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# Morphological Discretization

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One of the major goals of mathematical morphology is to provide a collection of algorithms suited for the automatic analysis of digital images on a computer. To a large extent, it relies upon mathematical disciplines such as stereology and integral geometry. Consequently, mathematical morphology is basically a continuous theory, that is, it deals with continuous images. So there is obviously a gap between the continuous mathematical tools which form the basis of morphology and the specific applications, mostly of a discrete nature, for which it has been designed. This paper attempts to initiate a solid theory of discretization which can bridge this gap.

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## 1. Introduction

Mathematical morphology is a methodology in the field of image processing whose main goal is the extraction of geometric information from an image. It derives its tools from such mathematical disciplines as integral geometry, stereology and stochastic geometry. As a consequence, mathematical morphology is essentially a continuous theory, that is, it applies to continuous images. However, morphological algorithms are usually developped for discrete (or digital) images, because the data at one's disposal are often discrete and because the machines on which these data have to be processed are digital. So there is definitely a need for a mathematically solid theory of discretization of continuous images and (morphological) operators.

A first step in any discretization procedure is to approximate a continuous image by some well-chosen mosaic of pixels and to ensure that consecutive refinement of this mosaic ultimately converges towards the original image. In this paper we shall present a formal definition of this intuitive idea, and work this out for a particular example. But approximation of a continuous

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image by a sequence of discrete images by itself is not sufficient. One also has to indicate how to approximate the continuous operators which are used to extract quantitative information from an image. This is a more delicate task which is the subject of a paper in preparation. Here we shall only give a formal description of this problem.

This paper contains the following sections. In Section 2 we will briefly discuss the concepts of mathematical morphology which we need in the rest of the paper. In Section 3 we discuss a sampling strategy: by sampling we mean here the transformation of a continuous image into a discrete image. This sampling strategy is used in Section 4 where we discuss a specific procedure for the discretization of continuous images. In Section 5 we briefly discuss the problem concerning the discretization of morphological operators and make some final comments.

We point out that the theory developped in this paper is closely related to some work of Serra reported in [8, Chapter VII]. However, the discretization procedure discussed here is slightly different.

## 2. Basic concepts from mathematical morphology

Nowadays, mathematical morphology is an important branch in image processing with a solid mathematical basis. For a comprehensive overview of the theory we refer to [6,8,9]. Originally mathematical morphology has been developped for continuous binary images, which can be represented mathematically as subsets of $\mathbb{R}^2$ (we restrict to the two-dimensional case only for simplicity). Later the theory has been extended to grey-level images, which can be modelled mathematically as functions on $\mathbb{R}^2$ [3,8,10]. Due to the discrete nature of digital computers there has been a practical need for discrete algorithms from the very beginning. So it is clear that a general theory of morphology should allow many different object spaces. This observation has been the main motivation for the development of a 'morphological theory' on complete lattices. Note that the object spaces mentioned before are just special examples of a complete lattice [1]. A rather complete account on mathematical morphology on complete lattices can be found in [4,7,9].

Let $\mathcal{L}_1, \mathcal{L}_2$ be two (possibly the same) complete lattices. The building blocks of mathematical morphology are dilation and erosion.

**Definition 1.** A mapping $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ is called a *dilation* if $\delta(\bigvee_{i\in I} X_i) = \bigvee_{i\in I} \delta(X_i)$ for any collection $X_i \in \mathcal{L}_1,\ i \in I$. A mapping $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$ is called an *erosion* if $\varepsilon(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \varepsilon(X_i)$ for any collection $X_i \in \mathcal{L}_2,\ i \in I$.

Dilations and erosions are increasing mappings. A mapping $\psi$ on the complete lattice $\mathcal{L}$ is called *increasing* if $X \leq Y$ implies $\psi(X) \leq \psi(Y)$ for any pair $X, Y \in \mathcal{L}$. It is clear that dilation and erosion form dual notions. Their roles are interchanged by reversing the ordering relations of the lattices involved.

Another important relation between dilations and erosions is the following. Let $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ and $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$ be two mappings on a complete lattice $\mathcal{L}$. The pair $(\varepsilon, \delta)$ is called an *adjunction* if

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y)$$

for all $X \in \mathcal{L}_1, Y \in \mathcal{L}_2$. If $(\varepsilon, \delta)$ is an adjunction it can be shown that $\delta$ is a dilation and $\varepsilon$ is an erosion. Furthermore, to every dilation $\delta$ there corresponds a unique corresponding erosion $\varepsilon$ such that $(\varepsilon, \delta)$ is an adjunction and vice versa. In this case $\varepsilon$ and and $\delta$ are called adjoints. Finally, if $(\varepsilon, \delta)$ froms an adjunction, then

$$\varepsilon\delta \geq \mathrm{id}_{\mathcal{L}_1}, \quad \delta\varepsilon \leq \mathrm{id}_{\mathcal{L}_2},$$

where $\mathrm{id}_{\mathcal{L}}$ denotes the identity mapping on $\mathcal{L}$, and

$$\varepsilon\delta\varepsilon = \varepsilon, \quad \delta\varepsilon\delta = \delta.$$

We consider the special case where $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{P}(\mathbb{R}^2)$, the power space of $\mathbb{R}^2$ (but the same definition applies if $\mathbb{R}^2$ is replaced by $\mathbb{Z}^2$). If $X \in \mathcal{P}(\mathbb{R}^2)$ and $h \in \mathbb{R}^2$ then we denote by $X_h$ the translate of $X$ along $h$, i.e., $X_h = \{x + h : x \in X\}$. The most important examples of dilation and erosion on $\mathcal{P}(\mathbb{R}^2)$ are respectively given by the Minkowski addition and subtraction:

$$\delta(X) = X \oplus A = \bigcup_{a \in A} X_a$$
$$\varepsilon(X) = X \ominus A = \bigcap_{a \in A} X_{-a},$$

where $X \in \mathcal{P}(\mathbb{R}^2)$. Here $A$ is a subset of $\mathbb{R}^2$ called the *structuring element*. It is easy to check that $\delta$ and $\varepsilon$ form an adjunction. Furthermore, these operators are invariant under translations, i.e., $\delta(X_h) = (\delta(X))_h$ for every $X \in \mathcal{P}(\mathbb{R}^2)$ and $h \in \mathbb{R}^2$, and the same for $\varepsilon$.

## 3. Morphological sampling

Many of the results of this section can also be found in [5], where we deal with the sampling of discrete grey-level images. Here we consider continuous binary images, although the extension to grey-level images is straightforward.

Let $\mathcal{P}(\mathbb{R}^2)$ be the complete Boolean lattice containing all subsets of $\mathbb{R}^2$. Let $S$ be a discrete 2-dimensional grid in $\mathbb{R}^2$, that is $S = \{ku + lv \mid k, l \in \mathbb{Z}\}$, where $u, v$ are two linearly independent vectors in $\mathbb{R}^2$. We call $S$ the *sampling grid*. Let $A \subseteq \mathbb{R}^2$. We assume that the collection of all translates $A_s$, $s \in S$ spans the whole space $\mathbb{R}^2$, that is

$$S \oplus A = \mathbb{R}^2. \tag{3.1}$$

Note that this assumption is equivalent with

$$\check{A}_x \cap S \neq \emptyset. \tag{3.2}$$

Here $\check{A}$ is the reflected set $\{-a : a \in A\}$. In Figure 1 we have depicted some sampling strategies which satisfy these equivalent assumptions.

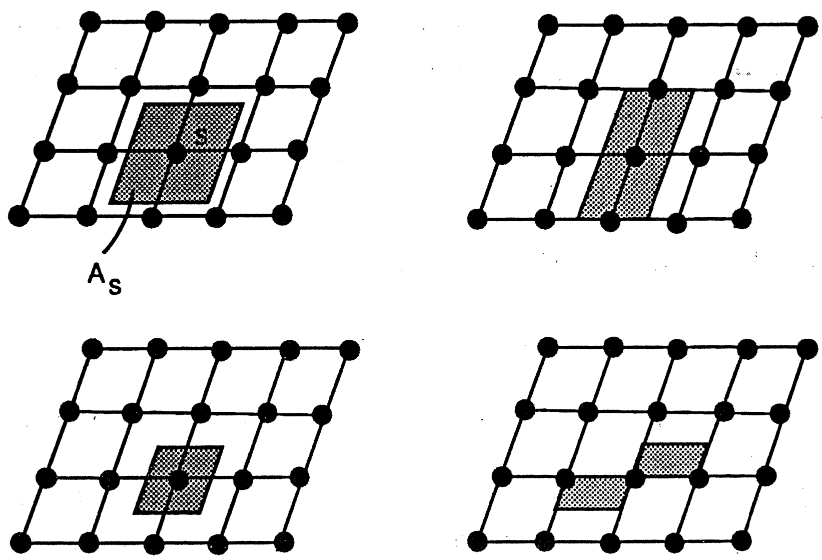


FIGURE 1. *Some sampling strategies.*

We call $A$ the *sampling element*. We can now define a sampling operator $\sigma : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(S)$ in the following way:

$$\sigma(X) = \{s \in S : A_s \cap X \neq \emptyset\} = (X \oplus \check{A}) \cap S. \tag{3.3}$$

It is obvious that $\sigma$ is a dilation, that is, $\sigma$ acts distributively over union. The adjoint erosion $\dot{\sigma} : \mathcal{P}(S) \to \mathcal{P}(\mathbb{R}^2)$ is given by

$$\dot{\sigma}(V) = \{x \in \mathbb{R}^2 : \check{A}_x \cap S \subseteq V\}, \tag{3.4}$$

for $V \subseteq S$. Now $\rho = \dot{\sigma}\sigma$ defines a closing on $\mathcal{P}(\mathbb{R}^2)$ (that is, $\rho^2 = \rho$ and $X \subseteq \rho(X)$), and $\sigma\rho = \sigma$. We call $\rho$ the reconstruction operator. From (3.3)-(3.4) together with (3.2) we find that

$$\begin{aligned}\rho(X) = \dot{\sigma}\sigma(X) &= \{x \in \mathbb{R}^2 : \check{A}_x \cap S \subseteq (X \oplus \check{A}) \cap S\} \\ &\subseteq \{x \in \mathbb{R}^2 : (\check{A}_x \cap S) \cap ((X \oplus \check{A}) \cap S) \neq \emptyset\} \\ &\subseteq \{x \in \mathbb{R}^2 : \check{A}_x \cap (X \oplus \check{A}) \neq \emptyset\} = (X \oplus \check{A}) \oplus A \\ &= X \oplus B,\end{aligned}$$

where

$$B = A \oplus \check{A}. \tag{3.5}$$

Summarizing, we have

$$X \subseteq \rho(X) \subseteq X \oplus B, \tag{3.6}$$

for every set $X \subseteq \mathbb{R}^2$. In [5] a number of other reconstruction operators are discussed. In figure 2 we have depicted an example. We point out that in this example the reconstructed set $\rho(X)$ is open.

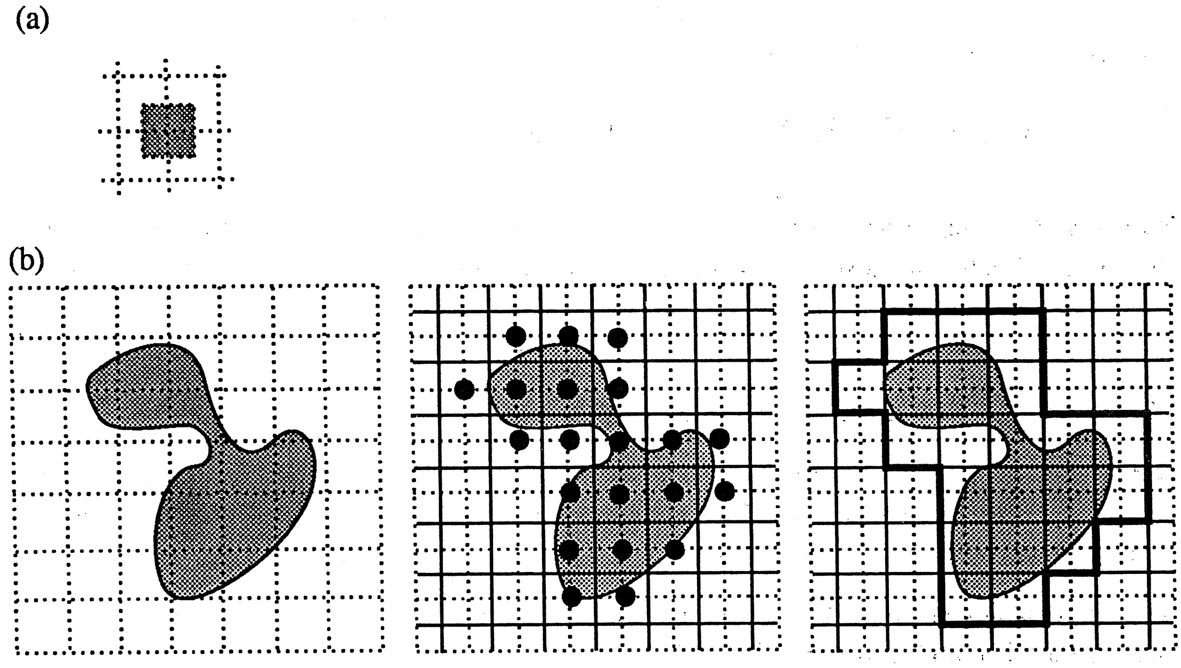


FIGURE 2. *(a) The sampling element A. (b) From left to right: the original set X and the grid S (dashed lines), the sampled set $\sigma(X)$ (black dots), the reconstruction $\rho(X)$ (thick polygon).*

## 4. Discretization of images

By a discretization of a binary image $X \subseteq \mathbb{R}^2$ we mean the approximation of $X$ by a sequence $X_n \subseteq \mathbb{R}^2$, $n \geq 1$, where $X_n$ has a discrete representation: see Definition 2 for a precise characterization. First of all, it is clear that such a characterization requires a suitable topology on the space of continuous binary images. It turns out that the entire space $\mathcal{P}(\mathbb{R}^2)$ is too large for our purposes. However, on the space $\mathcal{F}(\mathbb{R}^2)$, the complete lattice of all closed subsets of $\mathbb{R}^2$, there exists a topology which is well-suited for what we have in mind. In mathematical morphology this topology is known under the name *hit-or-miss topology* [6,8]. The space $\mathcal{F}(\mathbb{R}^2)$ endowed with this topology is compact, Hausdorff, and has a countable base. Here we shall not explicitly define the topology, but we shall only give a description of converging sequences (since the topology has a countable base, this specifies the topology). Let $X_n$ be a sequence in $\mathcal{F}(\mathbb{R}^2)$ and let $X \in \mathcal{F}(\mathbb{R}^2)$. Then $X_n \to X$ (i.e., $X_n$ converges towards $X$) if and only if for every open set $G \subseteq \mathbb{R}^2$ and every compact set $K \subseteq \mathbb{R}^2$ we have

$$X \cap G \neq \emptyset \implies X_n \cap G \neq \emptyset \quad \text{for } n \text{ sufficiently large} \tag{4.1}$$

$$X \cap K = \emptyset \implies X_n \cap K = \emptyset \quad \text{for } n \text{ sufficiently large.} \tag{4.2}$$

Note that this topology is compatible with the partial order $\subseteq$ in the following sense: if $U_n \subseteq X_n \subseteq V_n$ and $U_n \to X$, $V_n \to X$, then $X_n \to X$. We also need the notion of monotone convergence. Let $X, X_n \subseteq \mathbb{R}^2$, $n \geq 1$. We define

$$X_n \downarrow X \quad \text{if} \quad X_1 \supseteq X_2 \supseteq \ldots \supseteq X_n \supseteq X_{n+1} \supseteq \ldots \quad \text{and} \quad X = \bigcap_{n \geq 1} X_n. \tag{4.3}$$

Note that this convergence concept is not derived from some underlying topology. However, there exists the following relation with the hit-or-miss topology. Let $X, X_n \subseteq \mathbb{R}^2$, $n \geq 1$. Then

$$X_n \downarrow X \implies X_n \to X. \tag{4.4}$$

We now present a formal definition of a discretization.

**Definition 2.** By a discretization on $\mathcal{F}(\mathbb{R}^2)$ we mean a sequence $\{(\xi_n, \eta_n)\}_{n \geq 1}$ where every $\xi_n$ is an operator from $\mathcal{F}(\mathbb{R}^2)$ to $\mathcal{P}(\mathbb{Z}^2)$ and every $\eta_n$ is an operator from $\mathcal{P}(\mathbb{Z}^2)$ to $\mathcal{F}(\mathbb{R}^2)$ such that $\eta_n \xi_n(X) \to X$ as $n \to \infty$, for every $X \in \mathcal{F}(\mathbb{R}^2)$.

Note that, although $\eta_n \xi_n(X)$ lies in the original space $\mathcal{F}(\mathbb{R}^2)$, the operator $\eta_n \xi_n$ only takes values in a "discrete" subspace of $\mathcal{F}(\mathbb{R}^2)$.

In this section we will discuss a discretization procedure based on the sampling strategy described in the previous section. Let $A, S, \sigma, \rho$ be as before, and assume that (3.1) is satisfied.

**Proposition 3.** *If the sampling element $A$ is open, then $\rho(X)$ is closed for any $X \subseteq \mathbb{R}^2$.*

PROOF. We show that $\dot{\sigma}(V)$ is closed for every $V \subseteq S$. Let $x_n \in \dot{\sigma}(V)$ and $x_n \to x$. Then $\check{A}_{x_n} \cap S \subseteq V$. We show that $\check{A}_x \cap S \subseteq V$. Let $s \in \check{A}_x \cap S$ for some $s \in S$. We must show that $s \in V$. Then $x - s \in A$ and, since $A$ is open, $x_n - s \in A$ for $n$ large enough. Thus $s \in \check{A}_{x_n} \cap S \subseteq V$ for $n$ large enough. This concludes the proof.

∎

Note that in this result the initial set needs not to be closed.

We are now ready to define our discretization procedure. Throughout the rest of this section we make the following assumption.

**Assumption 4.** Let, respectively, $A_n, S_n$ be sequences of sampling elements and sampling grids such that:

- for every $n$ the relation (3.1) holds, that is $S_n \oplus A_n = \mathbb{R}^2$;
- every $A_n$ is an open set;
- the sequence $A_n$ is uniformly bounded, that is, $A_n \subseteq A$ for some bounded set $A$;
- $\overline{A_n} \to \{0\}$.

Let $B_n = A_n \oplus \check{A}_n$, then $B_n$ is also open and $0 \in B_n$ since $A_n$ is nonempty (take $y \in A_n$, then $-y \in \check{A}_n$ and $0 = y + -y \in B_n$).

**Example 5.** Let $a, b$ be two linearly independent vectors in $\mathbb{R}^2$ with length 1, and let $r_n$ be a sequence of real positive numbers converging to 0. Define $u_n = r_n a, v_n = r_n b$ and let

$S_n$ be the 2-dimensional grid spanned by $u_n, v_n$. Take $A_n = (-\frac{1}{2}cu_n, \frac{1}{2}cu_n) \oplus (-\frac{1}{2}cv_n, \frac{1}{2}cv_n)$ where $c > 1$. See Figure 3.

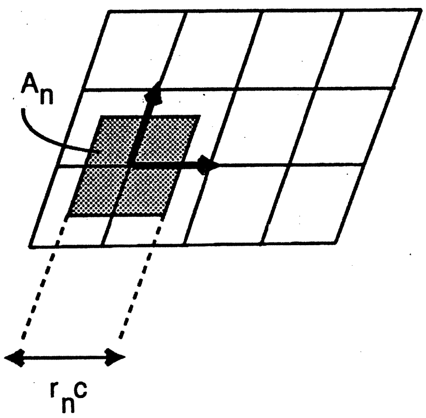


FIGURE 3. *Sampling strategy described in Example 5.*

If we choose $c = 2$ then the resulting discretization procedure coincides with the covering representation which will be discussed in the next section; see Figure 4 for an illustration. The proof of this fact is left as an exercise to the reader. It is easy to check that $A_n, S_n$ satisfy Assumption 4.

**Lemma 6.** *Under the given assumptions,* $\overline{B_n} \to \{0\}$.

PROOF. We must verify the conditions (4.1) and (4.2). The first condition is trivial since $0 \in B_n$, and we only prove the second one. Let $K$ be a compact set which does not contain 0. We must show that $B_n \cap K = \emptyset$ for $n$ sufficiently large. Suppose we have a subsequence $n_k$ such that $B_{n_k} \cap K \neq \emptyset$, say $b_{n_k} \in B_{n_k} \cap K$. Then $b_{n_k} = a_{n_k} - a'_{n_k}$, where $a_{n_k}, a'_{n_k} \in A_{n_k}$. Since $A_{n_k}$ is uniformly bounded, the sequence $\{a_{n_k}\}$ possesses a convergent subsequence. For simplicity we assume that the sequence $\{a_{n_k}\}$ is convergent with limit $a$. We can apply the same argument to the sequence $\{a'_{n_k}\}$, and may assume that $a_{n_k} \to a'$ as $k \to \infty$. One easily gets that both $a$ and $a'$ should belong to the limit set of $A_n$, that is $a = a' = 0$. Therefore, also $b_{n_k}$ converges towards 0. However, $b_{n_k} \in K$, hence $0 \in K$, which is a contradiction. This proves the result. ∎

Note that we may replace the last condition in Assumption 4 by $\overline{A_n} \to \{h\}$, where $h$ is an arbitrary element of $\mathbb{R}^2$. We are now ready to show that the approximations $\rho_n(X)$ of the continuous binary image $X \in \mathcal{F}(\mathbb{R}^2)$ converge to $X$ with respect to the hit-or-miss topology if $n$ becomes large.

**Proposition 7.** $\rho_n(X) \to X$ *as* $n \to \infty$, *for every* $X \in \mathcal{F}(\mathbb{R}^2)$.

PROOF. Let $X \in \mathcal{F}(\mathbb{R}^2)$. From (3.6) we know that

$$X \subseteq \rho_n(X) \subseteq X \oplus \overline{B_n}.$$

From the fact that $\overline{B_n}$ is compact we may conclude that $X \oplus \overline{B_n}$ is closed if $X$ is closed: see [6, p.19]. We show that $X \oplus \overline{B_n} \to X$. Since the hit-or-miss topology is compatible with the order $\subseteq$ this would also prove that $\rho_n(X) \to X$.

To prove that $X \oplus \overline{B_n} \to X$ we only verify condition (4.2) since condition (4.1) is trivially satisfied. Let $K$ be an arbitrary compact set and assume that $X \cap K = \emptyset$. We must show that $(X \oplus \overline{B_n}) \cap K = \emptyset$ for $n$ sufficiently large. Assume, on the contrary, that $y_{n_k} \in (X \oplus \overline{B_{n_k}}) \cap K$ for some subsequence $n_k$. Then $y_{n_k} = x_{n_k} + b_{n_k}$, where $x_{n_k} \in X$ and $b_{n_k} \in \overline{B_{n_k}}$. Just like in the proof of the previous lemma we may assume without loss of generality that $y_{n_k} \to y \in K$ and $b_{n_k} \to 0$ as $k \to \infty$. Then $x_{n_k} \to y$. Since $X$ is closed we have $y \in X$, hence $y \in X \cap K$. This contradicts our assumption that $X \cap K \neq \emptyset$.

∎

**Corollary 8.** *The sequence $(\sigma_n, \dot{\sigma}_n)$ defines a discretization on $\mathcal{F}(\mathbb{R}^2)$.*

Note that this discretization is somewhat special in the sense that a resampling (i.e., application of $\sigma_n$) of the approximation $\dot{\sigma}_n \sigma_n(X)$ leads to the same result as the original sampling $\sigma_n(X)$, since $\sigma_n \dot{\sigma}_n \sigma_n = \sigma_n$.

## 5. Concluding remarks

Discretization of continuous images is only a first step in settling the connection between continuous and discrete morphology. On the other hand it is quite obvious in practice how to carry over continuous morphological operators to the discrete space, since such operators often (but not always) only use the group structure of $\mathbb{R}^2$ and not the entire vector structure: see also [4,7]. The main question which remains to be answered is:

> *"which continuous morphological operators can be approximated by a sequence of discrete ones?"*

The following definition enables us to make this question more concrete.

**Definition 9.** Let $\psi$ be an operator on $\mathcal{F}(\mathbb{R}^2)$, and let $\{\xi_n, \eta_n\}_{n \geq 1}$ be a discretization of $\mathcal{F}(\mathbb{R}^2)$. By a discretization of $\psi$ we mean a sequence of discrete operators $\{\psi_n\}_{n \geq 1}$ on $\mathcal{P}(\mathbb{Z}^2)$ such that

$$\eta_n \psi_n \xi_n(X) \to \psi(X), \quad n \to \infty,$$

for $X \in \mathcal{F}(\mathbb{R}^2)$. The operator $\psi$ is called *discretizable* if it has a discretization.

This definition shows that the question posed above is intimately connected to the discretization of the space $\mathcal{F}(\mathbb{R}^2)$. In [8, Chapter VII] Serra discusses discretizability of operators with respect to a special kind of discretization of $\mathcal{F}(\mathbb{R}^2)$ which he calls the *covering representation*. This discretization procedure is best explained by Figure 4.

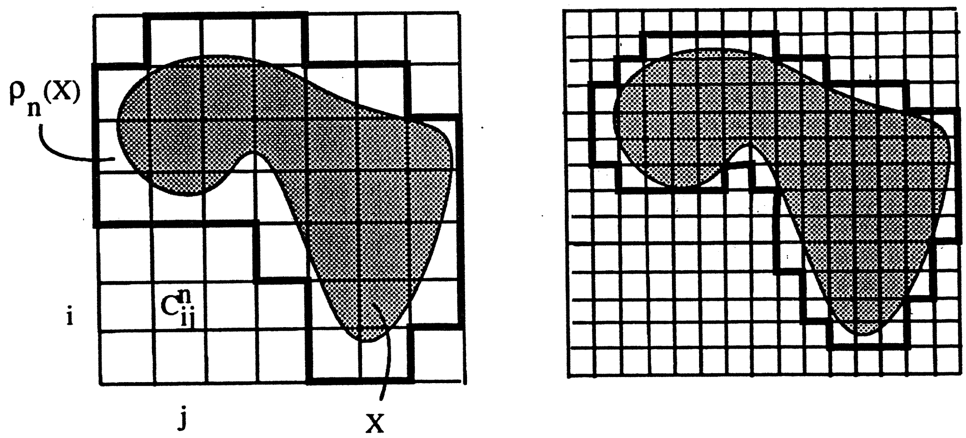


FIGURE 4. *The covering representation at two consecutive steps. Let $C_{ij}^n$ be the open cell at position $(i,j)$. Then $\rho_n(X) = \bigcup\{\overline{C_{ij}^n} : C_{ij}^n \cap X \neq \emptyset\}$.*

The reason why the question posed above is relatively easy to answer for the covering representation is because in this case the reconstructions $\rho_n(X)$ converge monotonically towards $X$, i.e., $\rho_n(X) \downarrow X$: see again Figure 4. In general we do not have this property and then the question is essentially more difficult. We will pursue this problem in a future publication.

In [2] one derives some error bounds which result if one computes certain continuously defined morphological measurements (e.g., size distributions) by discrete methods.

Finally we wish to point out that the ideas presented in this paper can be extended quite easily for grey-level functions. In that case we have to consider the space of upper-semicontinuous functions, that is, the functions $f$ for which the set $\{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : f(x) \geq t\}$ is closed.

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