\mathbf{z}_n$-clock is finite, null recurrent if it is infinite on this clock; similarly for a recurrent state of B in the $\mathbf{k}_m$-process on the $\mathbf{k}_m$-clock.

A sequence of hitting points all belonging to a same b_{10} - cycle of the reflecting $\mathbf{z}_n$-process may be considered to form a part of a realization of the hitting point process of the semi-bounded $\mathbf{w}_n$-process studied in section 4. It is actually that part of the hitting point process of the semi-bounded $\mathbf{w}_n$-process ($\mathbf{w}_n = (\mathbf{u}_n, \boldsymbol{\nu}_n)$), starting in $w_0 \in S \setminus B$, untill the first moment at which $\mathbf{u}_n$ becomes zero; and at this moment the $\mathbf{z}_n$-process starts a b_{00} - or b_{01}-cycle.

For a b_{00}-cycle in the $\mathbf{z}_n$-process we obviously have for $j = 1,2,\ldots$,

$$\Pr\{\mathbf{b}_{00} = j\} = [\Pr\{\mathbf{M}^{(32)}(\eta_0) < \mathbf{M}^{(31)}(\xi_0)\}]^j \Pr\{\mathbf{M}^{(32)}(\eta_0) \neq \mathbf{M}^{(31)}(\xi_0)\},$$

so that

$$\Pr\{\mathbf{b}_{00} < \infty\} = 1, \qquad \mathrm{E}\{\mathbf{b}_{00}\} < \infty. \tag{5.3}$$

A b_{10}-cycle of the $\mathbf{z}_n$-process when considered as a realization of the $\mathbf{w}_n$-process is a realization of a random walk, and as such it will consists of a finite number of hitting points if and only if the descending ladder variable of this random walk is finite with probability one. In a b_{10}-cycle of the $\mathbf{z}_n$-process the distances between successive hitting points are independent stochastic variables but not identically distributed. The distribution of two of its successive points, say $\mathbf{k}_m^{(1)}$ and $\mathbf{k}_{m+1}^{(1)}$ is that of, cf. (4.14),

$$\{\xi_{\mathbf{t}_m}^{(1)} - 1 + \Sigma_{m=1}^{\mathbf{n}_{m+1}} (\xi_{\mathbf{t}_m} - 1)\}(\mathbf{M}^{(32)}(\boldsymbol{\eta}_{\mathbf{t}_m}^{(1)}) < \mathbf{M}^{(31)}(\mathbf{k}_m^{(1)} + \xi_{\mathbf{t}_m}^{(1)} - 1)), \tag{5.4}$$

with, cf. (3.7),

$$\mathbf{n}_{m+1} = \min(\mathbf{M}^{(32)}(\boldsymbol{\eta}_{\mathbf{t}_m}^{(1)}),\ \mathbf{M}^{(31)}(\mathbf{k}_m^{(1)} + \xi_{\mathbf{t}_m}^{(1)} - 1)).$$

From (2.17) it follows that

$$\Pr\{\mathbf{M}^{(31)}(k) \geqslant k\} = 1,$$

and consequently for $k \to \infty$ and $\nu_3 \leqslant 1$,

$$\min(\mathbf{M}^{(32)}(\boldsymbol{\eta}^{(1)}_{\mathbf{t}_m}), k) = \mathbf{M}^{(32)}(\boldsymbol{\eta}^{(1)}_{t_m}) \text{ with prob. } 1,$$

since $\mathbf{M}^{(32)}(\boldsymbol{\eta}^{(1)}_{\mathbf{t}_m})$ is finite with probability one; and further for $k \to \infty$ and $\nu_3 < 1$,

$$\mathrm{E}\{\min(\mathbf{M}^{(32)}(\boldsymbol{\eta}^{(1)}_{t_m}), k)\} = E\{\mathbf{M}^{(32)}(\boldsymbol{\eta}^{(1)}_{\mathbf{t}_m})\} = \frac{\nu_1}{1-\nu_3}. \tag{5.5}$$

Consequently, it follows that if $\mathbf{k}^{(1)}_m$, $\mathbf{k}^{(1)}_{m+1}$ and $\mathbf{k}^{(1)}_{m+2}$ are successive hitting points belonging to a same b_{10}-cycle of the $\mathbf{z}_n$-process then for $\nu_3 < 1$,

$$\mathbf{k}^{(1)}_{m+2} - \mathbf{k}^{(1)}_{m+1} \text{ and } \mathbf{k}^{(1)}_{m+1} - \mathbf{k}^{(1)}_m,$$

are asymptotically independent, i.e. for $\mathbf{k}^{(1)}_m \to \infty$, and are identically distributed, and so we have, cf. (3.23), for $\mathbf{k}^{(1)}_m \to \infty$ and $\nu_3 < 1$,

$$\mathrm{E}\{\mathbf{k}^{(1)}_{m+1} - \mathbf{k}^{(1)}_m\} = r_1 = \mu_\cdot - 1 + \nu_1 \frac{\mu_3 - 1}{1 - \nu_0}. \tag{5.6}$$

Comparison of the b_{10}-cycles of the reflecting $\mathbf{z}_n$-process with the realizations of the semi-bounded $\mathbf{w}_n$-process as described above, shows by using (5.6) that the descending ladder variable of the hitting point process $\mathbf{h}_m$ of the semi-bounded $\mathbf{w}_n$-process determines the character of the b_{10}-cycles of the $\mathbf{z}_n$-process. So we have, by using (4.18) and (4.25) and remark 3.4, for a b_{10}-cycle of the $\mathbf{z}_n$-process if $\nu_3 < 1$,

$$\begin{array}{llll} r_1 & <0 \;\Rightarrow & \Pr\{\mathbf{b}_{10} < \infty\} = 1, & \mathrm{E}\{\mathbf{b}_{10}\} < \infty, \\ & =0 \;\Rightarrow & \qquad\qquad = 1, & \qquad = \infty, \\ & >0 \;\Rightarrow & \qquad\qquad < 1. & \end{array} \tag{5.7}$$

Analogous results hold for the b_{01}-cycles of the reflecting $\mathbf{z}_{10}$-process.

Theorem 5.1 below formulates the necessary and sufficient conditions for the $\mathbf{k}_m$-process (on the $\mathbf{k}_m$-clock) and for the $\mathbf{z}_n$-process (on the $\mathbf{z}_n$-clock) to be nonrecurrent, null recurrent or positive recurrent. We shall apply the following

NOTATION 5.1

$\mathbf{k}_m = -$:= *the* $\mathbf{k}_m$*-process is nonrecurrent,*

$\quad = 0$:= *the* $\mathbf{k}_m$*-process is null recurrent,*

$\quad = +$:= *the* $\mathbf{k}_m$*-process is positive recurrent,*

and similarly with $\mathbf{k}_m$ replaced by $\mathbf{z}_n$.

THEOREM 5.1.

i.	$\mu_3>1, \quad \nu_3>1$			$\Rightarrow$	$\mathbf{k}_m=-,$	$\mathbf{z}_n=-;$	(5.7)
ii.	$\mu_3>1, \nu_3=1$			$\Rightarrow$	$=-,$	$=-;$	
iii.	$\mu_3>1, \nu_3<1$						
	and	$r_1>0$		$\Rightarrow$	$=-,$	$=-,$	
		$=0$		$\Rightarrow$	$=0,$	$=0,$	
		<0		$\Rightarrow$	$=+,$	$=+;$	
iv.	$\mu_3=1, \nu_3>1$			$\Rightarrow$	$=-,$	$=-;$	
v.	$\mu_3=1, \nu_3=1$						
			see theorems 6.1 and 6.2	$\Rightarrow$	$=-,$	$=-,$	
			for the conditions,	$\Rightarrow$	$=0,$	$=0,$	
				$\Rightarrow$	$=+,$	$=0,$	
				$\Rightarrow$	$=+,$	$=+;$	
vi.	$\mu_3=1, \nu_3<1$						
	and	$r_1>0$		$\Rightarrow$	$=-,$	$=-,$	
		$=0$		$\Rightarrow$	$=0,$	$=0,$	
		<0		$\Rightarrow$	$=+,$	$=+;$	
vii.	$\mu_3<1, \nu_3>1$						
	and	$r_2>0$		$\Rightarrow$	$=-,$	$=-,$	
		$=0$		$\Rightarrow$	$=0,$	$=0,$	
		<0		$\Rightarrow$	$=+,$	$=+;$	
viii.	$\mu_3<1, \nu_3=1$						
	and	$r_2>0$		$\Rightarrow$	$=-,$	$=-,$	
		$=0$		$\Rightarrow$	$=0,$	$=0,$	
		<0		$\Rightarrow$	$=+,$	$=+;$	
ix.	$\mu_3<1, \nu_3<1$						
	and	$r_1>0$	or $r_2>0$	$\Rightarrow$	$=-,$	$=-,$	
		$r_1\leqslant 0,$	$r_2=0$ or $r_1=0, r_2\leqslant 0$	$\Rightarrow$	$=0,$	$=0,$	
		$r_1<0,$	$r_2<0$	$\Rightarrow$	$=+,$	$=+.$	

PROOF. Because the state space B of the $\mathbf{k}_m$-process and S of the $\mathbf{z}_n$-process are both irreducible and because the $\mathbf{k}_m$-process is part of the $\mathbf{z}_n$-process it follows immediately that

$$\mathbf{k}_m\text{-process is nonrec.} \Leftrightarrow \mathbf{z}_n\text{-process is nonrec.} \tag{5.8}$$

$$\mathbf{k}_m\text{-process is null rec.} \Rightarrow \mathbf{z}_n\text{-process is null rec.}$$

ad (5.7)i. This statement is equivalent with lemma 5.1.

ad (5.7)ii. In this case the $\mathbf{z}_n$-process hits $B_{00} \cup B_{10}$, i.o. with probability one, which implies that the $\mathbf{k}_m$-process consists of an infinite number of points, so its realisation consists of a sequence of b_{10}-, b_{00}- and b_{01}-cycles, cf. definition 3.2.

In our analysis we may and do discard the b_{00}-cycles, since, cf. (5.3),

$$\Pr\{\mathbf{b}_{00}=\infty\}=0. \tag{5.8}$$

From (3.23)ii and $\mu_3>1$, $\nu_3=1$ it follows that $r_1=\infty$. So the results concerning the $\mathbf{h}_m$-process, see the preceding section in particular (4.24), imply that the $\mathbf{k}_m$-process contains with probability one b_{10}-cycles. By noting that the descending ladder variable of the $\mathbf{h}_m$-process with $\mu_3>1$, $\nu_3=1$ is finite with probability less than one it follows that $\Pr\{\mathbf{b}_{10}=\infty\}>0$. Consequently, the $\mathbf{k}_m$-process is nonrecurrent, and (see (5.8)) so is the $\mathbf{z}_n$-process.

ad (5.7)iii. $\nu_3<1$ implies that the $\mathbf{z}_n$-process hits $B_{00}\cup B_{10}$, i.o. with probability one and, cf. (3.6), for finite $z_0\in S$,

$$\Pr\{\mathbf{n}(z_0)<\infty\}=1, \quad \mathrm{E}\{\mathbf{n}(z_0)\}<\infty. \tag{5.9}$$

Hence again we may consider the descending ladder variable of the $\mathbf{h}_m$-process which is for the present case finite with probability one if and only if $r_1\leqslant 0$; for $r_1<0$ it has a finite first moment whereas this moment is infinite for $r_1=0$, so by using remark 3.4 and (4.25),

$$\begin{aligned} r_1>0 &\Rightarrow \Pr\{\mathbf{b}_{10}<\infty\}<1,\\ \leqslant 0 &\Rightarrow \qquad\qquad\quad =1,\\ <0 &\Rightarrow \quad \mathrm{E}\{\mathbf{b}_{10}\}<\infty,\\ =0 &\Rightarrow \qquad\qquad\quad =\infty. \end{aligned} \tag{5.10}$$

Hence the $\mathbf{k}_m$-process is nonrecurrent iff $r_1>0$, and so is the $\mathbf{z}_n$-process.

Evidently (5.9) and (5.10) imply that the $\mathbf{k}_m$-process is null recurrent for $r_1=0$, and that it is positive recurrent for $r_1<0$.

Since $\mathrm{E}\{\mathbf{n}(z_0)\}<\infty$ it follows from (3.7) that

$$\mathrm{E}\{\mathbf{t}_1\}<\infty \text{ and } \mathrm{E}\{\mathbf{t}_{m+1}-\mathbf{t}_m\}<\infty, \quad m=1,2,..., \tag{5.11}$$

which implies that the first moments of return times when measured on the $\mathbf{z}_n$-clock and on the $\mathbf{k}_m$-clock have a finite ratio. Hence the $\mathbf{z}_n$-process is also null recurrent for $r_1=0$, and positive recurrent for $r_1<0$.

ad (5.7)iv. This case is symmetric with (5.7)ii.

□

ad (5.7)vi. In this case again (5.9) applies and by considering again the descending ladder variable of the $\mathbf{h}_m$-process the statements follow as for the case (5.7)iii.

ad (5.7)vii. This case is the symmetric one of (5.7)iii.

ad (5.7)viii. Symmetric with (5.7)vi.

ad (5.7)ix. Again the $\mathbf{z}_n$-process hits B i.o. with probability one and for $z_0\in S\setminus B$, cf. theorem 2.3,

$$\mathrm{E}\{\mathbf{n}(z_0)\}<\infty. \tag{5.12}$$

For $r_1>0$ the present statement is proved as in ad. (5.7)iii; for $r_2>0$ by symmetry. From the properties of the descending ladder variable of the $\mathbf{h}_m$-process we have for $r_1<0$, $r_2=0$ (use again a symmetry argument)

$$\Pr\{\mathbf{b}_{10}<\infty\}=\Pr\{\mathbf{b}_{01}<\infty\}=1, \quad \mathrm{E}\{\mathbf{b}_{10}\}<\infty, \quad \mathrm{E}\{\mathbf{b}_{01}\}=\infty. \tag{5.13}$$

The relation (5.10) implies that the $\mathbf{k}_m$-process is null recurrent and hence the $\mathbf{z}_n$-process also. By symmetry the same conclusion holds for $r_1=0$, $r_2<0$.

Since for $r_1<0$ the descending ladder index of the $\mathbf{h}_m$-process has a finite first moment, it follows by using a symmetry argument that for $r_1<0$, $r_2<0$,

$$\Pr\{\mathbf{b}_{10}<\infty\}=\Pr\{\mathbf{b}_{01}<\infty\}=1,\quad E\{\mathbf{b}_{10}\}<\infty,\quad E\{\mathbf{b}_{01}\}<\infty. \tag{5.14}$$

Consequently, the $\mathbf{k}_m$-process is positive recurrent, so the $\mathbf{z}_n$-process is recurrent. But (5.12) implies that in the present case

$$E\{\mathbf{t}_1\}=\infty,\quad E\{\mathbf{t}_{m+1}-\mathbf{t}_m\}<\infty. \tag{5.15}$$

Consequently for $r_1<0$, $r_2<0$ the $\mathbf{z}_n$-process is positive recurrent.

We conclude this section with some observations concerning generalisations of theorem 5.1.

REMARK 5.3. Theorem 5.1 has been proved for two-dimensional random walks which are continuous to the west, south-west and south (see below remark 4.2), i.e. the components $\xi_n^{(1)}-1$, $\eta_n^{(2)}-1$, $\xi_n^{(3)}-1$ and $\eta_n^{(3)}-1$ of the one-step displacement vectors, cf. (2.3), all have a support contained in $\{-1,0,1,2,...\}$ and their first moments do exist. The statements of theorem 5.1 have been listed according to the signs and zero values of the horizontal- and vertical drift μ_3-1 and ν_3-1. It is readily seen that with hardly any modifications theorem 5.1 can be proved along the same lines if the variables just mentioned have supports contained in $\{...,-1,0,1,...\}$, assuming, however, that their first moments exist. □

REMARK 5.4. By the same technique as used above the analogon of theorem 5.1 may be proved for the case where the variables $\xi_n^{(1)}-1$, $\eta_n^{(2)}-1$, $\xi_n^{(3)}-1$, $\eta_n^{(3)}-1$ have supports contained in $\{...,-1,0,1,...\}$ and where their first moments do not exist, as it may be seen from the following observations.
i. The analysis of the structure of the $\mathbf{z}_n$-process is based on that of its hitting point process $\{\mathbf{k}_m, m=1,2,...\}$ with the boundary B of S.
ii. The analysis of the structure of the $\mathbf{k}_m$-process is based on that of the hitting point process $\{\mathbf{h}_m, m=1,2,...\}$ of the semi-bounded random walk, see section 4.
iii. This $\mathbf{h}_m$-process is a one-dimensional random walk and its analysis has been based on results of Fluctuation Theory. Actually the results of Fluctuation Theory do not require left-continuity and the existence of the first moment of the one-step displacement vector of the $\mathbf{h}_m$-process, see remark 4.1, and consequently this is also not required for the $\mathbf{k}_m$-process.
iv. The relation between the $\mathbf{z}_n$- and the $\mathbf{k}_m$- process is based on the entrance time $\mathbf{n}(z_0)$ out from $z_0\in S\setminus B$ into B, cf. (2.9)ii and (3.6), in particular on the entrance time $\mathbf{M}_{x_0}^{(31)}$ in the component process $\mathbf{k}_n$ of $\mathbf{z}_n$ and on $\mathbf{M}_{y_0}^{(32)}$, see definition 2.3. This $\mathbf{k}_n$-component process has $\xi_n^{(3)}-1$ as its one-step displacement vector. In terms of Fluctuation Theory $\mathbf{M}_{x_0}^{(31)}$ is a descending ladder variable, and hence the entrance properties of the $\mathbf{z}_n$-process into B can be fully described in terms of properties of the ladder variables of the $\mathbf{k}_n$- and $\mathbf{y}_n$-component processes, i.e. in terms of the distribution of the one-step displacement vector (ξ_3,η_3).

6. THE CASE WITH ZERO DRIFT

In this section we shall discuss the conditions for non-recurrence, null- and positive recurrence of the reflecting $\mathbf{z}_n$-process an its hitting point process $\mathbf{k}_m$ for the case with zero drift i.e.

$$\mu_3 = \nu_3 = 1; \tag{6.1}$$

it will always be assumed that

$$E\{\xi_3^2\}<\infty,\quad E\{\eta_3^2\}<\infty. \tag{6.2}$$

Define

$$\alpha_1 := \mathrm{E}\{(\xi_3-1)^2\},\quad \alpha_{12} := \mathrm{E}\{(\xi_3-1)\},\ \alpha_2 := \mathrm{E}\{(\eta_3-1)^2\}, \tag{6.3}$$

$$\alpha_{12} < \alpha_1\alpha_2. \tag{6.4}$$

The hitting point process $\{\mathbf{k}_m, m=1,2,\cdots\}$ see section 3, is an aperiodic Markov chain with irreducible state space, cf. remark 2.2. Hence the following limits exist and are independent of z_0, cf. (3.12), for $|p|\leqslant 1$,

$$K_1(p) := \lim_{m\to\infty} \mathrm{E}\{p^{\mathbf{k}_m^{(1)}}(\mathbf{k}_m^{(1)}\geqslant 1,\ \mathbf{k}_m^{(2)}=0)|\mathbf{z}_0=z_0\}, \tag{6.5}$$

$$K_0 := \lim_{m\to\infty} \mathrm{E}\{\mathbf{k}_m^{(1)}=0,\ \mathbf{k}_m^{(2)}=0)|\mathbf{z}_0=z_0\},$$

$$K_2(p) := \lim_{m\to\infty} E\{p^{\mathbf{k}_m^{(2)}}(\mathbf{k}_m^{(1)}=0,\ \mathbf{k}_m^{(2)}\geqslant 1)|\mathbf{z}_0=z_0\}.$$

So by multiplying (3.13) by $1-t$ and letting $t\to 1$ it follows from a wellknown Abel theorem for power series that

$$[1-\frac{\hat{\phi}_1}{\hat{p}_1}]K_1(\hat{p}_1)+[1-\hat{\phi}_0]K_0+[1-\frac{\hat{\phi}_2}{\hat{p}_2}]K_2(\hat{p}_2)=0, \tag{6.6}$$

with

$$\hat{\phi}_j = \phi_j(\hat{p}_1,\hat{p}_2),\ j=0,1,2,$$

and $(\hat{p}_1,\hat{p}_2)$ a zero tuple of $Z(1-,p_1,p_2)$, cf. definition 3.1. We choose here the zero tuple, cf. (2.15) and (2.22),

$$\hat{p}_2 = P_2(\hat{p}_1) \equiv P_2(1,\hat{p}_2),\ |\hat{p}_1|=1, \tag{6.7}$$

note that (6.1) implies

$$P_2(1,\hat{p}_1)\to 1 \text{ for } \hat{p}_1\to 1. \tag{6.8}$$

Divide (6.6) with $(\hat{p}_1|=1$ by $1-\hat{p}_1$ and let $\hat{p}_1\to 1$, this leads to

$$[1-\mu_1-\nu_1\frac{dP_2(p_1)}{dp_1}|_{p_1=1}]K_1(1)+[(1-\nu_2)\frac{dP_2(p_1)}{dp_1}|_{p_1=1}-\mu_2]K_2(1)- \tag{6.9}$$

$$[\mu_0+\nu_0\frac{dP_2(p_1)}{dp_1}|_{p_1=1}]K_0=0.$$

By using (6.2) it is readily shown that $P_2(e^{i\phi})$, $\phi\in[-\pi_0,\pi]$ possesses a derivative with respect to $\phi,\phi\neq 0$, and that this derivative has a limit for $\phi\downarrow 0$ and also for $\phi\uparrow 0$. It has been shown in [17] (lemma 3.3) that

$$\frac{dP_2(1;p_1)}{dp_1}|_{p_1=1} = -\alpha_{12}\pm\frac{\sqrt{\alpha_{12}-\alpha_1\alpha_2}}{\alpha_2}. \tag{6.10}$$

The righthand side in (6.10) is obviously complex, cf. (6.4), so the imaginary part of (6.9) should be zero, this leads to

$$\nu_1 K_1(1)+(\nu_2-1)K_2(1) = -\nu_0 K_0, \tag{6.11}$$

$$(\mu_1-1)K_1(1)+\mu_2 K_2(1) = -\mu_0 K_0.$$

REMARK 6.1. The second relation in (6.11) follows from the first one by symmetry; equating to zero the real part of (6.9) leads to a relation which depends linearly on those of (6.11). In remark 6.2 another derivation of (6.11) will be presented. □

THEOREM 6.1. *For* $\mu_3=1$, $\nu_3=1$ *and* $E\{\xi_3^2\}<\infty$, $E\{\eta_3^2\}<\infty$ *(but cf. also corollary 6.1 and remark 6.2 below)*

$$
\begin{array}{llllll}
\text{i.} & \mu_1\geqslant 1, \text{ or } \quad \nu_2\geqslant 1 & & \Rightarrow & \mathbf{k}_m=-, & \mathbf{z}_n=-; \\
\text{ii.} & \mu_1<1,\ \nu_2<1 \quad \text{and} & & & & \\
 & (1-\mu_1)(1-\nu_2) < \nu_1\mu_2 & & \Rightarrow & =-, & =-, \\
 & \qquad = & & \Rightarrow & =0, & =0, \\
 & \qquad > & & \Rightarrow & =+, & =0\vee+, \\
 & \qquad > \text{ and } & \alpha_{12}\geqslant 0 & \Rightarrow & =+, & =0.
\end{array}
$$

PROOF. Obviously we have

$$1\geqslant K_0\geqslant 0,\ 1\geqslant K_1(1)\geqslant 0,\ 1\geqslant K_2(1)\geqslant 0, \tag{6.13}$$

and

$$\mathbf{k}_m\text{-process recurrent} \Rightarrow K_0+K_1(1)+K_2(1)=1. \tag{6.14}$$

If the condition in (6.12)i applies then obviously the null solution is the only solution of (6.11) so $\mathbf{k}_m$-process is nonrecurrent and also the $\mathbf{z}_n$-process, cf. (5.8), so (6.12)i has been proved.
Put

$$d:=(1-\mu_1)(1-\nu_2)-\mu_2\nu_1, \tag{6.15}$$

$$f:=\mu_0(1+\nu_1-\nu_2)+\nu_0(1+\mu_2-\mu_1)+d.$$

then

$$\mu_1<1,\ \nu_2<1 \text{ and } d\neq 0 \Rightarrow f\neq 0. \tag{6.16}$$

For $\mu_1<1$, $\nu_2<1$, $d\neq 0$, the equation in (6.10) and the norming condition in (6.13) have a unique solution viz.

$$K_0 = \frac{d}{f}. \tag{6.17}$$

$$K_1(1)=[\nu_0\mu_2+\mu_0(1-\nu_2)]f^{-1},\ K_2(1)=[\mu_0\nu_1+\nu_0(1-\mu_1)]f^{-1}.$$

If $\mu_1<1$, $\nu_2<1$ and $d>0$ then (6.17) implies

$$1>K_0>0,\quad 1>K_1(1)>0,\quad 1>K_2(1)>0,$$

so, since $K_0>0$, the $\mathbf{k}_m$-process is positive recurrent and consequently the $\mathbf{z}_n$-process is recurrent. From theorem 2.3iii and (6.3) it is seen that the first moment at the first entrance time of any point of $S\setminus B$ into B is infinite if and only if $\alpha_{12}\geqslant 0$, and so in the present case the $\mathbf{z}_n$-process cannot be positive recurrent if $\alpha_{12}\geqslant 0$; hence the third and fourth statement of (6.12) have been proved.

If $\mu_1<1$, $\nu_2<1$, $d<0$ then it is seen from (6.15) and (6.17) that $K_0>1$ and consequently (6.13) is violated; this proves the first statement of (6.12)ii. If $\mu_1<1$, $\nu_2<1$ and $d=0$ then (6.11) can only have a nonnull solution if

$$\frac{\nu_1}{\mu_1-1} = \frac{\nu_2-1}{\mu_2} = \frac{\nu_0}{\mu_0} \quad \text{or } K_0 = 0. \tag{6.18}$$

The first alternative in (6.18) contradicts $\mu_1<1$, $\nu_2<1$; the second alternative in conjunction with the norming condition leads to

$$K_0= 0, \tag{6.19}$$

$$K_1(1)= \frac{\mu_2}{1+\mu_2-\mu_1} = \frac{1-\nu_2}{1+\nu_1-\nu_2},$$

$$K_2(1) = \frac{\nu_1}{1+\nu_1-\nu_2} = \frac{1-\mu_1}{1+\mu_2-\mu_1};$$

and so the $\mathbf{K}_m$-process is null recurrent, which implies that the $\mathbf{z}_n$-process is also null recurrent; the second statement of (6.12)ii has been proved. □

For the further analysis of the $\mathbf{z}_n$-process we start from

$$\mu_1<1,\ \nu_2<1,\quad d\equiv(1-\mu_1)(1-\nu_2)-\nu_1\mu_2>0,\quad \alpha_{12}<0, \tag{6.20}$$

so that the hitting point process of the reflecting $\mathbf{z}_n$-process is positive recurrent. Let $(\mathbf{h}_1, \mathbf{h}_2)$ be a stochastic vector with distribution the stationary distribution of this hitting point process; so with probability one

$$\mathbf{h}_1\mathbf{h}_2 = 0, \tag{6.21}$$

and for $|p|\leqslant 1$, c.f. (6.5) ,

$$\begin{aligned} K_1(p) &= \mathrm{E}\{p^{\mathbf{h}_1}(\mathbf{h}_1>0,\mathbf{h}_2=0)\},\quad K_2(p)=\mathrm{E}\{p^{\mathbf{h}_2}(\mathbf{h}_2>0,\mathbf{h}_1=0)\}, \\ K_0 &= \mathrm{E}\{(\mathbf{h}_1=0,\ \mathbf{h}_2=0)\}. \end{aligned} \tag{6.22}$$

Obviously K_0, $K_1(1)$ and $K_2(1)$ are given by (6.17), and $K_1(p), K_0$ and $K_2(p)$ satisfy, cf. (6.6) and (6.7), for $|p|=1$,

$$[1-\frac{\phi_1(p,P)}{p}]K_1(p) + [1-\phi_0(p,P)]K_0+[1-\frac{\phi_2(p,P)}{P}]K_2(P), \tag{6.23}$$

with

$$P \equiv P_2(1,p),\quad |p|=1. \tag{6.24}$$

For the investigation of the conditions which guarantee that the $\mathbf{z}_n$-process is positive recurrent it will turn out that we need information concerning

$$\mathrm{E}\{\mathbf{h}_1\} = \frac{d}{dp}K_1(p)|_{p=1}\ ,\quad \mathrm{E}\{\mathbf{h}_2\} = \frac{d}{dp}K_2(p)|_{p=1}. \tag{6.25}$$

To obtain this information divide (6.23) by $1-p$, differentiate with respect to p and let $p\Rightarrow 1$. This leads to:

$$\begin{aligned} &K_1(1)\frac{d^2}{dp^2}\frac{\phi_1}{p}|_{p=1} + K_0\frac{d^2}{dp^2}\phi_0|_{p=1} + K_2(1)\frac{d^2}{dp^2}\frac{\phi_2}{P}|_{p=1} + \\ &2\{\frac{d}{dp}\frac{\phi_1}{p}|_{p=1}\}\frac{d}{dp}K_1(p)|_{p=1} + 2\{\frac{d}{dp}\frac{\phi_2}{P}|_{p=1}\}\frac{d}{dp}K_2(P)|_{p=1}=0. \end{aligned} \tag{6.26}$$

Put

$$a_1:=-\frac{\alpha_{12}}{\alpha_2},\quad b_1:=\frac{\{\alpha_1\alpha_2-\alpha_{12}^2\}}{\alpha_2^{1/2}},\quad a_2=-\frac{\alpha_{12}}{\alpha_1}, \tag{6.27}$$

so that, cf. (6.10),

$$\frac{dP}{dp}|_{p=1} = a_1\pm i b_1. \tag{6.28}$$

We have

$$\begin{aligned} &\{\frac{d}{dp}\cdots\}|_{p=1} = \{[\frac{\partial}{\partial p}+\frac{dP}{dp}\frac{\partial}{\partial P}]\cdots\}_{p=1}=\{[\frac{\partial}{dp}+(a_1\pm i b_1)\frac{\partial}{\partial P}]\cdots\}_{p=1}, \\ &\{\frac{d^2}{dp^2}\cdots\}_{p=1}=\{[\frac{\partial^2}{\partial p^2}+2a_1\frac{\partial^2}{\partial p\partial P}+(a_1^2-b_1^2)\frac{\partial^2}{\partial P^2}\pm 2ib[\frac{\partial^2}{\partial p\partial P}+a_1\frac{\partial^2}{\partial P^2}]]\cdots\}_{p=1}. \end{aligned} \tag{6.29}$$

By using (6.29) it is seen that the imaginary part of (6.26) should be zero, this leads to

$$\mathrm{E}\{\mathbf{h}_1\}\frac{\partial}{dp}\frac{\phi_2}{P}\Big|_{p=1} + \mathrm{E}\{\mathbf{h}_2\}[\frac{\partial}{\partial p}\frac{\phi_2}{P}+2a_1\frac{\partial}{\partial P}\frac{\phi_2}{P}]_{p=1} = -B_1, \tag{6.30}$$

$$B_1:=K_0\{\frac{\partial^2}{\partial p\partial P}+a_1\frac{\partial^2}{\partial P^2}\}\phi_0|_{p=1}\}+$$

$$K_1(1)\{\frac{\partial^2}{\partial p\partial P}+a_1\frac{\partial^2}{\partial P^2}\}\frac{\phi_1}{p}|_{p=1}\}+K_2(1)\{\frac{\partial^2}{\partial p\partial P}+a_1\frac{\partial^2}{\partial P^2}\}\frac{\phi_2}{P}|_{p=1}\}.$$

Assume that

$$\mathrm{E}\{\xi_j^2\}<\infty,\ \mathrm{E}\{\eta_j^2\}<\infty,\quad j=0,1,2, \tag{6.31}$$

then from (6.30) we obtain

$$\nu_1\mathrm{E}\{\mathbf{h}_1\}+\{\mu_2+2a_1(1-\nu_2)\}\mathrm{E}\{\mathbf{h}_2\}=-B_1, \tag{6.32}$$

$$\{\nu_1-2a_2(1-\mu_1)\}\ \mathrm{E}\{\mathbf{h}_1\}+\mu_2\mathrm{E}\{\mathbf{h}_2\}=-B_2,$$

with

$$B_1=K_0[\mathrm{E}\{\xi_0\eta_0\}-a_1\mathrm{E}\{\eta_0(\eta_0-1)\}]+ \tag{6.33}$$

$$K_1(1)[\mathrm{E}\{(\xi_1-1)\eta_1\}-a_1\mathrm{E}\{\eta_1(\eta_1-1)\}]+K_2(1)[\mathrm{E}\{\xi_2(\eta_2-1)\}-a_1\mathrm{E}\{(\eta_2-1)\}\{(\eta_2-2)\}],$$

$$B_2=K_0[\mathrm{E}\{\xi_0\eta_0\}-a_2\mathrm{E}\{\xi_0(\xi_0-1)\}]+$$

$$K_1(1)\mathrm{E}\{\eta_1(\xi_1-1)\}-a_2\mathrm{E}\{(\xi_1-1)(\xi_1-2)\}]+K_2(1)[\mathrm{E}\{\eta_2-1)\xi_2\}-a_2\mathrm{E}\{\xi_2(\xi_2-1)\}].$$

It is noted that the first equation in (6.32) follows from (6.30), the second one in (6.32) is obtained from the first one by symmetry, and taking the real part of (6.26) equal to zero leads to a relation which depends linearly on those in (6.32).

The solution of (6.32) reads, cf. (6.20),

$$\nu_1\ \mathrm{E}\{\mathbf{h}_1\}=\frac{B_1-B_2(1-2a_1\frac{1-\nu_2}{\mu_2})}{(1-2a_2\frac{1-\mu_1}{\nu_1})(1-2a_1\frac{1-\nu_2}{\mu_2})-1}, \tag{6.34}$$

$$\mu_2\ \mathrm{E}\{\mathbf{h}_2\}=\frac{B_2-B_1(1-2a_2\frac{1-\mu_1}{\nu_1})}{(1-2a_2\frac{1-\mu_1}{\nu_1})(1-2a_1\frac{1-\nu_2}{\mu_2})-1}.$$

THEOREM 6.2. *For* $\mu_3=1,\nu_3=1$, $\mathrm{E}\{\xi_3^2\}<\infty$, $\mathrm{E}\{\eta_3^2\}<\infty$ *the reflecting* $\mathbf{z}_n$*-process is positive recurrent if the following three conditions apply:*

i. $\mu_1<1,\ \nu_2<1,\quad d\equiv(1-\mu_1)(1-\nu_2)-\mu_2\nu_1>0,$ (6.35)

ii. $\alpha_{12}<0,$

iii. $\mathrm{E}\{\xi_j^2\}<\infty,\ \mathrm{E}\{\eta_j^2\}<\infty,\ j=0,1,2,$

iv. *the linear equations* (6.32) *have a positive, finite solution*;

if (6.35) i and ii do apply but (6.35) iii or iv do not then the $\mathbf{z}_n$*-process is null recurrent.*

PROOF. From theorem 6.1 it is seen that the conditions (6.35) i and ii are required.

Out from $\{0,0\}$ the first moment of the first return time into B on the $\mathbf{z}_n$ -clock, cf. remark 5.2, is given by, cf. theorem 2.3 iii and (6.3),

$$1+\frac{E\{\xi_0\eta_0\}}{-\alpha_{12}}, \tag{6.36}$$

which is finite if (6.35) ii and iii hold.

Out from a point, say $\mathbf{h}_1$, of B_{10} the first moment of the first return time to B is given by, cf. (2.3) ii and theorem 2.3. iii,

$$1+\frac{1}{-\alpha_{12}}E\{(\mathbf{h}_1+\xi_1-1)\eta_1|\mathbf{h}_1>0,\ \mathbf{h}_2=0\}= \tag{6.37}$$

$$1+\frac{1}{-\alpha_{12}}[\nu_1 E\{\mathbf{h}_1|\mathbf{h}_1>0,\ \mathbf{h}_2=0\}+E\{\xi_1\eta_1\}-\nu_1],$$

the probability of the conditioning event in (6.37) being given by $K_1(1)$; note that $\mathbf{h}_1$ and η_1 are independent.

If the $\mathbf{k}_m$-process is positive recurrent then necessarily $E\{\mathbf{h}_1|\mathbf{h}_1>0,\mathbf{h}_2=0\}\geqslant 1$ and so from (7.43)i, ii and iii it follows that the right hand side of (6.37) is finite if and only if $0\leqslant E\{\mathbf{h}_1\}<\infty$. Similarly for the first moment of the first return time to B out from a state of B_{01} with $0\leqslant E\{\mathbf{h}_2\}<\infty$. By definition $E\{\mathbf{h}_1\}$ and $E\{\mathbf{h}_2\}$ are nonnegative and if the $\mathbf{k}_m$-process is positive recurrent they are positive, possibly infinite. So if (6.34) has a non-positive solution then necessarily $E\{\mathbf{h}_1\}$ and $E\{\mathbf{h}_2\}$ are both infinite. (see also remark 6.5 below).

Consequently, the first moment of the first return time to B out from a state of B is given by

$$1+\frac{\Delta}{-\alpha_{12}}, \tag{6.38}$$

with

$$\Delta:=K_0 E\{\xi_0\eta_0\}+\nu_1 E\{\mathbf{h}_1\}+\mu_2 E\{\mathbf{h}_2\}+ \tag{6.39}$$

$$K_1(1)E\{(\xi_1-1)\eta_1\}+K_2(1)E\{(\eta_2-1)\xi_2\}.$$

Since (6.35) i implies that the $\mathbf{k}_m$-process is positive recurrent the states of B have a finite first return in the $\mathbf{k}_m$-process, i.e. on the $\mathbf{k}_m$-clock; hence it follows by using Wald's theorem and (6.38) that their first return times as states of the $\mathbf{z}_n$-process, i.e. on the $\mathbf{z}_n$-clock, have also a finite first moment; since the state space S is irreducible, this applies also for all states of the $\mathbf{z}_n$-process. Consequently, (6.35) implies that the $\mathbf{z}_n$-process is positive recurrent.

If (6.35) i and ii apply but (6.35) iii not then $\Delta=\infty$ and so the $\mathbf{z}_n$-process is null recurrent. □

REMARK 6.1. Suppose the $\mathbf{z}_n$-process is positive recurrent. Denote by $(\mathbf{x},\mathbf{y})$ a stochastic vector with distribution the stationary distribution of the $\mathbf{z}_n$-process. It then follows from (6.38) that

one time unit on the $\mathbf{k}_m$ $-$clock corresponds (6.40)

on the average with $1+\Delta(-\alpha_{12})$ time units on the $\mathbf{z}_n$-clock. □

REMARK 6.2. Suppose again that the $\mathbf{z}_n$-process is positive recurrent. Put with $(\mathbf{x},\mathbf{y})$ as defined in the preceding remark

$$\Phi(p_1,p_2):=E\{p_1^{\mathbf{x}}p_2^{\mathbf{y}}\},\ |p_1|\leqslant 1,\ |p_2|\leqslant 1. \tag{6.41}$$

A wellknown Abel theorem implies that

$$\Phi(p_1,p_2)=\lim_{r\to 1}(1-r)\Phi_{z_n}(r,p_1,p_2),\ |p_1|\leqslant 1,\ |p_2|\leqslant 1, \tag{6.42}$$

and hence it follows from (2.8) that for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$[p_1p_2-\phi_3]\Phi(p_1,p_2)=[p_1p_2\phi_0-\phi_3]\Phi(0,0)+ \tag{6.43}$$
$$[p_2\phi_1-\phi_3]\{\Phi(p_1,0)-\Phi(0,0)\}+[p_1\phi_2-\phi_3]\{\Phi(0,p_2)-\Phi(0,0)\},$$

with

$$\phi_j = \phi_j(p_1,p_2).$$

Take in (6.43) $p_1=1$, divide then by $1-p_2$ and let $p_2\to 1$, this leads to:

$$(1-\nu_0-\nu_3)\Phi(0,0)+(1+\nu_1-\nu_3)\{\Phi(1,0)-\Phi(0,0)\}+(\nu_2-\nu_3)\{\Phi(0,1)-\Phi(0,0)\}= \tag{6.44}$$
$$(1-\nu_3)\Phi_3(1,1)=(1-\nu_3)[\Phi(0,0)+\Phi(1,0)+\Phi(0,1)+\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}];$$

or

$$\nu_0\Phi(0,0)+\nu_1\{\Phi(1,0)-\Phi(0,0)\}=(1-\nu_2)\{\Phi(0,1)-\Phi(0,0)\}+(1-\nu_3)\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\} \tag{6.45}$$
$$\mu_0\Phi(0,0)+\mu_2\{\Phi(0,1)-\Phi(0,0)\}=(1-\mu_1)\{\Phi(1,0)-\Phi(0,0)\}+(1-\mu_3)\mathrm{E}\{\mathbf{x}>0,\mathbf{y}>0)\},$$

the second relation in (6.45) follows from the first one by symmetry.

Obviously, the first relation in (6.45) formulates the *balance equation* in the *vertical* direction for the stationary $\mathbf{z}_n$-process, the second one that in the *horizontal* direction, and as such these relations can be *directly derived* from *first principles* if the $\mathbf{z}_n$-process is positive recurrent, but also if it is null recurrent (then $\Phi(0,0)=0$).

Since the $\mathbf{z}_n$-process is assumed to be positive recurrent (and hence the $\mathbf{k}_m$-process also) we have

$$1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0\}>0,$$

and so it follows from (6.45),

$$\nu_0\frac{\Phi(0,0)}{1-\mathrm{E}\{\mathbf{x}>0,\mathbf{y}>0)\}}+\nu_1\frac{\Phi_1(1,0)-\Phi(0,0)}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}=$$
$$(1-\nu_2)\frac{\Phi(0,1)-\Phi(0,0)}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}+(1-\nu_3)\frac{\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>))\}},$$
$$\mu_0\frac{\Phi(0,0)}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}+\mu_2\frac{\Phi(0,1)-\Phi(0,0)}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}=$$
$$(1-\mu_1)\frac{\Phi(0,1)-\Phi(0,0)}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}+(1-\mu_3)\frac{\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}. \tag{6.46}$$

With $(\mathbf{k}_1,\mathbf{k}_2)$ a stochastic vector with distribution the stationary distribution of the $\mathbf{k}_m$-process we have

$$\Pr\{\mathbf{k}_1=\mathbf{k}_2=0\} = \frac{\Phi(0,0)}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}, \tag{6.47}$$
$$\Pr\{\mathbf{k}_1>0,\mathbf{k}_2=0\} = \frac{\Phi(1,0)-\Phi(0,0)}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}},$$
$$\Pr\{\mathbf{k}_1=0,\ \mathbf{k}_2>0\}=\frac{\Phi(0,1)-\Phi(0,0)}{1-\mathrm{E}\{(\mathbf{x}>0,\mathbf{y}>0)\}}.$$

Next take in (6.46) $\mu_3=\nu_3=1$ it then follows from (6.5), (6.46) and (6.47) that

$$\nu_0K_0+\nu_1K_1(1)=(1-\nu_2)K_2(1), \tag{6.48}$$
$$\mu_0K_0+\mu_2K_2(1)=(1-\mu_1)K_1(1).$$

These latter relations are identical with (6.11). So for the case $\mu_3=\nu_3=1$ the relations (6.11) represent the balance equations for the $\mathbf{k}_m$-process if the $\mathbf{z}_n$-process is positive recurrent. Actually, it is seen from first principles that these balance equations for the $\mathbf{k}_m$-process should also apply when the $\mathbf{k}_m$-process is recurrent. □

The considerations in the remark above lead to

COROLLARY 6.1. *The statements of theorem 6.1 concerning the* $\mathbf{k}_m$*-process also hold if* $E\{\xi_3^2\}=\infty, E\{\eta_3^3\}=\infty$.

PROOF. The corollary follows immediately from the fact that these statements are all based on the relations (6.11), the validity of which does not depend on $E\{\xi_3^2\}$ and $E\{\eta_3^2\}$. □

REMARK 6.3. Suppose again that the $\mathbf{z}_n$-process is positive recurrent. It then follows from (6.47) that

$$\frac{\Phi(0,0)}{K_0}=1-E\{(\mathbf{x}>0,\mathbf{y}>0)\}. \tag{6.49}$$

Since $\{\Phi(0,0)\}^{-1}$ is equal to the first moment of the first return time of the state $\{0,0\}$ in the $\mathbf{z}_n$-process, i.e. on the $\mathbf{z}_n$-clock, and K_0^{-1} is the first moment of the first return time of the state $\{0,0\}$ in the $\mathbf{k}_m$-process, i.e. on the $\mathbf{k}_m$-clock, it follows from (6.40), (6.49) and Wald's theorem that

$$\frac{\Phi(0,0)}{K_0}=\frac{1}{1+\Delta/(1-\alpha_{12})}. \tag{6.50}$$

Consequently, from (6.47), (6.49) and (6.50),

$$\begin{aligned} E\{(\mathbf{x}>0,\mathbf{y}>0)\} &= \frac{\Delta}{-\alpha_{12}+\Delta}, \\ \Phi(0,0) &= \frac{-\alpha_{12}}{-\alpha_{12}+\Delta}K_0, \\ \Phi(1,0)-\Phi(0,0) &= \frac{-\alpha_{12}}{-\alpha_{12}+\Delta}K_1(1), \\ \Phi(0,1)-\Phi(0,0) &= \frac{-\alpha_{12}}{-\alpha_{12}+\Delta}K_2(1), \end{aligned} \tag{6.51}$$

with $K_0, K_1(1)$ and $K_2(1)$ given by (6.17) and $\mu_1<1$, $\nu_2<1$, $(1-\mu_1)(1-\nu_2)-\mu_2\nu_1>0$. □

REMARK 6.4. Suppose again that the $\mathbf{z}_n$-process is positive recurrent. In the H.P.I., cf. remark 2.7, we take the starting point $z_0=(x_0,y_0)$ stochastic and such that

$$\mathbf{z}_0 = (\mathbf{x},\mathbf{y}),$$

with $(\mathbf{x},\mathbf{y})$ as defined in remark 6.1 above. It then follows from (2.25): for $|p_1|=1$,

$$\begin{aligned} &\Phi(p_1,P_2(1,p_1)) \equiv E\{p_1^{\mathbf{x}}P_2^{\mathbf{y}}(1,p_1)\}=E\{(\mathbf{k}_1=\mathbf{k}_2=0)\}+ \\ &E\{p_1^{\mathbf{k}_1}(\mathbf{k}_1>0,\ \mathbf{k}_2=0)\}+E\{P_2^{\mathbf{k}_2}(1,p_1)(\mathbf{k}_2>0,\mathbf{k}_1=0)\}, \end{aligned} \tag{6.52}$$

since

$$\mathbf{k}_1=\mathbf{k}_1(\mathbf{x},\mathbf{y}),\ \mathbf{k}_2=\mathbf{k}_2(\mathbf{x},\mathbf{y}).$$

Differentiation of (6.52) with respect to p_1 and letting $p_1\to 1$ leads to

$$E\{\mathbf{x}\}+\frac{dP_2(1,p_1)}{dp_1}\Big|_{p_1=1}E\{\mathbf{y}\}=E\{\mathbf{k}_1\}+E\{\mathbf{k}_2\}\frac{dP_2(1,p_1)}{dp_1}\Big|_{p_1=1}. \tag{6.53}$$

Since $\frac{dP_2(1,p_1)}{dp_1}\Big|_{p_1=1}$ is complex, see (6.28), it follows that

$$E\{\mathbf{x}\}=E\{\mathbf{k}_1\}\ ,\quad E\{\mathbf{y}\} = E\{\mathbf{k}_2\}. \tag{6.54}$$

with E{$\mathbf{k}_1$} and E{$\mathbf{k}_2$} given by (6.34) and (6.33). □

REMARK 6.5. By using (6.11) the expressions (6.33) may be rewritten as:

$$B_1 = [\mathrm{E}\{\xi_0\eta_0\} + a_1\,\mathrm{E}\{\eta_0^2\}]K_0 \qquad (6.55)$$
$$+[\mathrm{E}\{\xi_1\eta_1\} + a_1\,\mathrm{E}\{\eta_1^2\} - \nu_1]K_1(1) + [\mathrm{E}\{\xi_2\eta_2\} + a_1\mathrm{E}\{(\eta_2-1)^2\} - \mu_2]K_2(1).$$
$$B_2 = [\mathrm{E}\{\xi_0\eta_0\} + a_2\mathrm{E}\{\xi_0^2\}]K_0$$
$$+[\mathrm{E}\{\xi_1\eta_1\} + a_2\mathrm{E}\{(\xi_1-1)^2\} - \nu_1]K_1(1) + [\mathrm{E}\{\xi_2\eta_2\} + a_2\mathrm{E}\{\xi_2^2\} - \mu_2]K_2(1).$$

Obviously the terms with ν_1 and μ_2 are the only negative ones in the expressions for B_1 and B_2. If $\nu_1 = \mu_2 = 0$, which implies that $\eta_1 = 0$, $\xi_2 = 0$ with probability one then it is readily seen that for the conditions (6.35) i, ii and iii the system (6.32) has a finite positive solution and so the $\mathbf{z}_n$-process is positive recurrent. From (6.55) and (6.32) it is seen that the reflections (ν_1) and (μ_2) at the boundaries B_{10} and B_{01} do influence the character of the $\mathbf{z}_n$-process. □

REMARK 6.6. By using the same technique as in the preceding remark it is possible to derive expressions for the higher moments of ($\mathbf{k}_1$,$\mathbf{k}_2$) and (**x**,**y**); their finiteness require the finiteness of the higher moments of the vectors (ξ_k,η_n), $k = 0,1,2,3$; e.g. if the third moments of their vectors are finite then the second moments of ($\mathbf{k}_1$,$\mathbf{k}_2$) are finite. The explicit expressions are, however, quite complicated. □

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