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Descriptor Representations without Direct Feedthrough Term

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Descriptor representations are considered that are given by (E, A, B, C, D) with $D = 0$. Minimality under external equivalence is characterized in terms of the matrices E, A, B and C. Also, transformations are given by which minimal (E, A, B, C) representations are related under external equivalence. The results are compared with similar results for representations with $D \neq 0$; the transformations turn out to be more simple in the "$D = 0$" case. Finally, a realization procedure is given for obtaining a minimal (E, A, B, C) representation for a system that is given by autoregressive equations.

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1. Introduction and preliminaries

In this report we consider time invariant linear systems represented by

$$\begin{aligned} \sigma E\xi &= A\xi + Bu \\ y &= C\xi. \end{aligned} \tag{1.1}$$

Here σ denotes differentiation or shift, depending on whether one works in continuous or discrete time. The variables y, u and ξ take values in the output space Y, the input space U and the descriptor space X_d, respectively. The codomain of the mappings E and A will be denoted by X_e (equation space). It should be noted that the matrices E and A are not assumed to be square.

The above representation is a specific form of the so-called descriptor representation

$$\begin{aligned} \sigma E\xi &= A\xi + Bu \\ y &= C\xi + Du. \end{aligned} \tag{1.2}$$

The only difference between (1.1) and (1.2) is the absence of a direct feedthrough term D in (1.1). As we will see later, the direct feedthrough term plays an important role in descriptor representations, especially w.r.t. minimality issues. The reason for this is that the direct feedthrough term is related to the "non-dynamic" variables of the system as was first pointed out in [2].

The equivalence concept that we will use throughout the report is that of so-called "external equivalence". Systems are called externally equivalent if their induced "behaviours" are the same. Here the behaviour of a system consists of the time trajectories of the input and output variables (the "external variables") that arise from the system representation. For more details and motivation the reader is referred to [1, 3, 4]. It should be noted that the set of time trajectories of the output variables that are not influenced by the input variables (the "uncontrolled behaviour") remains invariant under external equivalence. This constitutes one of the main differences between external equivalence and so-called transfer equivalence where the invariant is the transfer function instead of the behaviour.

The aim of this report is the following. First, we consider the question under which conditions an (E, A, B, C) representation of a system is minimal among all other (E, A, B, C) representations that are externally equivalent. Here our definition of minimality is formulated in terms of three indices: the rank of E, the column defect of E (dim ker E) and the row defect of E (codim im E). We define a descriptor representation to be minimal if each of these three indices is minimal within the set of descriptor representations that correspond to the same behaviour. We will give a characterization of minimality in terms of the matrices E, A, B and C. We will also give the complete set of transformations by which minimal (E, A, B, C) representations that give rise to the same behaviour can be transformed into each other. Finally, we show how a minimal (E, A, B, C) realization can be constructed, starting from a system

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description in polynomial matrix fractional form.

Descriptor representations arise when dynamical systems are written as a combination of differential/difference equations and algebraic constraints. They have been found useful for instance in circuit models [9], econometric models [6] and system inversion [2]. In the literature of the past decade one can find many contributions on descriptor representations. Some of these are concerned with (E, A, B, C) representations, others with (E, A, B, C, D) representations. For example, two-point boundary-value descriptor systems (see [7]) are usually written in (E, A, B, C) form because of the time-reversible character of such systems: a symmetric representation is preferred to a non-symmetric one. As another example, the realization procedures of [14] and [1] lead from a transfer function to an (E, A, B, C) representation rather than an (E, A, B, C, D) representation. Both procedures are based on a decomposition of the transfer matrix into a strictly proper and a polynomial part. Separating finite and infinite frequencies in this way naturally leads to an (E, A, B, C) representation.

In [8] Rosenbrock defined the concept of *restricted system equivalence* for (E, A, B, C) representations:

DEFINITION ([8]) The descriptor representations (E, A, B, C) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ are *restricted system equivalent* if there exist matrices M and N with M and N invertible such that

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE-A & -B \\ C & 0 \end{bmatrix} = \begin{bmatrix} s\tilde{E}-\tilde{A} & -\tilde{B} \\ \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}. \tag{1.3}$$

With this equivalence concept dynamic and non-dynamic variables are treated alike. Verghese *et al.* ([11]) judged the above type of equivalence to be too restrictive: it did not allow for elimination of redundant non-dynamic variables. In order to define an alternative type of equivalence, they first had to incorporate the non-dynamic variables in a D term. The alternative type of equivalence (*strong equivalence*, see [11]) was then defined for (E, A, B, C, D) representations.

The following two algorithms will clarify the relation between (E, A, B, C) and (E, A, B, C, D) representations. The first algorithm gives a procedure for rewriting an (E, A, B, C, D) representation in (E, A, B, C) form while the reverse procedure is given by the second algorithm. In section 3 it will be proven that the algorithms lead to equivalent representations and that minimality properties are preserved.

ALGORITHM 1 Let a descriptor representation be given by (E, A, B, C, D).
Let $V = [V_1 \quad V_2]$ be an invertible matrix such that

$$DV = D[V_1 \quad V_2] = [D_1 \quad 0] \tag{1.4}$$

where D_1 is injective.
Let $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ be the inverse of V. Define $\xi_2 = T_1 u$. Then

$$\begin{aligned} Du &= D[V_1 \quad V_2] \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} u \\ &= [D_1 \quad 0] \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} u \\ &= D_1 T_1 u \\ &= D_1 \xi_2. \end{aligned}$$

Now define a representation $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ by

$$\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ -T_1 \end{bmatrix}, \quad \tilde{C} = [C \quad D_1]$$

ALGORITHM 2 Let a descriptor representation be given by (E, A, B, C). Decompose the descriptor space X_d as $X_{d1} \oplus X_{d2} \oplus X_{d3}$ where $X_{d3} = A^{-1}[\operatorname{im} E] \cap \ker E$ and $X_{d2} \oplus X_{d3} = \ker E$. Decompose the equation space X_e as $X_{e1} \oplus X_{e2}$ where $X_{e1} = \operatorname{im} E$. Accordingly write

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad C_2 \quad C_3]. \tag{1.5}$$

Then the matrix A_{22} is injective. Choose an appropriate basis in X_{e2}, such that w. r. t. this basis we have

$$A_{21} = \begin{bmatrix} A_{211} \\ A_{212} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}. \tag{1.6}$$

Now define a representation $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ by

$$\tilde{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_{11} - A_{12}A_{211} & A_{13} \\ A_{212} & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 - A_{12}B_{21} \\ B_{22} \end{bmatrix},$$

$$\tilde{C} = [C_1 - C_2A_{211} \quad C_3], \quad \tilde{D} = -C_2B_{21}. \tag{1.7}$$

In order to place this report into perspective we will now repeat some results for (E, A, B, C, D) representations. In [4] we gave a procedure for obtaining minimal (E, A, B, C, D) realizations under external equivalence for systems that are given in polynomial matrix fractional form. Necessary and sufficient conditions were derived for minimality under external equivalence in terms of E, A, B, C and D in [5]. Furthermore the transformations by which minimal equivalent (E, A, B, C, D) representations are related were presented in that paper. We will now repeat the last two results.

THEOREM 1.1 ([5]) *Let a descriptor representation be given by (E, A, B, C, D). The representation is minimal under external equivalence if and only if the following conditions hold:*

(i) $[E \quad B]$ *is surjective*
(ii) $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ *is injective*
(iii) $A[\ker E] \subset \operatorname{im} E$
(iv) $[sE^{\mathrm{T}} - A^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ *has full column rank for all* $s \in \mathbb{C}$.

THEOREM 1.2 ([5]) *Let (E, A, B, C, D) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be descriptor representations that are minimal under external equivalence. Then the two representations are externally equivalent if and only if there exist matrices M, N, X and Y with M and N invertible such that*

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}. \tag{1.8}$$

In the above results as well as in the development below, a prominent role is played by the so-called pencil representation:

$$\begin{aligned} \sigma G z &= F z \\ y &= H_y z \\ u &= H_u z. \end{aligned} \tag{1.9}$$

Here F and G are linear mappings from Z to X, where Z is the space of internal variables and X is the equation space. It is shown in [4, 13] that a minimal pencil representation can be realized directly from the behaviour of the system in a natural way (the representation is called minimal if both dim Z and dim X are minimal). We will now summarize some of the results on pencil representations. In these results the pencil representation is defined by taking the input and output variables together as the vector $w = [y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}}$ of external variables. A pencil representation is then written as

$$\sigma G z = F z$$
$$w = H z. \tag{1.10}$$

PROPOSITION 1.3 ([4]) *A pencil representation given by* (F, G, H) *is minimal under external equivalence if and only if the following conditions hold:*
(i) G *is surjective*
(ii) $[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *is injective*
(iii) $[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *has full column rank for all* $s \in \mathbb{C}$.

PROPOSITION 1.4 ([4, 12]) *Two minimal pencil representations* (F, G, H) *and* $(\tilde{F}, \tilde{G}, \tilde{H})$ *are externally equivalent if and only if there exist invertible matrices* S *and* T *such that*

$$\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} = \begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} T. \tag{1.11}$$

In [4, 5] it is shown that a close connection exists between a pencil representation and an (E, A, B, C, D) representation: an algorithm is given for rewriting a pencil representation in (E, A, B, C, D) form and vice versa in such a way that minimality is preserved. In the next section we present similar algorithms for rewriting a pencil representation in (E, A, B, C) form and vice versa. These algorithms are used for deriving minimality results and results on the transformation group in section 3. The results will be different from the results for (E, A, B, C, D) representations that were given above. In section 4 we will give a procedure for realization into (E, A, B, C) form. The procedure will be illustrated by an example.

2. RELATION WITH PENCIL FORM

In this section we present algorithms for obtaining an equivalent (E, A, B, C) representation from a pencil representation and vice versa. These algorithms will be used in the next section where known results on the minimality of pencil representations (see previous section) are exploited to derive results on the minimality of (E, A, B, C) representations. For that reason it is important that both algorithms preserve minimality. There is a trivial way to rewrite a pencil representation in (E, A, B, C) form. Starting from the pencil representation (F, G, H_y, H_u), we obtain an equivalent (E, A, B, C) representation that is given by

$$\sigma \begin{bmatrix} G \\ 0 \end{bmatrix} \xi = \begin{bmatrix} F \\ H_u \end{bmatrix} \xi + \begin{bmatrix} 0 \\ -I \end{bmatrix} u$$
$$y = H_y \xi. \tag{2.1}$$

However, it is not difficult to see that the minimality of the representation (F, G, H_y, H_u) does not necessarily imply that the representation (2.1) is minimal. The following algorithm does a better job at preserving minimality properties. It will be shown later that application of the algorithm to a minimal pencil representation leads to a minimal equivalent (E, A, B, C) representation.

ALGORITHM 3 Let a pencil representation be given by (F, G, H_y, H_u). Decompose the internal variable space Z as $Z_0 \oplus Z_1 \oplus Z_2 \oplus Z_3$ where $Z_3 = \ker G \cap \ker H$, $Z_2 \oplus Z_3 = \ker G \cap \ker H_y$, $Z_1 \oplus Z_2 \oplus Z_3 = \ker G$. Accordingly, write

$$G = [G_0 \quad 0 \quad 0 \quad 0], \qquad F = [F_0 \quad F_1 \quad F_2 \quad F_3],$$
$$H_y = [H_{y0} \quad H_{y1} \quad 0 \quad 0], \qquad H_u = [H_{u0} \quad H_{u1} \quad H_{u2} \quad 0]. \tag{2.2}$$

Then the matrices G_0 and H_{y1} have full column rank. Also, the matrix H_{u2} has full column rank, and by renumbering the u-variables if necessary, we can write

$$H_{u0} = \begin{bmatrix} H_{10} \\ H_{20} \end{bmatrix}, \quad H_{u1} = \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix}, \quad H_{u2} = \begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} \tag{2.3}$$

where H_{22} is invertible (or empty, if $\ker G \cap \ker H_y \subset \ker H_u$). Define descriptor matrices by

$$E = \begin{bmatrix} G_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \bar{F}_0 & \bar{F}_1 & F_3 \\ \bar{H}_{10} & \bar{H}_{11} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \bar{F}_2 \\ -I & \bar{H}_{12} \end{bmatrix},$$
$$C = [H_{y0} \quad H_{y1} \quad 0\,] \tag{2.4}$$

with

$$\begin{aligned} \bar{F}_0 &= F_0 - F_2 H_{22}^{-1} H_{20} \\ \bar{F}_1 &= F_1 - F_2 H_{22}^{-1} H_{21} \\ \bar{H}_{10} &= H_{10} - H_{12} H_{22}^{-1} H_{20} \\ \bar{H}_{11} &= H_{11} - H_{12} H_{22}^{-1} H_{21} \\ \bar{F}_2 &= F_2 H_{22}^{-1} \\ \bar{H}_{12} &= H_{12} H_{22}^{-1}. \end{aligned} \tag{2.5}$$

Vice versa, an (E, A, B, C) representation can be rewritten in pencil form by applying the algorithm of [5] which transforms (E, A, B, C, D) representations to equivalent pencil representations. In [5] it is shown that this algorithm preserves minimality: minimal (E, A, B, C, D) representations are transformed to minimal pencil representations. Surprisingly, the algorithm also preserves minimality for (E, A, B, C) representations, as will be shown later. We now repeat the algorithm of [5] when applied to (E, A, B, C) representations.

ALGORITHM 4 Let a descriptor representation be given by (E, A, B, C). Decompose the descriptor space X_d as $X_{d1} \oplus X_{d2}$ where $X_{d2} = \ker E$. Decompose the equation space X_e as $X_{e1} \oplus X_{e2} \oplus X_{e3}$ where $X_{e1} = \operatorname{im} E$ and $X_{e1} \oplus X_{e2} = \operatorname{im}[E \quad B]$. Accordingly write

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \quad C = [C_1 \quad C_2]. \tag{2.6}$$

Then the matrix B_2 is surjective. By renumbering the u-variables if necessary, we can write

$$\begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ 0 & 0 \end{bmatrix} \tag{2.7}$$

where B_{22} is invertible. Define pencil matrices as:

$$G = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_{11} \\ A_{31} & A_{32} & 0 \end{bmatrix}, \quad H_y = [C_1 \quad C_2 \quad 0\,],$$

$$H_u = \begin{bmatrix} 0 & 0 & I \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_{21} \end{bmatrix} \tag{2.8}$$

with

$$\begin{aligned} \bar{A}_{11} &= A_{11} - B_{12}B_{22}^{-1}A_{21} \\ \bar{A}_{12} &= A_{12} - B_{12}B_{22}^{-1}A_{22} \\ \bar{B}_{11} &= B_{11} - B_{12}B_{22}^{-1}B_{21} \\ \bar{A}_{21} &= -B_{22}^{-1}A_{21} \\ \bar{A}_{22} &= -B_{22}^{-1}A_{22} \\ \bar{B}_{21} &= -B_{22}^{-1}B_{21}. \end{aligned} \tag{2.9}$$

From [5] we have the following lemma.

LEMMA 2.1 ([5]) *Let (F, G, H_y, H_u) be a pencil representation that results from applying Algorithm 4 to a descriptor representation, given by (E, A, B, C). Then the two representations are externally equivalent and the following equalities hold:*

(i) $\text{rank}\, G = \text{rank}\, E$

(ii) $\text{codim im}\, G = \text{codim im}\, [E \quad B]$

(iii) $\dim\ker G = \dim\ker E + \dim B^{-1}[\text{im}\, E]$

(iv) $\dim\ker [G^T \quad H^T]^T = \dim(\ker E \cap \ker C \cap A^{-1}[\text{im}\, E])$

Moreover we have the following implications:

(v) $[E \quad B]$ *is surjective* $\Rightarrow$ G *is surjective*

(vi) $\ker E \cap \ker C \cap A^{-1}[\text{im}\, E] = \{0\} \Rightarrow [G^T \quad H^T]^T$ *is injective*

(vii) $[sE^T - A^T \quad C^T]^T$ *has full column rank for all* $s \in \mathbb{C} \Rightarrow [sG^T - F^T \quad H^T]^T$ *has full column rank for all* $s \in \mathbb{C}$.

We will now prove that applying algorithm 3 to a minimal pencil representation leads to an externally equivalent (E, A, B, C) representation that is minimal. For this purpose we first have to explore the concept of minimality under external equivalence for (E, A, B, C) representations. In the following we will present lower bounds for some of the indices that have to be minimized. These involve a certain invariant subspace $W^0 \subset W\, (= Y \oplus U)$ (see [4, 13]). Intuitively speaking, the subspace W^0 is spanned by the minimum number of "driving variables" of the system; when W^0 coincides with the input space we are dealing with a system with a strictly causal input-output structure. A definition of W^0 will be given in section 4. As already mentioned in the introduction, a pencil representation (F, G, H) can be obtained directly from the behaviour. It is therefore not surprising that W^0 can be easily expressed in terms of the matrices F, G and H. In [4] we proved the following proposition:

PROPOSITION 2.2 ([4]) *Assume that a pencil representation, given by (F, G, H), satisfies the following conditions:*

(i) G *is surjective*

(ii) $[G^T \quad H^T]^T$ *is injective.*

Then we have

$$W^0 = H[\ker G]. \tag{2.10}$$

Using the above proposition together with the properties of Algorithm 4, as expressed in the accompanying lemma, we are now able to express W^0 in terms of the matrices of an (E, A, B, C) representation.

LEMMA 2.3 *Let a descriptor representation be given by (E, A, B, C). Then necessary conditions for*

(E, A, B, C) to be minimal under external equivalence are
(i) $[E \quad B]$ *is surjective*
(ii) $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ *is injective.*
Moreover, if (i) and (ii) hold we have

$$\pi_Y W^0 = C[\ker E] \tag{2.11}$$

$$U \cap W^0 = B^{-1}[\operatorname{im} E]. \tag{2.12}$$

PROOF The fact that conditions (i) and (ii) are necessary for minimality is proved in [4]: Lemma 7.2 and Lemma 7.3 are also valid for (E, A, B, C) representations.
Next, assume that (i) and (ii) hold. Application of Algorithm 4 to our representation yields a pencil representation that, according to Lemma 2.1 satisfies the conditions of Prop. 2.2. We may therefore conclude that

$$W^0 = \operatorname{im} \begin{bmatrix} C_2 & 0 \\ 0 & I \\ -B_{22}^{-1}A_{22} & -B_{22}^{-1}B_{21} \end{bmatrix} \tag{2.13}$$

where the matrices are partitioned as in Algorithm 4. It now follows immediately that $\pi_Y W^0 = C[\ker E]$ and $U \cap W^0 = B^{-1}[\operatorname{im} E]$.

From the proofs of Lemma 7.2 and Lemma 7.3 in [4] we immediately have the following corollary.

COROLLARY 2.4 *Let a descriptor representation be given by (E, A, B, C). Then we have*
(i) $\dim \ker E \geqslant \dim(\pi_Y W^0)$
(ii) $\operatorname{codim} \operatorname{im} E \geqslant \operatorname{codim}(U \cap W^0)$.

We are now ready for the main theorems of this section.

THEOREM 2.5 *Let (E, A, B, C) be a descriptor representation that results from applying Algorithm 3 to a pencil representation, given by (F, G, H_y, H_u). Then the two representations are externally equivalent. Furthermore if (F, G, H_y, H_u) is minimal then (E, A, B, C) is also minimal.*

PROOF The only operations that are involved in Algorithm 3 are:
- choosing another basis for the internal variables
- reordering u-components

It is therefore immediate (see [10]) that the resulting descriptor representation is externally equivalent with (F, G, H_y, H_u). Further, the minimality of rank E follows immediately from the minimality of rank G since in both Algorithm 3 and Algorithm 4 we have

$$\operatorname{rank} E = \operatorname{rank} G.$$

According to Prop. 1.3 the minimality of (F, G, H_y, H_u) implies that G is surjective and $[G^{\mathrm{T}} \quad H_y^{\mathrm{T}} \quad H_u^{\mathrm{T}}]^{\mathrm{T}}$ is injective. Therefore $Z_3 = \{0\}$ in Algorithm 3 and it is easily seen that $[E \quad B]$ is surjective and $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ is injective. From Lemma 2.3 it now follows that the lower bounds in Cor. 2.4 are reached so that we can conclude that the representation (E, A, B, C) is minimal under external equivalence.

THEOREM 2.6 *Let (F, G, H_y, H_u) be a pencil representation that results from applying Algorithm 4 to an (E, A, B, C) representation. Then the two representations are externally equivalent. Furthermore if (E, A, B, C) is minimal then (F, G, H_y, H_u) is also minimal.*

PROOF The external equivalence of the representations has been proven in Lemma 2.2 of [5]. Next, by Lemma 2.3, the minimality of (E, A, B, C) implies that $[E \quad B]$ is surjective. We can then use Lemma 2.1

(v) to conclude that G is surjective. Furthermore the minimality of rank G follows immediately from the minimality of rank E since in both Algorithm 3 and Algorithm 4 we have

$$\operatorname{rank} G = \operatorname{rank} E.$$

In order to conclude that the pencil representation is minimal we still have to prove that dim ker G is minimal. Using Lemma 2.3 together with Lemma 2.1 (iii), we have

$$\begin{aligned}\dim\ker G &= \dim\ker E + \dim B^{-1}[\operatorname{im} E] \\ &= \dim(\pi_Y W^0) + \dim(U \cap W^0) \\ &= \dim W^0. \end{aligned} \tag{2.14}$$

From [4] we may now conclude that dim ker G is minimal and moreover that the pencil representation (F, G, H) is minimal.

3. Minimality and the transformation group

In this section we first derive necessary and sufficient conditions for the minimality under external equivalence of a (E, A, B, C) representation in terms of the matrices E, A, B and C.

THEOREM 3.1 *Let a descriptor representation be given by (E, A, B, C). The representation is minimal under external equivalence if and only if the following conditions hold:*

(i) $[E \quad B]$ *is surjective*

(ii) $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ *is injective*

(iii) $[sE^{\mathrm{T}} - A^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ *has full column rank for all* $s \in \mathbb{C}$.

PROOF From Lemma 2.3 it follows immediately that conditions (i) and (ii) should hold for a minimal (E, A, B, C) representation. In order to prove (iii) we apply Algorithm 4 to the representation. According to Theorem 2.6 the pencil representation (F, G, H) that is obtained in this way is minimal. This implies that $[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ should have full column rank for all $s \in \mathbb{C}$ (Prop. 1.3). Condition (iii) now easily follows from the matrix equality

$$\begin{bmatrix} sI-\bar{A}_{11} & -\bar{A}_{12} & -\bar{B}_{11} \\ C_1 & C_2 & 0 \\ 0 & 0 & I \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_{21} \end{bmatrix} = \\ = \begin{bmatrix} I & -B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & B_{22}^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} sI-A_{11} & -A_{12} & -B_{11} \\ -A_{21} & -A_{22} & -B_{21} \\ C_1 & C_2 & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{3.1}$$

Conversely, when Algorithm 4 is applied to an (E, A, B, C) representation for which conditions (i)-(iii) hold, Lemma 2.1 yields that the resulting pencil representation satisfies the conditions of Prop. 1.3 and is therefore minimal. From this it follows that rank E is minimal. Furthermore since conditions (i) and (ii) are assumed to be satisfied we can use Lemma 2.3 to derive

$$\begin{aligned}\dim\ker E &= \dim C[\ker E] \\ &= \dim \pi_Y W^0 \end{aligned} \tag{3.2}$$

and

$$\operatorname{codim}\operatorname{im} E = \operatorname{codim} B^{-1}[\operatorname{im} E]$$

$$= \operatorname{codim} U \cap W^0. \tag{3.3}$$

By Cor. 2.4 this proves that the (E, A, B, C) representation is minimal.

REMARK 3.2 In the above theorem there is no requirement on the absence of non-dynamic variables as in the analogous theorem for (E, A, B, C, D) representations (see Theorem 1.1). This is not surprising since for (E, A, B, C) representations the non-dynamic variables cannot be eliminated: there is no D term in which they can be incorporated.

Using the above theorem, we can easily prove that Algorithm 1 and Algorithm 2 preserve minimality.

THEOREM 3.3 *Let $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ be a descriptor representation that results from applying Algorithm 1 to an (E, A, B, C, D) representation. Then the two representations are externally equivalent. Furthermore if (E, A, B, C, D) is minimal then $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ is also minimal.*

PROOF The operations that are involved in the algorithm clearly do not affect the behaviour: variables are merely written in another way. From this the external equivalence of the two representations follows. Next, from the minimality of (E, A, B, C, D) it follows by Theorem 1.1 that we have that $[E \quad B]$ is surjective and $[E^T \quad C^T]^T$ is injective. Together with the surjectivity of T_1 and the injectivity of D_1 this implies that $[\tilde{E} \quad \tilde{B}]$ is surjective and $[\tilde{E}^T \quad \tilde{C}^T]^T$ is injective. Further, it is easily seen that the matrix $[s\tilde{E}^T - \tilde{A}^T \quad \tilde{C}^T]^T$ has full column rank for all $s \in \mathbb{C}$ since the same holds for the matrix $[sE^T - A^T \quad C^T]^T$. From Theorem 3.1 it now follows that $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ is minimal under external equivalence.

THEOREM 3.4 *Let $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be a descriptor representation that results from applying Algorithm 2 to an (E, A, B, C) representation. Then the two representations are externally equivalent. Furthermore if (E, A, B, C) is minimal then $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is also minimal.*

PROOF It is easily seen that the two representations are externally equivalent (see e.g. the proof of Lemma 2.1 in [5] Next, from the minimality of (E, A, B, C) it follows by Theorem 3.1 that we have that $[E \quad B]$ is surjective and $[E^T \quad C^T]^T$ is injective. It is easily checked that this leads to the surjectivity of $[\tilde{E} \quad \tilde{B}]$ and the injectivity of $[\tilde{E}^T \quad \tilde{C}^T]^T$. Further, note that by construction we have that $\tilde{A}[\ker \tilde{E}] \subset \operatorname{im} \tilde{E}$. Finally, $[s\tilde{E}^T - \tilde{A}^T \quad \tilde{C}^T]^T$ has full column rank for all $s \in \mathbb{C}$ since the fact that $[sE^T - A^T \quad C^T]^T$ is injective for all $s \in \mathbb{C}$ implies that the matrix

$$\begin{bmatrix} sI-(A_{11}-A_{12}A_{211}) & -A_{13} \\ 0 & 0 \\ -A_{212} & 0 \\ C_1 - C_2A_{211} & C_3 \end{bmatrix} = \begin{bmatrix} sI - A_{11} & -A_{12} & -A_{13} \\ -A_{211} & -I & 0 \\ -A_{212} & 0 & 0 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{211} & 0 \\ 0 & I \end{bmatrix}$$

has full column rank for all $s \in \mathbb{C}$. It now follows from Theorem 1.1 that $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is minimal under external equivalence.

Next, we ask ourselves by which transformations minimal equivalent (E, A, B, C) representations are related. We have the following theorem.

THEOREM 3.5 *Let (E, A, B, C) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ be descriptor representations that are minimal under external equivalence. Then the two representations are externally equivalent if and only if they are restricted system equivalent, i. e. there exist invertible matrices M and N such that*

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE-A & -B \\ C & 0 \end{bmatrix} = \begin{bmatrix} s\tilde{E}-\tilde{A} & -\tilde{B} \\ \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}. \tag{3.4}$$

PROOF See Appendix.

REMARK 3.6 The above theorem tells us that for minimal (E, A, B, C) representations the only operations that are allowed under external equivalence are "change of basis" in X_d and in X_e. Therefore the situation for (E, A, B, C) representations parallels the standard state space case: minimal equivalent representations are related by similarity. Note that this does not hold for minimal (E, A, B, C, D) representations: see Theorem 1.2.

4. A REALIZATION PROCEDURE

In this section we present a method for obtaining a minimal (E, A, B, C) representation for a system described by equations of the form

$$R_1(\sigma)y + R_2(\sigma)u = 0 \tag{4.1}$$

where $R_1(s)$ and $R_2(s)$ are polynomial matrices of sizes $k \times p$ and $k \times m$, respectively. Throughout this section it will be assumed that the matrix $[R_1(s) \quad R_2(s)]$ has full row rank, which is of course no restriction.

In [4, 13] an abstract procedure is given for realizing (4.1) into minimal pencil form; the pencil form turns out to be a natural first order form for systems described by equations of the above type. We will repeat this result here. Following the terminology of J. C. Willems ([13]), we shall refer to (4.1) as a set of *autoregressive equations*. Taking inputs and outputs together as external variables, the equation is rewritten as

$$R(\sigma)w = 0 \tag{4.2}$$

where $R(s) = [R_1(s) \quad R_2(s)]$ and $w = [y^T \quad u^T]^T$ is the vector of external variables. In the following $\lambda^{-1}W[[\lambda^{-1}]]$ denotes the space of strictly proper rational W-valued functions. The symbol π_- will be used for the natural projection of $X(\lambda)$ (where X is any vector space) onto $\lambda^{-1}X[[\lambda^{-1}]]$. For an element $w(\lambda)$ of $\lambda^{-1}W[[\lambda^{-1}]]$, the value of $sw(s)$ at infinity will be denoted by w_{-1}.

THEOREM 4.1 ([4]) *Let a system be given in AR form (4.2), with $R(s) \in \mathbb{R}^{k \times q}(s)$ of full row rank. Consider the following spaces of rational vector functions in a formal parameter λ:*

$$X^R = \{w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \mid \pi_- R(\lambda)w(\lambda) = 0\} \tag{4.3}$$

$$X_R = \{p(\lambda) \in \mathbb{R}^k[\lambda] \mid \exists w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \text{ s.t. } p(\lambda) = R(\lambda)w(\lambda)\} \tag{4.4}$$

$$N^R = \{w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \mid R(\lambda)w(\lambda) = 0\}. \tag{4.5}$$

The following mappings (G and F from $X^R/\lambda^{-1}N^R$ to X_R, H from $X^R/\lambda^{-1}N^R$ to W) are well-defined:

$$G: w(\lambda) \bmod \lambda^{-1}N^R \mapsto R(\lambda)w(\lambda) \tag{4.6}$$

$$F: w(\lambda) \bmod \lambda^{-1}N^R \mapsto R(\lambda)\pi_-(\lambda w(\lambda)) \tag{4.7}$$

$$H: w(\lambda) \bmod \lambda^{-1}N^R \mapsto w_{-1}. \tag{4.8}$$

With these definitions, $(X^R/\lambda^{-1}N^R, X_R, W; F, G, H)$ is a minimal pencil representation of the behavior given by (4.2).

As mentioned before, the above theorem gives an abstract realization procedure. A choice of bases for the spaces that appear in the procedure still has te be made. In [4] we chose bases in such a way that the pencil realization could be trivially rewritten as an (E, A, B, C, D) representation. Here our aim is different: we want to end up with an (E, A, B, C) representation. We will therefore choose different bases here.

The first step is to take $[R_1(s) \quad R_2(s)]$ to row proper form. This can be done by unimodular operations ([3, p. 386]); so, we may assume that $R(s)$ is row proper to start with. This means that we can write

$$[R_1(s) \quad R_2(s)] = \Delta(s)[B_1(s) \quad B_2(s)] \tag{4.9}$$

where $[B_1(s) \quad B_2(s)]$ is right bicausal, and

$$\Delta(s) = \text{diag}\,(s^{\kappa_1}, \cdots, s^{\kappa_k}). \tag{4.10}$$

We note here that $[B_1(\infty) \quad B_2(\infty)]$ is the "leading row coefficient matrix" of $[R_1(s) \quad R_2(s)]$ and that the subspace W^0 can be defined as ([4])

$$W^0 = \ker[B_1(\infty) \quad B_2(\infty)]. \tag{4.11}$$

Now, let $\tilde{B}(s)$ be any matrix such that $\hat{B}(s) = [B^{\mathrm{T}}(s) \quad \tilde{B}^{\mathrm{T}}(s)]^{\mathrm{T}}$ is bicausal. It will be discussed later how to make a suitable choice for $\tilde{B}(s)$. As in [4] we choose as a basis for the space $X^R/\lambda^{-1}N^R$ the columns of the following matrix of size $q \times (n+q-k)$ (modulo $\lambda^{-1}N^R$):

$$\hat{B}^{-1}(\lambda)\begin{bmatrix} \lambda^{-1} \cdot \cdot \lambda^{-\kappa_1} & & & & \\ & \cdots & & & \\ & & \lambda^{-1} \cdot \cdot \lambda^{-\kappa_k} & & \\ & & & \lambda^{-1} & \\ & & & & \cdot \\ & & & & & \lambda^{-1} \end{bmatrix}. \tag{4.12}$$

A basis matrix for X_R is given by the following matrix of size $k \times n$:

$$\begin{bmatrix} \lambda^{\kappa_1 - 1} \cdot \cdot \lambda \;\; 1 & 0 & & \\ 0 & \lambda^{\kappa_2 - 1} \cdot \cdot & & \\ \cdot & 0 & & \\ \cdot & \cdot & & \\ 0 & \cdot & \lambda^{\kappa_k - 1} \cdot \cdot \lambda \;\; 1 \end{bmatrix}. \tag{4.13}$$

With respect to these bases, G has the form $[I \quad 0]$. The matrix of H now has the form

$$H = \hat{B}(\infty)^{-1}\begin{bmatrix} 1 \cdot \cdot 0 & & & & \\ & \cdots & & & \\ & & 1 \cdot \cdot 0 & & \\ & & & 1 & \\ & & & & \cdot \\ & & & & & 1 \end{bmatrix}. \tag{4.14}$$

Finally, if we let $G(\lambda)$ denote the matrix whose columns are the images under G in $\mathbb{R}^k[\lambda]$ of the basis elements for $X^R/\lambda^{-1}N^R$ displayed above, then a similar matrix for F is given by the formula

$$F(\lambda) = \lambda G(\lambda) - R(\lambda)H, \tag{4.15}$$

which follows from the definitions of F, G, and H. Because of the simple basis we chose for X_R the constant matrix F can now be easily obtained.

We will now describe how $\tilde{B}(s)$ should be chosen in order to arrive at a (E, A, B, C) realization. First note that we have

$$U \cap W^0 = \ker B_2(\infty). \tag{4.16}$$

This enables us to decide which u-variables are driving variables. By renumbering the inputs if necessary, we may assume that

$$B_2(\infty) = [B_2^1(\infty) \quad B_2^2(\infty)] \tag{4.17}$$

where $B_2^1(\infty)$ has full column rank, and the columns of $B_2^2(\infty)$ depend linearly on those of $B_2^1(\infty)$. Let $B_2^2(\infty)$ have m_2 columns; note that $m_2 \leqslant q-k$ and that we have $m_2 = 0$ when $\dim\ker B_2(\infty) = 0$. It is easily verified that a matrix $\tilde{B}$ which completes $B(\infty)$ to an invertible matrix may be found of the form

$\begin{bmatrix} \bullet & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$. We then have that $\hat{B}(\infty)^{-1}$ is of the form

$$\hat{B}(\infty)^{-1} = \begin{bmatrix} \star & \star & 0 \\ \star & \star & \star \\ 0 & 0 & I \end{bmatrix} \tag{4.18}$$

where the partitioning is $(p+m_1+m_2)\times(k+(q-k-m_2)+m_2)$ (m_1 is the number of columns of $B_2^1(\infty)$). We therefore obtain equations of the following form:

$$\sigma z_0 = A_0 z_0 + B_1 z_1 + B_2 z_2 \tag{4.19}$$

$$y = H_{00} z_0 + H_{01} z_1 \tag{4.20}$$

$$u_1 = H_{10} z_0 + H_{11} z_1 + H_{12} z_2 \tag{4.21}$$

$$u_2 = z_2. \tag{4.22}$$

This can obviously be rewritten as

$$\sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} A_0 & B_1 \\ H_{10} & H_{11} \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ -I & H_{12} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.23}$$

$$y = [H_{00} \quad H_{01}] \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}. \tag{4.24}$$

We have that rank E (= rank G) = n is minimal where n denotes the sum of the minimal row indices (κ_i) of $[R_1(s) \quad R_2(s)]$. By checking the other dimensions as well, it can be easily verified that the above (E, A, B, C) realization is minimal.

REMARK 4.2 Let us assume that $R_1(s)$ and $R_2(s)$ are left coprime so that we can compare our realization procedure with procedures under transfer equivalence, such as in [14] and [1]. We are then essentially starting from a transfer function $T(s) = R_1^{-1}(s)R_2(s)$ given as a polynomial left coprime factorization. In both [14] and [1] a minimal (E, A, B, C) representation is obtained by splitting the finite and infinite frequencies. In the resulting (E, A, B, C) representation the matrix E has the form $\begin{bmatrix} I & 0 \\ 0 & \bullet \end{bmatrix}$ (the matrix A has the form $\begin{bmatrix} * & 0 \\ 0 & I \end{bmatrix}$). In contrast, our procedure does not split finite and infinite frequencies (leading to a matrix E of the form $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$) thus providing a more direct link between polynomial and state space representations. This makes it easier to translate system properties from polynomial terms to state space terms and vice versa. Note that it follows from Theorem 3.5 that the two representations are restricted system equivalent.

We conclude this section with an example.

EXAMPLE 4.3 Take

$$[R_1(s) \quad R_2(s)] = \begin{bmatrix} s+2 & 0 & s+1 & 0 & 1 & 0 \\ 0 & s-1 & 0 & 3 & s & s+4 \\ s & 0 & 0 & s^2 & 0 & 0 \end{bmatrix} \tag{4.25}$$

corresponding to 3 outputs and 3 inputs. The leading row coefficient matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

has full row rank, so that the given matrix $R(s)$ is already row reduced; also, $m_2 = 1$ and the inputs need not be renumbered. We see that the sum of the minimal row indices of $R(s)$ is 4 and that the rank of $B_2(\infty)$ (formed by the last three columns of the matrix above) is 2; so, a descriptor representation (E, A, B, C) will be minimal if and only if the matrix E has size 6×6 and rank 4.

Applying the above procedure we take

$$\tilde{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.26}$$

which leads to

$$\hat{B}(\infty)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.27}$$

Consequently, we get

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.28}$$

The matrix of F is computed from

$$F(\lambda) = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} \lambda+2 & 0 & \lambda+1 & 0 & 1 & 0 \\ 0 & \lambda-1 & 0 & 3 & \lambda & \lambda+4 \\ \lambda & 0 & 0 & \lambda^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -3 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & \lambda & -\lambda & 0 & 0 \end{bmatrix}. \tag{4.29}$$

This gives

$$F = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -3 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.30}$$

Of course, $G = [I_4 \;\; 0]$. Re-organizing the pencil equations as described above, one obtains the (E, A, B, C) realization

$$\sigma \begin{bmatrix} I_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4.31}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}. \tag{4.32}$$

REMARK 4.4 In the above example the column defect of $B_1(\infty)$ is equal to 1. Therefore (see [4]) a minimal (E, A, B, C, D) realization would have an E-matrix of size 5×5 (and again rank 4). Indeed, there is one non-dynamic variable in the above (E, A, B, C) realization, namely the variable z_2 which can be incorporated in a D term.

5. CONCLUSIONS

In this report we have characterized the minimality of an (E, A, B, C) representation in terms of the matrices E, A, B and C. The conditions that are necessary and sufficient for minimality under external equivalence can be summarized as:

- controllability and observability at infinity
- observability at finite modes.

Thus for minimality of (E, A, B, C) representations there is no requirement on the absence of non-dynamic modes, as is the case for (E, A, B, C, D) representations.

We also gave a procedure for realizing systems given by autoregressive equations to minimal (E, A, B, C) form. It is proven that such a realization is unique up to operations of restricted system equivalence. Thus the transformation group consists of isomorphisms and is therefore more simple than in the case of (E, A, B, C, D) representations. In some situations this might be a reason to use an (E, A, B, C) representation rather than an (E, A, B, C, D) representation.

6. APPENDIX

PROOF OF THEOREM 3.5

Multiplying the system from the right by

$$\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}$$

can be considered as a "change of internal variables" while left multiplication by

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$

is a "reformulation of constraints" in the terminology of [10]. It now follows from [10] that the resulting representation is externally equivalent to the original one.
Conversely, it should be noted that we can arrive at (E, A, B, C) representations of the form (2.6) by using operations of restricted system equivalence. For that reason we can assume that our descriptor representations (E, A, B, C) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ are already of that form. Now decompose the input space U as $U \cap W^0 \oplus U_2$. Accordingly write

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}.$$

It follows from Lemma 2.3 that both B_{22} and $\tilde{B}_{22}$ are invertible. Next apply Algorithm 4 to both representations. This yields externally equivalent pencil representations (F, G, H) and $(\tilde{F}, \tilde{G}, \tilde{H})$ that are minimal by Theorem 2.6. We can now use existing knowledge on the transformation group for externally equivalent minimal pencil representations. According to Prop. 1.4 there exist invertible matrices S and T such that

$$\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} = \begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} T. \tag{6.1}$$

Writing this in further detail gives:

$$\begin{bmatrix} S & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & B_{22}^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} & -B_{11} \\ -A_{21} & -A_{22} & -B_{21} \\ C_1 & C_2 & 0 \\ 0 & 0 & I \end{bmatrix} =$$

$$= \begin{bmatrix} I & -\tilde{B}_{12}\tilde{B}_{22}^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & \tilde{B}_{22}^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} sI - \tilde{A}_{11} & -\tilde{A}_{12} & -\tilde{B}_{11} \\ -\tilde{A}_{21} & -\tilde{A}_{22} & -\tilde{B}_{21} \\ \tilde{C}_1 & \tilde{C}_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{bmatrix}. \tag{6.2}$$

It now follows immediately that

$$\begin{aligned} T_1 &= S, \\ T_2 = T_3 &= T_7 = T_8 = 0, \\ T_9 &= I \end{aligned} \tag{6.3}$$

and T_5 is invertible. Comparing the (2,3)-elements on both sides gives

$$0 = \tilde{C}_2 T_6. \tag{6.4}$$

The injectivity of $\tilde{C}_2$ now implies that $T_6 = 0$. As a result we have

$$\begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -B_{12}B_{22}^{-1} & 0 \\ 0 & 0 & I \\ 0 & B_{22}^{-1} & 0 \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} & -B_{11} & -B_{12} \\ -A_{21} & -A_{22} & -B_{21} & -B_{22} \\ C_1 & C_2 & 0 & 0 \end{bmatrix} = \tag{6.5}$$

$$= \begin{bmatrix} I & -\tilde{B}_{12}\tilde{B}_{22}^{-1} & 0 \\ 0 & 0 & I \\ 0 & \tilde{B}_{22}^{-1} & 0 \end{bmatrix} \begin{bmatrix} sI - \tilde{A}_{11} & -\tilde{A}_{12} & -\tilde{B}_{11} & -\tilde{B}_{12} \\ -\tilde{A}_{21} & -\tilde{A}_{22} & -\tilde{B}_{21} & -\tilde{B}_{22} \\ \tilde{C}_1 & \tilde{C}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} S & 0 & 0 & 0 \\ T_4 & T_5 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Multiplying from the left by

$$\begin{bmatrix} I & -\tilde{B}_{12}\tilde{B}_{22}^{-1} & 0 \\ 0 & 0 & I \\ 0 & \tilde{B}_{22}^{-1} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 & \tilde{B}_{12} \\ 0 & 0 & \tilde{B}_{22} \\ 0 & I & 0 \end{bmatrix} \tag{6.6}$$

yields:

$$\begin{bmatrix} S & -SB_{12}B_{22}^{-1} + \tilde{B}_{12}B_{22}^{-1} & 0 \\ 0 & \tilde{B}_{22}B_{22}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} sI-A_{11} & -A_{12} & -B_{11} & -B_{12} \\ -A_{21} & -A_{22} & -B_{21} & -B_{22} \\ C_1 & C_2 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} sI-\tilde{A}_{11} & -\tilde{A}_{12} & -\tilde{B}_{11} & -\tilde{B}_{12} \\ -\tilde{A}_{21} & -\tilde{A}_{22} & -\tilde{B}_{21} & -\tilde{B}_{22} \\ \tilde{C}_1 & \tilde{C}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} S & 0 & 0 & 0 \\ T_4 & T_5 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{6.7}$$

Defining

$$M = \begin{bmatrix} S & -SB_{12}B_{22}^{-1} + \tilde{B}_{12}B_{22}^{-1} \\ 0 & \tilde{B}_{22}B_{22}^{-1} \end{bmatrix}, \quad N = \begin{bmatrix} S & 0 \\ T_4 & T_5 \end{bmatrix}, \tag{6.8}$$

we see that M and N are invertible and

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE-A & -B \\ C & 0 \end{bmatrix} = \begin{bmatrix} s\tilde{E}-\tilde{A} & -\tilde{B} \\ \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}. \tag{6.9}$$

This proves that the representations are restricted system equivalent.

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