1991

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Department of Operations Research, Statistics, and System Theory Report BS-R9101 January

CWI, nationaal instituut voor onderzoek op het gebied van wiskunde en informatica

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A New Device for the Synthesis Problem of Optimal Control of Admission to an M/M/c Queue

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The problem of finding an optimal admission policy to an M/M/c queue with customers with deadlines is addressed in this paper. There are two streams of customers (customers of class 1 and 2) that are generated according to independent Poisson processes with constant arrival rates. All service times are exponentially distributed with a class independent service rate. Upon arrival a class 1 customer may be admitted or rejected, while incoming class 2 customers are always admitted. Associated with the n-th class 1 customer is a deadline, D_n, by which it must complete service where $\{D_n\}_1^\infty$ is a sequence of i.i.d. random variables. We are interested in the throughput of class 1 customers that complete their service before their deadline (usually referred to as goodput), and we wish to determine an admission control that maximizes this throughput. At each decision epoch we assume that the controller has available to it the complete history of the total queue length process as well as all of the past decisions that have been made up to that epoch. We show that, for a large class of deadline distributions, there exists a stationary admission policy of a threshold type that maximizes a discounted cost function (for small discount factors) that corresponds to the discounted goodput of customers that make their deadline. The proof relies on a new device that consists in a partial construction of the solution of the dynamic programming equation. In addition, we show that there also exists a threshold admission control that maximizes the long-run average problem (i.e., maximizes the goodput). The proposed method is of independent interest and should apply to many queueing control problems.

1980 Mathematics Subject Classification: 60K25, 90B22.

Keywords & Phrases: Control of Queues; Optimal Control of Admission; Markov Decision Processes; Dynamic Programming; Deadlines.

*This author was supported in part by ONR under contract N00014-87-K-0304 and NSF under grant ASC-8802764

Report BS-9101
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1 Introduction

We consider an M/M/c queueing system ($1 \leq c < \infty$) fed by two independent Poisson streams of customers, with intensities $\lambda_1 > 0$ and $\lambda_2 \geq 0$. Customers of stream i will be referred to as class i customers, $i = 1, 2$. The buffer has unlimited capacity and the customers are served according to the first-in-first-out service discipline. The customer service times are independent and exponentially distributed random variables, with class-independent service rate $\mu > 0$. The n-th arriving class 1 customer has a *deadline* D_n associated with it, where $\{D_n\}_1^\infty$ is a sequence of i.i.d. nonnegative random variables. The arrival, service time and deadline sequences are assumed to be mutually independent.

Our objective is to find an *admission control* (policy) for class 1 customers that maximizes the fraction of class 1 customers that complete their service *before* their deadline has expired. We assume that the controller has only available to it the *total* number of customers currently in the queue at the time a new class 1 customer arrives, as well as the history of the queue length process and the history of the decisions that have been made (i.e., rejection or admittance). We also assume that all class 1 customers admitted into the queue are served regardless of whether they make their deadline or not.

This study is motivated by the new services available in current communication networks (ISDN's). In this context, class 1 customers represent *interactive traffic* (e.g., packetized voice) and class 2 customers *non interactive* traffic (e.g., batch file transfers), where both traffic types compete for the use of limited resources (e.g., telephone exchanges). Therefore, a natural objective is to seek an admission policy that maximizes the *throughput of customers that complete by their deadlines*, i.e. the *goodput* of this system. If all of the class 1 customers are accepted in the system, then the queue will build up and they will be more likely to miss their deadline; on the other hand, if these are all rejected then the goodput with respect to class 1 customers will be zero. Consequently, it is reasonable to expect the existence of an optimal policy that will tradeoff these two effects.

Throughout the years, several authors have studied problems of control of arrivals (or flow control problems) in the context of queueing systems, and a comprehensive discussion can be found in the survey paper by Stidham [13]. A standard approach in the control of queueing systems consists in formulating the optimization problem at hand as a Markov (see e.g., Serfozo [12], Lin and Kumar [8]) or Semi-Markov (see e.g., Lippman [10]) decision problem, from which the functional equation of dynamic programming can be derived (Heyman and Sobel [6]). Then, the policy improvement algorithm (see e.g., Lin and Kumar [8]) or the propagation of "good" properties, such as convexity, concavity, (sub)modularity, monotonicity properties in the induction of dynamic programming (see e.g., Hajek [5], Johansen and Stidham [7], Ma and Makowski [11]) may be used to determine the optimal policy (e.g., threshold policy, switch-over policy).

Let us now give a short description of the optimization problem model that will be considered in this paper, as well as the results that will be obtained. When a new class 1 customer is admitted, it earns a reward $g(k+1)$ if there are k customers in the queue upon its arrival, where the properties of g are given in Section 2. These properties will be chosen so as to place the model in the communication network framework that has been discussed earlier. The aim is to find an admission policy that maximizes the total average discounted reward earned over an infinite horizon (discounted reward problem) and then to derive from this result the control that maximizes the long-run average reward (average reward problem). We will establish the existence of an optimal *threshold* admission policy for both the discounted reward problem (in the case of small discount factors) and the average reward problem.

Because of the particular assumptions we place on the reward function (in particular, that it is neither monotone nor convex, see Section 2), none of the classical techniques listed above applies to our problem. We therefore employ a new device, first proposed by de Waal [3], based on a partial construction of the optimal value function, from which the optimal control can be determined. This method is of independent interest and should apply to many similar optimization problems.

In Section 2 the problem is formulated as a Markov decision problem. Section 3 addresses the discounted reward control problem in the case where $\lambda_2 = 0$, which will turn out to be considerably simpler to analyse than the case where $\lambda_2 > 0$ of which the analysis is given in Section 4. The existence of an optimal threshold policy for the average reward control problem is shown in Section 5.

2 The Model

All of the random variables considered in this paper are defined on some fixed probability triple $(\Omega, \mathcal{F}, \mathrm{P})$. $\mathbb{N}$ will denote the set of nonnegative integers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}^+$ the set of strictly positive real numbers. The optimization problem described in Section 1 is now formulated as a Markov decision problem. This formulation closely follows that of Lippman in [9] (an alternative formulation can be found in de Waal [4], pp. 46-67).

Let a_n^i, $0 \leq a_1^i < a_2^i < \cdots$, be the arrival time of the n-th class i customer, $i = 1, 2$, and let d_n be the departure time of the n-th customer that is served, $n \geq 1$. Define $\{t_n\}_1^\infty := \{a_n^1\}_1^\infty \cup \{a_n^2\}_1^\infty \cup \{d_n\}_1^\infty$, $0 \leq t_1 < t_2 < \cdots$. In order to cast our problem into the framework of Markov Decision Processes (MDP's) [6], we specify the decision epochs, the state of the system and the action space.

We assume that the n-th decision $U_n \in \{0, 1\}$ is made at time t_n, $n \geq 1$. If $t_n \notin \{a_j^1\}_1^\infty$, then the decision is irrelevant since only the stream of class 1 customers is controlled. In that case, we shall assume by convention that $U_n = 0$. If $t_n \in \{a_j^1\}_1^\infty$, then the decision may be either to accept $(U_n = 1)$ or to reject $(U_n = 0)$ the new class 1 customer.

Let Q_t be the total number of customers in the system at time t, $t \geq 0$, including the customers in service, if any. At time t_n, $n \geq 1$, the state of the system is represented by the process $Z_n = (Q_n, X_n) \in \mathbb{N} \times \{0, 1\}$, where $Q_n := Q_{t_n} \mathbf{1}\{t_n \notin \{a_j^1\}_1^\infty\} + Q_{t_n^-} \mathbf{1}\{t_n \in \{a_j^1\}_1^\infty\}$ and $X_n := \mathbf{1}\{t_n \in \{a_j^1\}_1^\infty\}$. Here, $Q_{t_n^-}$ stands for the number of customers in the system just *before* the occurrence of the n-th event. Let $A_{k,x}$ be the action space when the state of the system is (k, x), $k \in \mathbb{N}$, $x \in \{0, 1\}$. It follows from the above definitions that $A_{k,0} = \{0\}$ and $A_{k,1} = \{0, 1\}$.

The process $\mathbf{Z} := \{Z_n, n \geq 1\}$ is a MDP, since clearly

$$\mathrm{P}\,(Z_{n+1} = (l_{n+1}, y_{n+1}) \,|\, Z_1 = (l_1, y_1), U_1 = u_1, \ldots, Z_n = (l_n, y_n), U_n = u_n)$$

$$= \mathrm{P}\left(Z_{n+1} = (l_{n+1}, y_{n+1}) \,|\, Z_n = (l_n, y_n), U_n = u_n\right),$$

for all $l_i \geq 0$, $y_i, u_i \in \{0,1\}$, $i = 1, 2, \ldots, n+1$. We let $\mathbb{H}_n = (Z_1, U_1, Z_2, U_2, \ldots, Z_{n-1}, U_{n-1}, Z_n)$ denote the history of $\mathbf{Z}$ up to time t_n, $n \geq 1$.

At the n-th decision epoch, a nonnegative discounted reward $\exp(-\alpha t_n)\, r(Z_n; U_n)$ is earned, where

$$r(k, x; u) := g(k+1)\, \mathbf{1}\{u = 1,\, x = 1\}, \tag{2.1}$$

for all $k \in \mathbb{N}$, $x, u \in \{0,1\}$ and where g is a mapping $\mathbb{N} \to \mathbb{R}^+$. We will restrict ourselves to mappings that are *modified unimodal* as defined below.

DEFINITION 2.1 *The mapping $g : \mathbb{N} \to \mathbb{R}^+$ is called modified unimodal with mode c and exponential factor Ψ, where $c = 1, 2, \ldots$, and $0 < \Psi < 1$, if*

$$k\, g(k) \leq (k+1)\, g(k+1), \quad \textit{for } k = 1, 2, \ldots, c-1; \tag{2.2}$$

$$g(k+1) \leq \Psi g(k), \quad \textit{for } k = c, c+1, \ldots \tag{2.3}$$

An *admissible control* U is a sequence $\{u_n\}_1^\infty$ of $[0,1]$-valued random variables such that u_n is measurable with respect to the σ-field $\mathcal{F}_n := \sigma(\mathbb{H}_n)$, with the interpretation that for $n \geq 1$, $u_n = \mathrm{P}\left(U_n = 1 \,|\, \mathcal{F}_n\right)$,

Let $\mathcal{U}$ be the set of admissible controls. A control is said to be *stationary* if u_n only depends on the value of Z_n and if it is non-randomized, i.e., $u_n \in \{0,1\}$, $n \geq 1$.

Let us now introduce the so-called *value function* for the discounted problem. Let $0 < \alpha < +\infty$

$$V_\alpha(k, x; U) := \mathrm{E}\left[\sum_{n \geq 1} e^{-\alpha t_n}\, r(Z_n; U_n) \,\middle|\, Z_1 = (k, x)\right], \tag{2.4}$$

be the total average discounted reward gained over an infinite horizon starting from state $(k, x) \in \mathbb{N} \times \{0,1\}$, when policy $U \in \mathcal{U}$ is used. It is easily seen from definitions (2.1)-(2.4) that the value function $V_\alpha(k, x; U)$ is uniformly bounded in k, x and U. More precisely,

$$0 \leq V_\alpha(k, x; U) \leq \left(\frac{\alpha + \lambda_1}{\alpha}\right) c\, g(c), \tag{2.5}$$

for all $k \in \mathbb{N}$, $x \in \{0,1\}$, $U \in \mathcal{U}$.

Our first objective is to find an admissible control that maximizes (2.4) over the set of admissible controls $\mathcal{U}$ for all $(k, x) \in \mathbb{N} \times \{0,1\}$. Let

$$V_\alpha^*(k, x) := \sup_{U \in \mathcal{U}} V_\alpha(k, x; U), \tag{2.6}$$

for all $(k, x) \in \mathbb{N} \times \{0,1\}$.

Define the total reward $W_T(k,x;U)$ gained under policy $U \in \mathcal{U}$ over a horizon of length $T > 0$, given the initial state is $(k,x) \in \mathbb{N} \times \{0,1\}$, as

$$W_T(k,x;U) := \mathrm{E}\left[\sum_{\{n\,:\,0 \le t_n < T\}} r(Z_n;U_n) \,\middle|\, Z_1 = (k,x) \right], \tag{2.7}$$

Our second objective is to find an admissible control that maximizes the long-run average reward gained over an infinite horizon defined as

$$W(k,x;U) := \liminf_{T\uparrow\infty} \frac{1}{T} W_T(k,x;U), \tag{2.8}$$

for all $U \in \mathcal{U}$, $k \in \mathbb{N}$, $x \in \{0,1\}$.

Theorem 2.1 gives the *Dynamic Programming* (DP) equation that is satisfied by the optimal value function V_α^*.

THEOREM 2.1 *There exists an optimal stationary policy for the discounted problem. In addition, $V_\alpha^*(k,x)$ is the unique uniformly bounded solution in (k,x) to the DP equation*

$$V_\alpha^*(k,x) = \max_{u \in A_{k,x}} \left\{ r(k,x;u) + \frac{\theta_{k,x}(u)}{\alpha + \theta_{k,x}(u)} \sum_{l \in \mathbb{N},\, y \in \{0,1\}} Q(l,y \,|\, k,x;u)\, V_\alpha^*(l,y) \right\}, \tag{2.9}$$

for all $k \in \mathbb{N}$, $x \in \{0,1\}$, where

- *$Q(\cdot,\cdot \,|\, k,x;u)$ is the one-step probability transition of the process $\mathbf{Z}$ given the current state is (k,x) and that the action u is chosen;*
- *$\theta_{k,x}(u)$ is the transition rate out of state (k,x) given that action u is chosen.*

Furthermore, the stationary control which selects an action maximizing the right-hand side of (2.9) for all $k \in \mathbb{N}$, $x \in \{0,1\}$, is optimal.

PROOF. The proof follows from Theorem 1 in Lippman [10]. □

For $k \in \mathbb{N}$, $x \in \{0,1\}$, introduce

$$\begin{aligned} \lambda &:= \lambda_1 + \lambda_2; \\ \mu_k &:= \mu \min(k,c); \\ \nu_k &:= \lambda + \mu_k; \\ \theta_{k,x}(u) &:= \nu_k \mathbf{1}\{u=0\} + \nu_{k+1}\mathbf{1}\{u=1\}. \end{aligned}$$

By observing from the model description that

$$Q(l,y \,|\, k,1;1) = \begin{cases} \lambda_1/\nu_{k+1}, & \text{if } l = k+1,\ y = 1; \\ \lambda_2/\nu_{k+1}, & \text{if } l = k+2,\ y = 0; \\ \mu_{k+1}/\nu_{k+1}, & \text{if } l = k,\ y = 0; \\ 0, & \text{otherwise;} \end{cases} \tag{2.10}$$

$$Q(l,y\,|\,k,1;0) = Q(l,y\,|\,k,0;\bullet) = \begin{cases} \lambda_1/\nu_k, & \text{if } l=k,\ y=1; \\ \lambda_2/\nu_k, & \text{if } l=k+1,\ y=0; \\ \mu_k/\nu_k, & \text{if } k\geq 1,\ l=k-1,\ y=0; \\ 0, & \text{otherwise,} \end{cases} \tag{2.11}$$

routine manipulations of (2.9) yield

$$(\alpha+\nu_k)V_\alpha^*(k,0) = \lambda_1 V_\alpha^*(k,1) + \lambda_2 V_\alpha^*(k+1,0) + \mu_k V_\alpha^*(k-1,0)\,\mathbf{1}\{k\geq 1\}, \tag{2.12}$$

$$V_\alpha^*(k,1) = \max\{g(k+1) + V_\alpha^*(k+1,0); V_\alpha^*(k,0)\}, \tag{2.13}$$

for all $k \in \mathbb{N}$.

As a consequence of the last statement of Theorem 2.1 and (2.13), the optimal action when the state of the system is $(k,1)$ (denoted as u_k^*) is the one that maximizes $[g(k+1) + V_\alpha^*(k+1,0) - V_\alpha^*(k,0)] \vee 0$, that is

$$u_k^* = \mathbf{1}\{V_\alpha^*(k+1,0) - V_\alpha^*(k,0) + g(k+1) > 0\}, \tag{2.14}$$

for all $k \in \mathbb{N}$.

Define $V_\alpha^*(k) := V_\alpha^*(k,0)$, $k \in \mathbb{N}$ (from now on, the subscript α in $V_\alpha^*(k)$ will be omitted). By substituting the r.h.s. of (2.13) for $V_\alpha^*(k,1)$ in (2.12), we finally obtain, by using (2.14), that $V^*(k)$, $k \in \mathbb{N}$, is the unique uniformly bounded solution in k to the DP equation

$$\begin{aligned} 0 &= -\alpha V^*(k) + \lambda_1\, u_k^*\,[V^*(k+1) - V^*(k) + g(k+1)] \\ &\quad + \lambda_2\,[V^*(k+1) - V^*(k)] - \mu_k\,[V^*(k) - V^*(k-1)]\,\mathbf{1}\{k\geq 1\}, \end{aligned} \tag{2.15}$$

for $k \in \mathbb{N}$, with $u_k^* = \mathbf{1}\{V^*(k+1) - V^*(k) + g(k+1) > 0\}$.

The following equivalent form of equation (2.15) will be frequently used. Define $x^* : \mathbb{N} \to \mathbb{R}$ as

$$x^*(k) = \begin{cases} V^*(0), & k=0; \\ V^*(k) - V^*(k-1), & k\geq 1. \end{cases} \tag{2.16}$$

Then, the DP equation (2.15) can be rewritten as

$$0 = -\alpha \sum_{i=0}^{k} x^*(i) + \lambda_1\, u_k^*\,[x^*(k+1) + g(k+1)] + \lambda_2\, x^*(k+1) - \mu_k x^*(k)\,\mathbf{1}\{k\geq 1\}, \quad k\geq 0, \tag{2.17}$$

with

$$u_k^* = \mathbf{1}\{x^*(k+1) + g(k+1) > 0\}. \tag{2.18}$$

The DP equation (2.15)/(2.17) will play a central role in the characterization of the optimal admission policy (Sections 3-5).

To conclude this section, let us briefly motivate the choice of the function g. As indicated in Section 1, our objective is to maximize the fraction of class 1 customers that complete before their deadline. Therefore, a natural candidate for the reward function g is to let $g(k)$ be the probability that a new class 1 customer meets its deadline given there are k customers in the system, including itself, upon its arrival.

With this definition of g, it is easily seen that condition (2.2) of Definition 2.1 is satisfied, since $g(k) = \mathrm{P}(S < D)$, $k = 1, 2, \ldots, c$, where S (resp. D) is a generic random variable for the service time (resp. deadline) of a customer.

In the case that $\mathrm{P}(D \leq x) = 1 - \exp(-\gamma x)$ for $x \geq 0$, $\gamma > 0$ (exponentially distributed deadlines), it is easily seen that

$$g(k) = \begin{cases} \mu/(\mu+\gamma), & k = 1, 2, \ldots, c; \\ (\mu/(\mu+\gamma))\,(c\mu/(c\mu+\gamma))^{k-c}, & k = c+1, c+2, \ldots \end{cases}$$

It is easy to show that the above expression for $g(k)$ satisfies the definition of a modified unimodal mapping with $\Psi = c\mu/(c\mu+\gamma)$.

In the general case where where the deadline distribution function is arbitrary, then $g(k)$ cannot be computed under closed form. However, many interesting deadline distribution functions give rise to modified unimodal mappings. More precisely, we have the following result:

PROPOSITION 2.1 *The mapping g is modified unimodal if one of the three following conditions is fulfilled:*

1. *the deadlines are deterministic;*
2. *the deadlines have a failure rate that is bounded away from 0 by a strictly positive constant;*
3. *the deadlines have an Erlang distribution.*

The proof of Proposition 2.1 can be found in de Waal [4]. Note that condition 2 in Proposition 2.1 is satisfied by a large class of distributions, including the exponential distribution, subsets of the class of Gamma distributions, and truncated normal distributions (see Barlow and Proschan [1], Sec. 5).

3 Discounted Reward Problem: the Single-Stream Case

This section is devoted to the analysis of the single-stream discounted problem (i.e., $\lambda_2 = 0$). In that case the DP equation (2.17) reads:

$$0 = -\alpha \sum_{i=0}^{k} x^*(i) + \lambda_1\, u_k^* \,[x^*(k+1) + g(k+1)] - \mu_k x^*(k)\, \mathbf{1}\{k \geq 1\}, \;\; k \geq 0. \qquad (3.1)$$

The main result of this section is:

PROPOSITION 3.1 *Let $g : \mathbb{N} \to \mathbb{R}^+$ be a modified unimodal mapping with mode c and exponential factor Ψ. Let $u^* : \mathbb{N} \to \{0,1\}$ be the optimal stationary control for the discounted problem with $\lambda_2 = 0$. Define $u_k^* := u^*(k)$. Then,*

1. $u_k^* = 1$ *for* $k = 0, 1, \ldots, c$;

2. *if* $0 < \alpha < c\mu(1 - \Psi)/\Psi$ *and if there is an* $l \in \mathbb{N}$ *such that* $u_l^* = 0$, *then* $u_k^* = 0$ *for* $k = l, l+1, \ldots$

PROOF. We first prove 1. by induction. Substituting $k = 0$ and $k = 1$ into equation (3.1) yields

$$0 = -\alpha x^*(0) + \lambda_1 u_0^* (x^*(1) + g(1)), \tag{3.2}$$

$$0 = -\alpha(x^*(0) + x^*(1)) + \lambda_1 u_1^* (x^*(2) + g(2)) - \mu_1 x^*(1). \tag{3.3}$$

Subtracting (3.2) from (3.3) yields

$$\lambda_1 u_1^*(x^*(2) + g(2)) - \lambda_1 u_0^*(x^*(1) + g(1)) = (\alpha + \mu_1)x^*(1).$$

If we assume that $u_0^* = 0$, then since $u_1^* (x^*(2) + g(2)) \geq 0$ we can write

$$(\alpha + \mu_1)x^*(1) \geq 0. \tag{3.4}$$

However, because $u_0^* = 0$, it follows that $0 \geq x^*(1) + g(1) > x^*(1)$. But according to (3.4), $x^*(1)$ is nonnegative which results in a contradiction and therefore $u_0^* = 1$.

Assume now that $u_0^* = u_1^* = \cdots = u_{l-1}^* = 1$ for $l < c$ and let us show that $u_l^* = 1$. Substituting $k = l$ and $k = l + 1$ into equation (3.1) and subtracting the first equation from the second one, yields

$$\lambda_1 u_{l+1}^*(x^*(l+2) + g(l+2)) - \lambda_1 u_l^*(x^*(l+1) + g(l+1)) = (\alpha + \mu_{l+1})x^*(l+1) - \mu_l x^*(l). \tag{3.5}$$

If we assume that $u_l^* = 0$, we then deduce from (3.5) that

$$(\alpha + \mu_{l+1})x^*(l+1) - \mu_l x^*(l) \geq 0,$$

since $u_{l+1}^*(x^*(l+2) + g(l+2)) \geq 0$, or equivalently that

$$(\alpha + \mu_{l+1})(x^*(l+1) + g(l+1)) - \mu_l(x^*(l) + g(l)) - g(l+1)(\alpha + \mu_{l+1}) + \mu_l g(l) \geq 0. \tag{3.6}$$

By noting now that $x^*(l+1) + g(l+1) \leq 0$ (since $u_l^* = 0$ by assumption), $-(x^*(l) + g(l)) < 0$ (since $u_{l-1}^* = 1$ by assumption) and $-g(l+1)\mu_{l+1} + \mu_l g(l) \leq 0$ (since g is modified unimodal) we see that the left-hand side of equation (3.6) is strictly negative, which gives a contradiction. Therefore $u_l^* = 1$.

We also prove 2. by induction. Fix α such that $0 < \alpha < c\mu(1 - \Psi)/\Psi$. Let $l \in \mathbb{N}$, $l \geq c$ be such that $u_l^* = 0$. This implies that $x^*(l+1) \leq -g(l+1)$.

Define $x : \mathbb{N} \to \mathbb{R}$ as

$$x(k) = \begin{cases} x^*(k), & k = 0, 1, \ldots, l; \\ -\alpha \sum_{i=0}^{k-1} x(i)/(\alpha + c\mu), & k = l+1, l+2, \ldots \end{cases} \tag{3.7}$$

Note that $x(k)$, $k > l$, is the recursion we get from (3.1) if we choose $u_k^* = 0$ for $k > l$.

We prove that $x(k) \leq -g(k)$ for $k > l$ by induction on k.

Basis step. Let $k = l + 1$. From the definition of x we have

$$
\begin{aligned}
x(l+1) &= -\alpha \sum_{i=0}^{l} x(i)/(\alpha + c\mu), \\
&= -\alpha \sum_{i=0}^{l} x^*(i)/(\alpha + c\mu), \\
&= (\alpha x^*(l+1) - \lambda_1 u_{l+1}^*(x^*(l+2) + g(l+2)) + c\mu x^*(l+1))/(\alpha + c\mu), \\
&\leq x^*(l+1), \\
&\leq -g(l+1).
\end{aligned}
$$

The last two steps follow from the fact that $u_k^*(x^*(k+1) + g(k+1)) \geq 0$ for all $k \in \mathbb{N}$ and the fact that $u_l^* = 0$.

Inductive step. We assume that $x(k') \leq -g(k')$ for $k' = l+1, l+2, \ldots, k$. We show that $x(k+1) \leq -g(k+1)$.

$$
\begin{aligned}
x(k+1) &= \left[-\alpha x(k) - \alpha \sum_{i=1}^{k-1} x(i) \right] /(\alpha + c\mu), \\
&= -[\alpha x(k) - (\alpha + c\mu) x(k)]/(\alpha + c\mu), \\
&= c\mu\, x(k)/(\alpha + c\mu), \\
&\leq -c\mu\, g(k)/(\alpha + c\mu), \\
&\leq -g(k+1),
\end{aligned}
$$

by the induction hypothesis, the assumptions on g and the condition on α. We have thus found a sequence of real numbers that when substituted for $x^*(k)$ in equation (3.1) satisfies that equation.

Define now $V(k) := \sum_{i=0}^{k} x(i)$, $k \in \mathbb{N}$. First observe that V satisfies the DP equation (2.15). Let us show that $V(k)$ is uniformly bounded in k. Since by construction $V(k) = V^*(k)$ for $0 \leq k \leq l$, $V(k)$ is bounded for $0 \leq k \leq l$. On the other hand, the definitions of x and V imply that

$$
V(k) - V(k-1) = \frac{-\alpha}{\alpha + c\mu} V(k-1), \quad \text{for } k \geq l, \tag{3.8}
$$

which, in turn, entails that

$$
V(k) = \left(\frac{c\mu}{\alpha + c\mu} \right)^{k-l} V^*(l), \quad \text{for } k \geq l, \tag{3.9}
$$

which shows that $V(k)$ is uniformly bounded in k. Hence, $V \equiv V^*$ since (2.15) has a unique solution that is uniformly bounded in k, which in turn yields $x \equiv x^*$. Therefore, $x^*(k) + g(k) \leq 0$ for $k > l$, or equivalently, $u_k^* = 0$ for $k \geq l$, which concludes the proof. □

The methodology used in the proof of Proposition 3.1 does not fall into any of the categories that were reported in Section 1. Therefore, we remark that the importance of the result lies not as much in the optimality of threshold policies (which is what we would intuitively expect) but rather in the method of proof. The next section shows that this method also applies to the case where $\lambda_2 > 0$, although this case differs from the single-stream case in an essential way: in the two-stream case, the number of customers in the system is *never* bounded from above regardless of the admission policy for class 1 customers. This fact makes the analysis of the two-stream case much more involved.

4 Discounted Reward Problem: the Two-Stream Case

This section presents the analysis of the discounted problem with two streams of customers. Recall that only the stream of class 1 customers is controlled. Again, our objective is to find an admission policy that maximizes the discounted cost function (2.4).

To illustrate the basic difference between both cases $\lambda_2 = 0$ and $\lambda_2 > 0$, let us briefly come back to the single-stream case. Denote by $\mathbf{S}(l)$ the system of $l+1$ equations obtained from the DP equations (2.15) by setting $u_k^* = 0$ for $k = 0, 1, \ldots, l-1$ and $u_l^* = 0$.

Consider first the case where $\lambda_2 = 0$. Fix α as indicated in Proposition 3.1, and let $l^* < +\infty$ be the smallest integer such that $u_{l^*}^* = 0$ (the boundedness of l^* will be discussed later on, see Remark 4.2). Then, $u_k^* = \mathbf{1}\{k < l^*\}$ by Proposition 3.1, and the optimal threshold l^* is the smallest integer $l \geq c$ such that the (unique) solution x to the system $\mathbf{S}(l)$ of $l+1$ unknown variables and $l+1$ equations satisfies the inequalities $x(k) + g(k) > 0$ for $1 \leq k < l$ and $x(l+1) + g(l+1) \leq 0$. Note that this procedure defines a computational algorithm for the determination of the optimal threshold (see the end of this section where a much simpler numerical approach is given).

Assume now that $\lambda_2 > 0$, and suppose that there exists a threshold policy that solves the discounted problem — which is what we would intuitively expect. Then, a glance at $\mathbf{S}(l)$ indicates that this set of equations now contains $l+2$ unknown variables and only $l+1$ equations. Therefore, an equation is missing in order to compute the optimal threshold.

The next lemma addresses the problem of determining this missing equation in the case where a threshold policy is optimal. This will yield the construction of the optimal value function (Propositions 4.1 and 4.2), and subsequently the optimal control (Proposition 4.3).

Let us first introduce some notation. Let

$$\beta_1 := \frac{\alpha + c\mu + \lambda_2 - \sqrt{(\alpha + c\mu + \lambda_2)^2 - 4\,\lambda_2\, c\mu}}{2\,\lambda_2}; \tag{4.1}$$

$$\beta_2 := \frac{\alpha + c\mu + \lambda_2 + \sqrt{(\alpha + c\mu + \lambda_2)^2 - 4\,\lambda_2\, c\mu}}{2\,\lambda_2}, \tag{4.2}$$

be the two roots of the polynomial (in z) $\lambda_2\, z^2 - (\alpha + c\mu + \lambda_2)\, z + c\mu$. Note that $0 < \beta_1 < 1 < \beta_2$ for all $\alpha > 0$.

The following result will also be used. Assume that $\lambda_2 < c\mu$. Then, by observing that $\beta_1 = 1$ when $\alpha = 0$, and that the mapping $\alpha \to \beta_1$ is strictly decreasing in $[0, +\infty)$, it is seen that there exists $\alpha_0 > 0$ such that

$$\beta_1 > \Psi, \tag{4.3}$$

for $0 < \alpha < \alpha_0$, where Ψ has been introduced in (2.3).

The lemma below determines the missing equation in the case where $u_k^* = 0$ for $k \geq l$, and therefore strongly suggests what has to be the analog to (3.7) in the case where $\lambda_2 > 0$ (see Proposition 4.1).

LEMMA 4.1 *Assume that there exists $c \leq l < +\infty$ such that $u_k^* = 0$ for $k \geq l$. Then,*

$$V^*(k) = \beta_1 V^*(k-1), \quad \textit{for } k \geq l, \tag{4.4}$$

and

$$x^*(k) = \beta_1 x^*(k-1), \quad \textit{for } k > l. \tag{4.5}$$

PROOF. If $u_k^* = 0$ for $k \geq l \geq c$, then from (2.15)

$$V^*(k+1)\lambda_2 - (\alpha + c\mu + \lambda_2)V^*(k) + c\mu\, V^*(k-1) = 0, \tag{4.6}$$

for $k \geq l$. Let us show that any solution $(V(k))_{k \geq l-1}$ of (4.6) can be expressed as

$$V(k) = a\,\beta_1^k + b\,\beta_2^k, \tag{4.7}$$

for $k \geq l-1$, where a and b are arbitrary numbers.

First observe that both functions $k \to \beta_i^k$, $i = 1, 2$, satisfy (4.7), which therefore implies that $k \to a\,\beta_1^k + b\,\beta_2^k$ also satisfies (4.6) for any pair (a, b) of real numbers. Conversely, let $(V(k))_{k \geq l-1}$ be a solution of (4.6) and let us show that there exists two numbers a and b such that $V(k) = a\,\beta_1^k + b\,\beta_2^k$, $k \geq l-1$.

Since $\beta_1 \neq \beta_2$, $\beta_1 \neq 0$, $\beta_2 \neq 0$, it is seen that the following system of equations in (x, y):

$$V(l-1) = \beta_1^{l-1}\,x + \beta_2^{l-1}\,y; \tag{4.8}$$

$$V(l) = \beta_1^l\,x + \beta_2^l\,y, \tag{4.9}$$

has a unique solution (a, b). Using now the fact that $(V(k))_{k \geq l-1}$ is a solution of (4.6) together with (4.8), (4.9), it is readily seen by induction that $V(k) = a\,\beta_1^k + b\,\beta_2^k$ for $k \geq l-1$, which proves (4.7).

Assume now that $V^*(l-1)$ and $V^*(l)$ are known numbers. Then, cf. (4.6), (4.7),

$$a = \frac{-V^*(l) + \beta_2\,V^*(l-1)}{\beta_1^{l-1}\,(\beta_2 - \beta_1)}; \tag{4.10}$$

$$b = \frac{V^*(l) - \beta_1\,V^*(l-1)}{\beta_2^{l-1}\,(\beta_2 - \beta_1)}. \tag{4.11}$$

If $b \neq 0$, then we deduce from (4.7) that $|\lim_{k\uparrow\infty} V^*(k)| = +\infty$ since $\beta_1 < 1$ and $\beta_2 > 1$ for $\alpha > 0$ (cf. (4.1), (4.2)), which contradicts the fact that $V^*(k)$ is uniformly bounded in k (cf. Section 2). Therefore, $b = 0$, or equivalently, cf. (4.11), $V^*(l) = \beta_1 V^*(l-1)$, which in turn yields from (4.7) that $V^*(k) = \beta_1 V^*(k-1)$ for $k \geq l$. Using now (2.16) and (4.4), we readily obtain (4.5), which completes the proof. □

From now on, we shall assume that:

- $g : \mathbb{N} \to \mathbb{R}^+$ is modified unimodal with mode c and exponential factor Ψ;
- $g(k) \geq g(k+1)$, for $k = 1, 2, \ldots, c-1$ (see Remark 4.1);
- $\lambda_2 < c\mu$;
- α lies in $(0, \alpha_0)$.

Note that the case where $\lambda_2 < c\mu$ is the only case of interest for the average reward problem (which is the criterion we are really interested in), since otherwise the long-run average cost function (2.8) is zero under any admission policy, which in turn implies that all admission policies are equivalent (see Section 5).

We now state the key result of this section:

PROPOSITION 4.1 *If there exists a finite integer $m \geq 0$ such that the set of equations*

$$0 = -\alpha \sum_{i=0}^{k} y(i) + \lambda_1 (y(k+1) + g(k+1)) + \lambda_2 y(k+1) - \mu_k y(k), \quad 0 \leq k < m+c; \tag{4.12}$$

$$0 = -\alpha \sum_{i=0}^{m+c} y(i) + \lambda_2 y(m+c+1) - c\mu y(m+c); \tag{4.13}$$

$$0 = y(m+c+1) - \beta_1 y(m+c), \tag{4.14}$$

has a solution y that satisfies

$$y(k) + g(k) > 0, \quad \textit{for } 1 \leq k \leq m+c; \tag{4.15}$$

$$y(m+c+1) + g(m+c+1) \leq 0, \tag{4.16}$$

then

1. *$x^*(k) = y(k)$ for $0 \leq k \leq m+c+1$ and $x^*(k) = \beta_1 x^*(k-1)$ for $k > m+c+1$;*
2. *$u_k^* = \mathbf{1}\{k < m+c\}$ for $k \in \mathbb{N}$.*

PROOF. Let $m \geq 0$ be such that $(y(k))_{k=0}^{m+c+1}$ satisfies the set of equations (4.12)-(4.14) and inequalities (4.15), (4.16). Define $x : \mathbb{N} \to \mathbb{R}$ as

$$x(k) = \begin{cases} y(k), & k = 0, 1, \ldots, l; \\ \beta_1 x(k-1), & k = l+1, l+2, \ldots, \end{cases} \tag{4.17}$$

with $l := m + c$. Note that $x(k)$, $k > l$, is the recursion we get from Lemma 4.1 if we choose $u_k^* = 0$ for $k > l$.

We prove that $x(k) + g(k) \leq 0$ for $k > l$ by induction on k.

Basis step. Let $k = l + 1$. From the definition of x we have

$$\begin{aligned} x(l+1) &= \beta_1 y(l), \\ &= y(l+1), \text{ from (4.14)}, \\ &\leq -g(l+1), \end{aligned}$$

from (4.16).

Inductive step. We assume that $x(k') + g(k') \leq 0$ for $k' = l+1, l+2, \ldots, k$. We show that $x(k+1) + g(k+1) \leq 0$. We have, cf. (4.17),

$$\begin{aligned} x(k+1) &= \beta_1 x(k), \\ &\leq -\beta_1 g(k), \quad \text{from the induction hypothesis}, \\ &\leq -\Psi g(k), \quad \text{from (4.3)}, \\ &\leq -g(k+1), \end{aligned}$$

since g is modified unimodal. Consequently,

$$x(k) + g(k) \leq 0, \quad \text{for } k > l. \tag{4.18}$$

By combining this result together with the definitions of y, x and β_1, it is easily seen that x satisfies the DP equation (2.17).

Let $V(k) := \sum_{i=0}^{k} x(i)$, $k \in \mathbb{N}$. Then, V satisfies the DP equation (2.15), and further it is easily seen from the properties of x that V is uniformly bounded in k. Therefore, $V^* \equiv V$ since (2.15) has a unique uniformly bounded solution (Theorem 2.1), which implies that $x^* \equiv x$. Consequently, cf. (4.15), (4.17) and (4.18), $u_k^* = \mathbf{1}\{k < l\}$ for $k \in \mathbb{N}$. □

The next result establishes the existence of a solution to (4.12)-(4.16).

PROPOSITION 4.2 *There exists a finite integer $m = m^*$, $m^* \geq 0$, such that the (unique) solution to the set of equations (4.12)- (4.14) satisfies the constraints (4.15), (4.16). Further, m^* is uniformly bounded as $\alpha \downarrow 0$.*

The proof of Proposition 4.2 relies upon the following three lemmas, whose proofs are given in Appendix A. We introduce the following notation: for any $m \geq 0$, $(x_{m+c}(k))_{k=0}^{r+c+1}$ will denote the unique solution to the set of equations (4.12)-(4.14) (the uniqueness of the solution is discussed at the beginning of Appendix A).

LEMMA 4.2 *The unique solution $(x_c(k))_{k=0}^{c+1}$ to the set of equations (4.12)-(4.14) when $m = 0$ is such that*

$$x_c(k) + g(k) > 0, \quad \textit{for } 1 \leq k \leq c. \tag{4.19}$$

LEMMA 4.3 *Let C_m, $m \geq 0$, be the condition on the model parameters λ_1, λ_2, μ, c, α, $(g(k))_{k=1}^{m+1+c}$, which is equivalent to $x_{m+c}(m+1+c)+g(m+1+c) \leq 0$. If none of the conditions $C_0, C_1, \ldots, C_{m-1}$ holds, $m \geq 1$, then $x_{m+c}(k)+g(k) > 0$ for $k = 1, 2, \ldots, m+c$.*

LEMMA 4.4 *There exists m, $0 \leq m < +\infty$, such that $x_{m+c}(m+c+1)+g(m+c+1) \leq 0$. Moreover, m is uniformly bounded as $\alpha \downarrow 0$.*

PROOF(of Proposition 4.2).

Fix α such that $0 < \alpha < \alpha_0$. Let m^* be the smallest nonnegative integer such that $x_{m^*+c}(m^*+c+1) + g(m^*+c+1) \leq 0$, where the existence of m^* is ensured by Lemma 4.4. If $m^* = 0$ then the proposition follows from Lemma 4.2, whereas if $m^* > 0$ the proposition follows from Lemma 4.3. The second part follows from the second statement of Lemma 4.4. □

Combining Propositions 4.1 and 4.2 yields the following final result:

PROPOSITION 4.3 *Assume $\lambda_2 < c\mu$. Let g be modified unimodal with mode c and exponential factor Ψ. If g is nonincreasing in $[1, c]$, then the threshold policy $u_k^* = \mathbf{1}\{k < m^* + c\}$, $k \in \mathbb{N}$, is optimal for $\alpha \in (0, \alpha_0)$. Moreover, $m^* < +\infty$ and m^* is uniformly bounded as $\alpha \downarrow 0$.*

Before concluding this section, let us briefly address the numerical computation of the optimal threshold $m^* + c$. This computation can be performed very easily (i.e., without solving systems of linear equations) from (A.51) by using the recursions (A.23) and (A.24) in Appendix A. More precisely, m^* will be the smallest nonnegative integer such that (A.51) holds.

REMARK 4.1 The assumption that g is nonincreasing in $[1, c]$ is used in the proof of Lemma 4.2. It is clearly fulfilled by the function g that has been defined at the end of Section 2. We conjecture that Lemma 4.2 holds without this extra assumption on g (we have checked it for $c = 2$ and $c = 3$, which implies, in particular, that Proposition 4.3 holds for $c \leq 3$ without this assumption).

REMARK 4.2 Since $\lim_{\lambda_2 \downarrow 0} \beta_1 = c\mu/(c\mu+\alpha)$, it follows that relations (4.4) reduce to relations (3.9) that were obtained in the single-stream case. Hence, as expected, the results of the single-stream case may be deduced from the analysis of the two-stream case by letting $\lambda_2 \downarrow 0$. In particular, this implies the finiteness of the optimal threshold in the single-stream case.

5 THE AVERAGE REWARD CONTROL PROBLEM

In this section we shall discuss the long-run average reward control problem for the M/M/c queue with $\lambda_1 > 0$ and $\lambda_2 \geq 0$.

Since $V_\alpha(k, x; U)$ is well defined for all $k \in \mathbb{N}$, $x \in \{0, 1\}$, $U \in \mathcal{U}$ (see (2.4), (2.5)), we know from a Tauberian theorem (Widder [14], pp. 181-182) that

$$W(k, x; U) = \liminf_{T \uparrow \infty} \frac{W_T(k, x; U)}{T} \leq \liminf_{\alpha \downarrow 0} \alpha V_\alpha(k, x; U), \tag{5.1}$$

for all $k \in \mathbb{N}$, $x \in \{0,1\}$, $U \in \mathcal{U}$. Further, if $\lim_{T\uparrow\infty} W_T(k,x;U)/T$ exists then $\lim_{\alpha\downarrow 0} \alpha V_\alpha(k,x;U)$ exists as well, and

$$W(k,x;U) = \lim_{\alpha\downarrow 0} \alpha\, V_\alpha(k,x;U), \tag{5.2}$$

for all $k \in \mathbb{N}$, $x \in \{0,1\}$, $U \in \mathcal{U}$.

Assume first that $\lambda_2 < c\mu$ and let k and x be fixed numbers in $\mathbb{N}$ and $\{0,1\}$, respectively. For any $\alpha \in (0,\alpha_0)$, let $u^*_\alpha : \mathbb{N} \to \{0,1\}$ be the optimal control for the discounted reward problem, that is from Proposition 4.3,

$$u^*_\alpha(j) = \mathbf{1}\{j \le c + m^*_\alpha\},$$

for $j \ge 0$ with $0 \le m^*_\alpha < +\infty$. Consequently, for $\alpha \in (0,\alpha_0)$,

$$\alpha\, V_\alpha(k,x;U) \le \alpha\, V_\alpha(k,x;u^*_\alpha), \tag{5.3}$$

for all $U \in \mathcal{U}$.

Let $\{\alpha_i\}_1^\infty$ be a sequence in $(0,\alpha_0)$ such that $\alpha_i \downarrow 0$ as $i \uparrow \infty$. Since the mapping $\alpha \to m^*_\alpha$ is uniformly bounded as $\alpha \downarrow 0$ (cf. Proposition 4.3) and since m^*_α is an integer, there exists $J < +\infty$ and a subsequence of $\{\alpha_i\}_1^\infty$, denoted as $\{\alpha_j\}_1^\infty$, such that $m^*_{\alpha_j}$ is a constant (denoted as m^*) for all $j \ge J$. Define $u^*(j) := \mathbf{1}\{j < c + m^*\}$, $j \ge 0$.

If we now take the limit in (5.3) along α_j, $j \uparrow \infty$, we get from (5.1) that for any policy $U \in \mathcal{U}$

$$W(k,x;U) \le \liminf_{j\uparrow\infty} \alpha\, V_\alpha(k,x;u^*). \tag{5.4}$$

Under the stationary threshold policy u^* the Markov chain $\{Z_n\}_1^\infty$ is clearly ergodic. Therefore, $\lim_{T\uparrow\infty} W_T(k,x;u^*)/T$ exists (Chung [2], Section I.15) which implies from (5.2) and (5.4) that

$$W(k,x;U) \le W(k,x;u^*),$$

for all $U \in \mathcal{U}$.

For $\lambda_2 \ge c\mu$, it should be clear from (2.1), together with the fact that $\lim_{k\uparrow\infty} g(k) = 0$, cf. (2.3), that $W(k,x;U) = 0$ for all $U \in \mathcal{U}$, $k \in \mathbb{N}$, $x \in \{0,1\}$.

The results of this section are collected in the following proposition:

PROPOSITION 5.1 *Let g be modified unimodal with mode c and exponential factor Ψ such that $g(k) \ge g(k+1)$ for $1 \le k \le c-1$. If $\lambda_2 < c\mu$, then there exists a stationary threshold policy with finite threshold that is average optimal over the class of all admissible policies. If $\lambda_2 \ge c\mu$, then all admissible policies are average optimal.*

ACKNOWLEDGEMENT. The authors would like to thank P. Mussi for a useful discussion about Lemma 4.1.

A Appendix

We first introduce some notation and establish some intermediate results.

Define the matrices

$$M_k^j(a_0,b_0,c_0) := \begin{pmatrix} a_0 & \alpha & \alpha & \cdots & \alpha & \alpha & b_0 \\ -\mu_j & \alpha+\lambda & \alpha & \cdots & \alpha & \alpha & b_0 \\ 0 & -\mu_{j+1} & \alpha+\lambda & & & & \\ 0 & 0 & -\mu_{j+2} & & & & \\ & & & & \vdots & \vdots & \vdots \\ & & & & \alpha & & \\ & & & & \alpha+\lambda & \alpha & \\ 0 & 0 & 0 & & -\mu_{k-1} & \alpha+\lambda & b_0 \\ 0 & 0 & 0 & \cdots & 0 & -\mu_k & c_0 \end{pmatrix}, \tag{A.1}$$

for $1 \le j < k$, and

$$M_k^k(a_0,b_0,c_0) := \begin{pmatrix} a_0 & b_0 \\ -\mu_k & c_0 \end{pmatrix}, \tag{A.2}$$

for $k \ge 1$, where a_0, b_0 and c_0 are arbitrary constants (recall that $\mu_j = \mu \min(j,c)$)

Let $|M|$ be the determinant of any matrix M, with the convention that $|x| = x$ if x is a scalar number. It is easily seen by using an induction argument that

$$|M_k^j(a_0,b_0,c_0)| > 0, \tag{A.3}$$

when $a_0 > 0$, $b_0 > 0$ and $c_0 > 0$, for $1 \le j \le k$.

Let us show that the set of equations (4.12)-(4.14) has a unique solution for all $m \ge 0$. Let $x_{m+c}(k) := y(k)$, $k = 0, 1, \ldots, m+c+1$, in Proposition 4.1.

By substituting (4.14) into (4.13), by eliminating $x_{m+c}(0)$ from (4.12) by using (4.13), and finally by using the definition of β_1, we obtain the matrix equation:

$$A_{m+c}\,\mathbf{x}_{m+c} = -\lambda_1\,\mathbf{g}_{m+c}, \tag{A.4}$$

where

$$\begin{aligned} \mathbf{x}_{m+c} &:= (x_{m+c}(1), \ldots, x_{m+c}(m+c))^{\mathrm{T}}; \\ \mathbf{g}_{m+c} &:= (g(1), \ldots, g(m+c))^{\mathrm{T}}; \\ A_{m+c} &:= \begin{cases} M^1_{m+c-1}(\alpha+\lambda, c\mu/\beta_1 - \lambda_2, \lambda_1 + c\mu/\beta_1), & \text{if } m+c > 1; \\ \lambda_1 + c\mu/\beta_1, & \text{if } m+c = 1. \end{cases} \end{aligned}$$

By noting that $c\mu/\beta_1 - \lambda_2 > 0$ (since $\beta_1 < 1$, cf. (4.1) for $\alpha > 0$, and $\lambda_2 < c\mu$) we see from (A.3) that $|A_{m+c}| > 0$ for $m \ge 0$. Therefore, the set of equations (4.12)-(4.14) has a unique solution for all $m \ge 0$.

We start with the proof of Lemma 4.2.

PROOF(of Lemma 4.2). For $m = 0$ rewrite the equation (A.4) as

$$A_c\,[\mathbf{x}_c + \mathbf{g}_c] = [A_c - \lambda_1\, I_c]\,\mathbf{g}_c := \mathbf{h}_c,$$

where I_c stands for the identity matrix and $\mathbf{h}_c := (h_c(1), \ldots, h_c(c))^{\mathrm{T}}$. It follows readily that

$$h_c(k) = -\mu_{k-1}\, g(k-1)\, \mathbf{1}\{k > 1\} + \alpha \sum_{i=k}^{c-1} g(i) + \lambda_2\, g(k) + \left(\frac{c\mu}{\beta_1} - \lambda_2\right) g(c), \tag{A.5}$$

for $k = 1, 2, \ldots, c$.

By developing the determinant which forms the numerator of $x_c(j) + g(j)$ to the j-th column we get after a tedious but easy computation

$$\begin{aligned}(x_c(j) + g(j))\,|A_c| &= |A_c^{j+1}| \sum_{k=1}^{j} h_c(k)\,|\Lambda_{k-1}|\frac{(j-1)!}{(k-1)!}\,\mu^{j-k} \\ &- \sum_{k=1}^{j} |\Lambda_{k-1}|\,\frac{(j-1)!}{(k-1)!}\,\mu^{j-k} \sum_{i=j+1}^{c} h_c(i)\,\lambda^{i-j-1}\,|V_c^i|,\end{aligned} \tag{A.6}$$

for $j = 1, 2, \ldots, c$, where

$$\begin{aligned}
A_c^j &:= \begin{cases} M_{c-1}^j(\alpha+\lambda, c\mu/\beta_1 - \lambda_2, \lambda_1 + c\mu/\beta_1), & \text{if } 1 \le j \le c-1; \\ \lambda_1 + c\mu/\beta_1, & \text{if } j = c; \\ 1, & \text{if } j = c+1; \end{cases} \\
V_c^j &:= \begin{cases} M_{c-1}^j(\alpha, c\mu/\beta_1 - \lambda_2, \lambda_1 + c\mu/\beta_1), & \text{if } 1 \le j \le c-1; \\ c\mu/\beta_1 - \lambda_2, & \text{if } j = c; \end{cases} \\
\Lambda_k &:= \begin{cases} M_{k-1}^1(\alpha+\lambda, \alpha, \alpha+\lambda), & \text{if } k > 1; \\ \alpha+\lambda, & \text{if } k = 1; \\ 1, & \text{if } k = 0. \end{cases}
\end{aligned} \tag{A.7}$$

With the above definitions the following recursions can easily be established for $j = 1, 2, \ldots, c-1$,

$$|A_c^j| = (\alpha+\lambda)\,|A_c^{j+1}| + j\mu\,|V_c^{j+1}|, \tag{A.8}$$

$$|V_c^j| = \alpha\,|A_c^{j+1}| + j\mu\,|V_c^{j+1}|. \tag{A.9}$$

Repeated application of the recursions (A.8) and (A.9) leads to

$$|A_c^{j+1}| = \lambda^{c-j} + \sum_{i=j+1}^{c} \lambda^{i-j-1}\,|V_c^i|, \;\; j = 0, 1, \ldots, c-1. \tag{A.10}$$

Introducing (A.10) into (A.6) finally yields for $j = 1, 2, \ldots, c$,

$$(x_c(j) + g(j))\,|A_c| = \sum_{k=1}^{j} \frac{(j-1)!}{(k-1)!}\,|\Lambda_{k-1}|\mu^{j-k}\left\{h_c(k)\,\lambda^{c-j} + \sum_{i=j+1}^{c}\,[h_c(k) - h_c(i)]\,\lambda^{i-j-1}\,|V_c^i|\right\}.$$

As we have seen before $|A_c| > 0$, $|\Lambda_{k-1}| > 0$ for $k = 0, 1, \ldots, c-1$, $|V_c^i| > 0$ for $i = 1, 2, \ldots, c$. Further, the assumptions on g imply that $h_c(k) > 0$ for $k = 1, 2, \ldots, c$ and that $h_c(k) - h_c(i) > 0$ for $k < i$. Hence, $x_c(j) + g(j) > 0$ for $j = 1, 2, \ldots, c$, which concludes the proof of Lemma 4.2. □

PROOF(of Lemma 4.3). Let us show that the lemma is true if

$$x_{m+c}(0) > x_{m-1+c}(0), \tag{A.11}$$

when condition C_{m-1}, $m \geq 1$, is not satisfied. We use an induction argument.

First notice that, cf. (4.12), (4.13),

$$x_{m+c}(k) = \frac{\alpha\sum_{i=0}^{k-1} x_{m+c}(i) - \lambda_1\, g(k) + \mu_{k-1}\, x_{m+c}(k-1)}{\lambda}, \quad 1 \leq k \leq m+c; \tag{A.12}$$

$$x_{m+c}(m+c+1) = \frac{\alpha\sum_{i=0}^{m+c} x_{m+c}(i) + \mu_{m+c}\, x_{m+c}(m+c)}{\lambda_2}, \tag{A.13}$$

for all $m \geq 0$, $c \geq 1$.

Basis step. Assume that C_0 is not satisfied and let us show that $x_{1+c}(k) + g(k) > 0$ for $1 \leq k \leq 1 + c$. From (A.12) and the inequality (A.11) (with $m = 1$), it is readily seen that

$$x_{1+c}(k) > x_c(k), \quad \text{for } 0 \leq k \leq c, \tag{A.14}$$

which implies from Lemma 4.2 that

$$\begin{aligned} x_{1+c}(k) + g(k) &> x_c(k) + g(k), \\ &> 0, \end{aligned}$$

for $k = 1, 2, \ldots, c$.

Further, cf. (A.12), (A.13), (A.14),

$$\begin{aligned} x_{1+c}(1+c) &= \frac{\alpha\sum_{i=0}^{c} x_{1+c}(i) - \lambda_1 g(1+c) + \mu_c x_{1+c}(c)}{\lambda}, \\ &> \frac{\alpha\sum_{i=0}^{c} x_c(i) - \lambda_1 g(1+c) + \mu_c x_c(c)}{\lambda}, \\ &= \frac{\lambda_2 x_c(1+c) - \lambda_1 g(1+c)}{\lambda}, \end{aligned}$$

and so

$$\begin{aligned} x_{1+c}(1+c) + g(1+c) &> \frac{\lambda_2}{\lambda}\,(x_c(1+c) + g(1+c)), \\ &> 0, \end{aligned} \tag{A.15}$$

from the assumption on C_0.

Inductive step. Assume that none of the conditions $C_0, C_1, \dots, C_{m-2}$, $m \geq 2$, is satisfied and that

$$x_{m-1+c}(k) + g(k) > 0, \quad \text{for } 1 \leq k \leq m-1+c. \tag{A.16}$$

Let us show that $x_{m+c}(k) + g(k) > 0$ for $1 \leq k \leq m+c$ if C_{m-1} is not satisfied. It is easily seen from (A.11), (A.12), (A.13), that

$$\begin{aligned} x_{m+c}(k) &> x_{m-1+c}(k), \quad \text{for } 1 \leq k \leq m-1+c; & \text{(A.17)} \\ x_{m+c}(m+c) &> \frac{\lambda_2 x_{m-1+c}(m+c) - \lambda_1 g(m+c)}{\lambda}. & \text{(A.18)} \end{aligned}$$

Consequently, $x_{m+c}(k)+g(k) > 0$ for $1 \leq k \leq m-1+c$ from (A.16) and (A.17), and $x_{m+c}(m+c)+g(m+c) > 0$ from (A.18) and the assumption on C_{m-1}.

We are therefore left to prove that (A.11) holds if the condition C_{m-1} is not satisfied, $m \geq 1$. More precisely, we shall show that

$$[x_{m+c}(0) - x_{m-1+c}(0)]\,|A_{m+c}| = \frac{\lambda_1}{\alpha}\,\lambda^{m-1+c}\left(\frac{c\mu}{\beta_1} - \lambda_2\right)[x_{m-1+c}(m+c) + g(m+c)], \tag{A.19}$$

for all $m \geq 1$, if C_{m-1} is not satisfied (i.e., $x_{m-1+c}(m+c) + g(m+c) > 0$), which will prove (A.11) since $|A_{m+c}| > 0$ and $c\mu/\beta_1 - \lambda_2 > 0$. The proof decomposes into 3 steps:

Step 1: Computation of $x_{m-1+c}(m+c) + g(m+c)$

Recall the definition of Λ_k (cf. (A.7)). Let

$$Y_k := \begin{cases} M^1_{k-1}(\alpha+\lambda, \mu_k, \mu_k), & \text{if } k > 1; \\ \mu, & \text{if } k = 1; \\ 0, & \text{if } k = 0. \end{cases} \tag{A.20}$$

With these definitions, we easily obtain that

$$\begin{aligned} |\Lambda_k| &= (\alpha+\lambda)\,|\Lambda_{k-1}| + \alpha\,|Y_{k-1}|; & \text{(A.21)} \\ |Y_k| &= \mu_k\,(|\Lambda_{k-1}| + |Y_{k-1}|), & \text{(A.22)} \end{aligned}$$

for $k \geq 1$, from which we deduce that

$$\begin{aligned} \lambda\,\mu_k\,|\Lambda_{k-1}| &= \mu_k\,|\Lambda_k| - \alpha\,|Y_k|; & \text{(A.23)} \\ \lambda\,\mu_k\,|Y_{k-1}| &= (\alpha+\lambda)\,|Y_k| - \mu_k\,|\Lambda_k|, & \text{(A.24)} \end{aligned}$$

for $k \geq 1$.

By means of the recursion

$$x_{m+c}(m+c)\,|A_{m+c}| = -\lambda_1 g(m+c)\,|\Lambda_{m-1+c}| + \mu_{m-1+c}\,x_{m+c-1}(m-1+c)\,|A_{m-1+c}|, \quad m \geq 0,$$

that follows from (A.4) we obtain for $m \geq 0$,

$$x_{m+c}(m+c)\,|A_{m+c}| = -\lambda_1 \sum_{j=1}^{m+c} g(j)\,|\Lambda_{j-1}| \prod_{i=j}^{m-1+c} \mu_i, \tag{A.25}$$

$$= -\lambda_1\,(c\mu)^m \sum_{j=1}^{c-1} g(j)\frac{(c-1)!}{(j-1)!}\,\mu^{c-j}\,|\Lambda_{j-1}|$$

$$- \lambda_1 \sum_{j=0}^{m} g(j+c)(c\mu)^{m-j}\,|\Lambda_{c+j-1}|, \tag{A.26}$$

where by convention we have assumed that $\sum_{j=1}^{0} \cdot = 0$.

Next, we introduce the new quantities

$$\tilde{A}_{m+1} := \begin{cases} M^c_{m+c-1}(\alpha+\lambda, c\mu/\beta_1 - \lambda_2, \lambda_1 + c\mu/\beta_1), & \text{if } m \geq 1; \\ \lambda_1 + c\mu/\beta_1, & \text{if } m = 0; \end{cases}$$

$$\tilde{V}_{m+1} := \begin{cases} M^c_{m+c-1}(\alpha, c\mu/\beta_1 - \lambda_2, \lambda_1 + c\mu/\beta_1), & \text{if } m \geq 1; \\ c\mu/\beta_1 - \lambda_2, & \text{if } m = 0. \end{cases}$$

Then, for $m \geq 0$, $c \geq 1$,

$$|A_{m+c}| = |\Lambda_{c-1}|\,|\tilde{A}_{m+1}| + |Y_{c-1}|\,|\tilde{V}_{m+1}|. \tag{A.27}$$

For $c = 1$, the proof of (A.27) is trivial by noting that $\tilde{A}_{m+1} = A_{m+1}$ when $c = 1$, $\Lambda_0 = 1$ (cf. (A.7)) and $Y_0 = 0$ (cf. (A.20)). For $c \geq 2$, the proof follows from Lemma A.1.

For these new matrices, it is easily seen that for $m \geq 0$,

$$|\tilde{A}_{m+1}| = (\alpha+\lambda)\,|\tilde{A}_m| + c\mu\,|\tilde{V}_m|; \tag{A.28}$$

$$|\tilde{V}_{m+1}| = \alpha\,|\tilde{A}_m| + c\mu\,|\tilde{V}_m|, \tag{A.29}$$

from which it follows that for $m \geq 0$,

$$|\tilde{V}_{m+1}| = |\tilde{A}_{m+1}| - \lambda\,|\tilde{A}_m|; \tag{A.30}$$

$$|\tilde{A}_{m+2}| = (\alpha+\lambda+c\mu)\,|\tilde{A}_{m+1}| - c\mu\,\lambda|\tilde{A}_m|, \tag{A.31}$$

provided $|\tilde{A}_0| = 1$ and $|\tilde{V}_0| = 1 - \lambda_2\beta_1/(c\mu)$.

Further, (A.27) and (A.30) imply that for $m \geq 0$,

$$|A_{m+c}| = (|\Lambda_{c-1}| + |Y_{c-1}|)\,|\tilde{A}_{m+1}| - \lambda\,|Y_{c-1}|\,|\tilde{A}_m|. \tag{A.32}$$

Introducing the matrix

$$\tilde{\Lambda}_{m+1} = \begin{cases} M^c_{m+c-1}(\alpha+\lambda, \alpha, \alpha+\lambda), & \text{if } m \geq 1; \\ \alpha+\lambda, & \text{if } m = 0, \end{cases}$$

we have similarly to (A.32)

$$|\Lambda_{m+c}| = (|\Lambda_{c-1}| + |Y_{c-1}|)\,|\tilde{\Lambda}_{m+1}| - \lambda\,|Y_{c-1}|\,|\tilde{\Lambda}_m|, \tag{A.33}$$

for $m \geq 0$, with the convention that $|\tilde{\Lambda}_0| = 1$.

Using (A.33), (A.26) and the relation $x_{m-1+c}(m+c) = \beta_1\, x_{m-1+c}(m-1+c)$, cf. (4.14), we finally obtain for $m \geq 1$,

$$\begin{aligned}
&[x_{m-1+c}(m+c) + g(m+c)]\,|A_{m-1+c}| = \\
&\quad g(m+c)\,|A_{m-1+c}| - \lambda_1\,\beta_1\,(c\mu)^{m-1}\sum_{j=1}^{c-1} g(j)\,\frac{(c-1)!}{(j-1)!}\,\mu^{c-j}\,|\Lambda_{j-1}| - \\
&\quad \lambda_1\,\beta_1 \sum_{j=0}^{m-1} g(j+c)(c\mu)^{m-1-j}\left[(|\Lambda_{c-1}| + |Y_{c-1}|)\,|\tilde{\Lambda}_j| - \lambda\,|Y_{c-1}|\,|\tilde{\Lambda}_{j-1}|\right].
\end{aligned} \tag{A.34}$$

Step 2: Computation of $x_{m+c}(0) - x_{m-1+c}(0)$

In order to describe $x_{m+c}(0)$, we introduce the matrix

$$V^j_{m+c} := \begin{cases} M^j_{m+c-1}(\alpha, c\mu/\beta_1 - \lambda_2, \lambda_1 + c\mu/\beta_1), & \text{if } 1 \leq j < m+c; \\ c\mu/\beta_1 - \lambda_2, & \text{if } j = m+c. \end{cases}$$

By straightforward manipulations with the determinants, it can be shown that

$$x_{m+c}(1)\,|A_{m+c}| = -\lambda_1\, g(1)\,\frac{|A_{m+c}| - |V^1_{m+c}|}{\lambda} + \lambda_1 \sum_{j=2}^{m+c} g(j)\,\lambda^{j-2}\,|V^j_{m+c}|, \quad m \geq 0.$$

From this relation, it follows readily by using (A.12) with $k = 1$ that

$$x_{m+c}(0)\,|A_{m+c}| = \frac{\lambda_1}{\alpha}\sum_{j=1}^{m+c} g(j)\,\lambda^{j-1}\,|V^j_{m+c}|, \quad m \geq 0. \tag{A.35}$$

From the definitions of the matrices V^j_{m+c} and $\tilde{V}_{m+1+c-j}$ we have

$$V^j_{m+c} = \tilde{V}_{m+1+c-j}, \tag{A.36}$$

for $c \leq j \leq m+c$. Further, by applying again Lemma A.1 to V^j_{m+c} it is easily seen that

$$|V^j_{m+c}| = |W^j_{c-1}|\,|\tilde{A}_{m+1}| + |Z^j_{c-1}|\,|\tilde{V}_{m+1}|, \tag{A.37}$$

for $1 \leq j \leq c-1$, $m \geq 0$, where

$$W^j_{c-1} := \begin{cases} M^j_{c-2}(\alpha, \alpha, \alpha+\lambda), & \text{if } 1 \leq j \leq c-2; \\ \alpha, & \text{if } j = c-1; \end{cases}$$

$$Z^j_{c-1} := \begin{cases} M^j_{c-2}(\alpha, \mu_{c-1}, \mu_{c-1}), & \text{if } 1 \leq j \leq c-2; \\ \mu_{c-1}, & \text{if } j = c-1. \end{cases}$$

These matrices satisfy the recursions

$$|W_c^j| \;=\; (\alpha+\lambda)\,|W_{c-1}^j| + \alpha\,|Z_{c-1}^j|; \qquad \text{(A.38)}$$

$$|Z_c^j| \;=\; c\mu\left(|W_{c-1}^j| + |Z_{c-1}^j|\right), \qquad \text{(A.39)}$$

for $j = 1, 2, \ldots, c-1$, $c \geq 2$, while $|W_c^c| = \alpha$ and $|Z_c^c| = c\mu$ for $c \geq 1$. With the aid of relations (A.36) and (A.37) we may rewrite (A.35) as

$$\begin{aligned} x_{m+c}(0)\,|A_{m+c}| \;&=\; \frac{\lambda_1}{\alpha}\sum_{j=1}^{c-1} g(j)\,\lambda^{j-1}\left[|W_{c-1}^j|\,|\tilde{A}_{m+1}| + |Z_{c-1}^j|\,|\tilde{V}_{m+1}|\right] \\ &+\; \frac{\lambda_1}{\alpha}\sum_{j=0}^{m} g(c+j)\,\lambda^{c+j-1}\,|\tilde{V}_{m+1-j}|, \;\; m \geq 0. \end{aligned} \qquad \text{(A.40)}$$

Hence, for $m \geq 1$,

$$\begin{aligned} &\frac{\alpha}{\lambda_1}\left[x_{m+c}(0) - x_{m-1+c}(0)\right]|A_{m+c}|\,|A_{m-1+c}| = \\ &\quad \left(|\tilde{A}_{m+1}|\,|A_{m-1+c}| - |\tilde{A}_m|\,|A_{m+c}|\right)\sum_{j=1}^{c-1} g(j)\,\lambda^{j-1}\,|W_{c-1}^j| \\ &+\; \left(|\tilde{V}_{m+1}|\,|A_{m-1+c}| - |\tilde{V}_m|\,|A_{m+c}|\right)\sum_{j=1}^{c-1} g(j)\,\lambda^{j-1}\,|Z_{c-1}^j| \\ &+\; g(m+c)\,\lambda^{m-1+c}\,|\tilde{V}_1|\,|A_{m-1+c}| \\ &+\; \sum_{j=0}^{m-1} g(j+c)\,\lambda^{j-1+c}\left(|\tilde{V}_{m+1-j}|\,|A_{m-1+c}| - |\tilde{V}_{m-j}|\,|A_{m+c}|\right). \end{aligned} \qquad \text{(A.41)}$$

It can be shown by induction on m (and by using (A.31)) that for $m > 1$ and $j = 1, 2, \ldots, m-1$ or $m = 1$ and $j = 0$,

$$|\tilde{A}_{m+1}|\,|\tilde{V}_{m-j}| - |\tilde{A}_m|\,|\tilde{V}_{m+1-j}| = \lambda_1\beta_1\,(c\mu)^{m-1-j}\,\lambda^{m-j}\,|\tilde{V}_1|\,|\tilde{\Lambda}_j|. \qquad \text{(A.42)}$$

By applying (A.27) and (A.42) to the first two terms in the right-hand side of (A.41), as well as (A.30) and (A.42) to the last one, it is straightforward to reduce (A.41) for $m \geq 1$ to

$$\begin{aligned} &\frac{\alpha}{\lambda_1}\left[x_{m+c}(0) - x_{m-1+c}(0)\right]|A_{m+c}|\,|A_{m-1+c}| = \\ &\quad \lambda_1\beta_1(c\mu)^{m-1}\,\lambda^m\,|\tilde{V}_1|\sum_{j=1}^{c-1} g(j)\,\lambda^{j-1}\left(|Y_{c-1}|\,|W_{c-1}^j| - |\Lambda_{c-1}|\,|Z_{c-1}^j|\right) + \end{aligned}$$

$$g(m+c)\,\lambda^{m-1+c}\,|\tilde{V}_1|\,|A_{m-1+c}| + \sum_{j=0}^{m-1} g(j+c)\,\lambda_1\beta_1\,(c\mu)^{m-1-j}\,\lambda^{m-1+c}\,|\tilde{V}_1| \times$$

$$\left(\lambda\,|Y_{c-1}|\,|\tilde{\Lambda}_{j-1}| - [|\Lambda_{c-1}| + |Y_{c-1}|]\,|\tilde{\Lambda}_j|\right). \tag{A.43}$$

Step 3: Proof of (A.19)

We are now in position to prove (A.19). For $c = 1$, it is seen from (A.34) and (A.43) that (A.19) is true.

For $c \geq 2$, it follows from (A.34) and (A.43) that the relation

$$\sum_{j=1}^{c-1} g(j)\,\lambda^{j-1}\left(|Y_{c-1}|\,|W^j_{c-1}| - |\Lambda_{c-1}|\,|Z^j_{c-1}|\right) = -\lambda^{c-1}\sum_{j=1}^{c-1} g(j)\,\frac{(c-1)!}{(j-1)!}\,\mu^{c-j}\,|\Lambda_{j-1}|, \tag{A.44}$$

has to be proved in order to establish (A.19).

For $c = 2$, this relation reads

$$|Y_1|\,|W^1_1| - |\Lambda_1|\,|Z^1_1| = -\lambda\,\mu\,|\Lambda_0|,$$

which is true since $|Y_1| = \mu$, $|\Lambda_0| = 1$, $|\Lambda_1| = \alpha + \lambda$, $|W^1_1| = \alpha$ and $|Z^1_1| = \mu$.

Now suppose that (A.44) holds for some fixed c, $c \geq 2$. Then,

$$\lambda^c \sum_{j=1}^{c} g(j)\,\frac{c!}{(j-1)!}\,\mu^{c+1-j}\,|\Lambda_{j-1}| = \lambda^c\,g(c)\,c\mu\,|\Lambda_{c-1}|$$

$$+\quad \lambda\,c\mu \sum_{j=1}^{c-1} g(j)\,\lambda^{j-1}\left(|\Lambda_{c-1}|\,|Z^j_{c-1}| - |Y_{c-1}|\,|W^j_{c-1}|\right). \tag{A.45}$$

On the other hand, using the recursions (A.21), (A.22), (A.38) and (A.39), it follows that

$$\sum_{j=1}^{c} g(j)\,\lambda^{j-1}\left(|\Lambda_c|\,|Z^j_c| - |Y_c|\,|W^j_c|\right) = \lambda^{c-1}\,g(c)\,(c\mu\,|\Lambda_c| - \alpha\,|Y_c|) +$$

$$\lambda\,c\mu \sum_{j=1}^{c-1} g(j)\,\lambda^{j-1}\left(|\Lambda_{c-1}|\,|Z^j_{c-1}| - |Y_{c-1}|\,|W^j_{c-1}|\right). \tag{A.46}$$

Finally, by using (A.23) with $k = c$, it is seen that (A.45) and (A.46) are equivalent, so that (A.44) holds for c instead of $c - 1$, which concludes the proof. □

The notation introduced in the proof of Lemma 4.3 will be used in the remainder of this appendix.

PROOF(of Lemma 4.4). By using (A.25) and the identity $x_{m+c}(m+c+1) = \beta_1\,x_{m+c}(m+c)$, it follows that the condition C_m, $m \geq 0$, can be expressed as

$$g(m+1+c) \leq \frac{\lambda_1\beta_1}{c\mu} \sum_{j=1}^{m+c} g(j)\,\frac{|\Lambda_{j-1}|}{|A_{m+c}|} \prod_{i=j}^{m+c} \mu_i. \tag{A.47}$$

On the other hand, it is easily seen from the definition of the matrices A_{m+c}, Y_{m+c} and Λ_{m+c} that for $m \geq 0$

$$|A_{m+c}| = \left(\lambda_1 + \frac{c\mu}{\beta_1}\right) |\Lambda_{m-1+c}| + \left(\frac{c\mu}{\beta_1} - \lambda_2\right) |Y_{m-1+c}|, \qquad \text{(A.48)}$$

which implies, together with (A.21) and (A.22), that for $m \geq 1$

$$|A_{m+c}| = \lambda_1 |\Lambda_{m-1+c}| + \frac{c\mu}{\beta_1} |A_{m-1+c}|. \qquad \text{(A.49)}$$

Repeated applications of (A.49) give for $m \geq 1$

$$|A_{m+c}| = \left(\frac{c\mu}{\beta_1}\right)^m |A_c| + \lambda_1 \sum_{j=c+1}^{m+c} \left(\frac{c\mu}{\beta_1}\right)^{m+c-j} |\Lambda_{j-1}|. \qquad \text{(A.50)}$$

Note that (A.50) trivially holds for $m = 0$.

With (A.50) it is easily seen that (A.47) reduces to

$$g(m+1+c) \left[\left(\frac{c\mu}{\beta_1}\right)^m |A_c| + \lambda_1 \sum_{j=c+1}^{m+c} \left(\frac{c\mu}{\beta_1}\right)^{m+c-j} |\Lambda_{j-1}|\right]$$
$$\leq \lambda_1 \beta_1 \sum_{j=1}^{c} \mu^{m+c-j} c^m \frac{(c-1)!}{(j-1)!} |\Lambda_{j-1}| \, g(j) + \lambda_1 \beta_1 \sum_{j=c+1}^{m+c} (c\mu)^{m+c-j} |\Lambda_{j-1}| \, g(j), \qquad \text{(A.51)}$$

for $m \geq 0$.

Because α is such that

$$\Psi < \beta_1 < 1, \qquad \text{(A.52)}$$

and generally (see (2.3))

$$g(m+1+c) \leq \Psi^{m+1-j+c} g(j), \qquad \text{(A.53)}$$

for $j \geq c$, it follows that

$$g(m+1+c) \sum_{j=c+1}^{m+c} \left(\frac{c\mu}{\beta_1}\right)^{m+c-j} |\Lambda_{j-1}| < \beta_1 \sum_{j=c+1}^{m+c} (c\mu)^{m+c-j} |\Lambda_{j-1}|, g(j). \qquad \text{(A.54)}$$

Further, (A.52) and (A.53) implies that

$$\lim_{m \uparrow \infty} \frac{g(m)}{\beta_1^m} = 0,$$

so that there must exist an M (that clearly depends on α), $0 \leq M < +\infty$, such that for all $m \geq M$, $1 \leq j \leq c$,

$$\frac{g(m+1+c)}{\beta_1^m} |A_c| \leq \lambda_1 \beta_1 \mu^{c-j} \frac{(c-1)!}{(j-1)!} |\Lambda_{j-1}| \, g(j). \qquad \text{(A.55)}$$

By combining (A.51), (A.54) and (A.55), we finally see that C_m is satisfied for all $m \geq M$.

The proof is concluded by observing that such an M also exists for $\alpha = 0$, since $\beta_1 = 1$ if $\alpha = 0$, cf. Section 4, and since $\lim_{m\uparrow\infty} g(m) = 0$. □

LEMMA A.1 *Let $c \geq 2$ and define*

$$\tilde{A}_{m+1}(b_0, c_0) := \begin{cases} M^c_{m+c-1}(\alpha+\lambda, b_0, c_0), & \text{if } m \geq 1; \\ c_0, & \text{if } m = 0; \end{cases}$$

$$\tilde{V}_{m+1}(b_0, c_0) := \begin{cases} M^c_{m+c-1}(\alpha, b_0, c_0), & \text{if } m \geq 1; \\ b_0, & \text{if } m = 0; \end{cases}$$

$$W^j_{c-1}(a_0) := \begin{cases} M^j_{c-2}(a_0, \alpha, \alpha+\lambda), & \text{if } 1 \leq j \leq c-2; \\ a_0, & \text{if } j = c-1; \end{cases}$$

$$Z^j_{c-1}(a_0) := \begin{cases} M^j_{c-2}(a_0, \mu_{c-1}, \mu_{c-1}), & \text{if } 1 \leq j \leq c-2; \\ \mu_{c-1}, & \text{if } j = c-1. \end{cases}$$

Then,

$$|M^j_{m+c-1}| = |W^j_{c-1}(a_0)|\,|\tilde{A}_{m+1}(b_0, c_0)| + |Z^j_{c-1}(a_0)|\,|\tilde{V}_{m+1}(b_0, c_0)|, \tag{A.56}$$

for $1 \leq j \leq c-1$, $m \geq 0$, and for any constants a_0, b_0 and c_0.

PROOF. We use an induction argument. Assume that $m = 0$. Then, it is easily seen from definitions (A.1) and (A.2) that (A.56) holds for $j = 1, 2, \ldots, c-1$ and for arbitrary constants a_0, b_0 and c_0. Assume that (A.56) holds for $m = 1, 2, \ldots, n$, $j = 1, 2, \ldots, c-1$ and for arbitrary constants a_0, b_0 and c_0. Let a_0, b_0 and c_0 be fixed arbitrary constants and let $j \in \{1, 2, \ldots, c-1\}$. Then, cf. (A.1),

$$\begin{aligned} |M^j_{n+c}(a_0, b_0, c_0)| &= c_0\,|M^j_{n+c-1}(a_0, \alpha, \alpha+\lambda)| + \mu_{n+c}\,|M^j_{n+c-1}(a_0, b_0, b_0)|, \\ &= |W^j_{c-1}(a_0)| \left(c_0\,|\tilde{A}_{n+1}(\alpha, \alpha+\lambda)| + \mu_{n+c}\,|\tilde{A}_{n+1}(b_0, b_0)|\right) \\ &+ |Z^j_{c-1}(a_0)| \left(c_0\,|\tilde{V}_{n+1}(\alpha, \alpha+\lambda)| + \mu_{n+c}\,|\tilde{V}_{n+1}(b_0, b_0)|\right), \end{aligned}$$

by the induction hypothesis. The proof is concluded by observing that

$$\begin{aligned} |\tilde{A}_{n+2}(b_0, c_0)| &= c_0\,|\tilde{A}_{n+1}(\alpha, \alpha+\lambda)| + \mu_{n+c}\,|\tilde{A}_{n+1}(b_0, b_0)|; \\ |\tilde{V}_{n+2}(b_0, c_0)| &= c_0\,|\tilde{V}_{n+1}(\alpha, \alpha+\lambda)| + \mu_{n+c}\,|\tilde{V}_{n+1}(b_0, b_0)|, \end{aligned}$$

from the definition of the matrices $\tilde{A}_{m+1}(\cdot,\cdot)$ and $\tilde{V}_{m+1}(\cdot,\cdot)$. □

REFERENCES

[1] R. E. BARLOW AND F. PROSCHAN (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart and Winston, Inc., New York.

[2] K. L. CHUNG (1967). *Markov Chains with Stationary Transition Probabilities.* 2nd. Ed., Springer Verlag, New York.

[3] P. R. DE WAAL (1987). Performance analysis and optimal control of an M/M/1/k queueing system with impatient customers, in *Proc. Messung, Modellierung und Bewertung von Rechensystemen 4. GI/ITG-Fachtagung*, pp. 28-40, U. Herzog and M. Paterok, Eds., Springer-Verlag, Berlin.

[4] P. R. DE WAAL (1990). *Overload Control of Telephone Exchanges.* PhD. thesis, CWI, Amsterdam, 1990.

[5] B. HAJEK (1984). Optimal control of two interacting service stations. *IEEE Trans. Automat. Contr.*, vol. AC-29, no. 6, pp. 491-498.

[6] D. P. HEYMAN AND M. J. SOBEL (1982). *Stochastic Models in Operations Research*, Vol II. McGraw-Hill, New York.

[7] S. G. JOHANSEN AND S. S. STIDHAM (1980). Control of arrivals to a stochastic input-output system. *Adv. Appl. Prob.*, vol. 12, pp. 972-999.

[8] W. LIN AND P. R. KUMAR (1984). Optimal control of a queueing system with two heterogeneous servers. *IEEE Trans. Automat. Contr.*, vol. AC-29, no. 8, pp. 696-703.

[9] S. A. LIPPMAN (1975). Applying a new device in the optimization of exponential queueing systems. *Operat. Res.*, vol. 23, no. 4, pp. 687-710.

[10] S. A. LIPPMAN (1975). On dynamic programming with unbounded rewards. *Management Sci.*, vol. 21, no. 11, pp. 1225-233.

[11] D.-J. MA AND A. M. MAKOWSKI (1987). Optimality results for a simple flow control problem, in *Proc. 26th Conf. on Decision Contr.*, pp. 1852-1857, Los Angeles.

[12] R. SERFOZO (1981). Optimal control of random walks, birth and death processes, and queues. *Adv. Appl. Prob.*, vol. 13, pp. 61-83.

[13] S. S. STIDHAM (1985). Optimal control of admission to a queueing system. *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 705-713.

[14] D. V. WIDDER (1941). *The Laplace Transform.* Princeton University Press, Princeton.