1991

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Department of Operations Research, Statistics, and System Theory Report BS-R9106 February

CWI, nationaal instituut voor onderzoek op het gebied van wiskunde en informatica

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Induced Circuits in Planar Graphs

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Abstract. In [4] we give a polynomial-time algorithm for the problem of finding an induced circuit traversing two given vertices of a planar graph. We give a combinatorial algorithm and a min-max relation for the problem of finding a maximum number of paths connecting two given vertices in a planar graph so that each pair of these paths form an induced circuit.

1985 *Mathematics Subject Classification:* 05C50,05C75.

Key Words and Phrases: Planar graph, universal covering surface, induced circuit.

Let $G = (V, E)$ be an undirected graph without loops, and let s, t be distinct vertices. We call two $s - t$ paths P', P'' *separate* if they form an induced circuit. We consider the problem of finding a maximum number of pairwise separate $s - t$ paths. For general graphs this is an NP-hard problem: this follows from Bienstock [1] in which it is shown that the problem of deciding if there exists an induced circuit containing s and t, is NP-complete.

We show that the problem can be solved in polynomial time for planar graphs. Assume that G is embedded in the 2-sphere S_2. Moreover, we give a good characterization, based on the following concepts. Let C be a closed curve in S_2, not traversing s or t. The *winding number* $w(C)$ of C is, roughly speaking, the number of times that C separates s and t. More precisely, consider any curve P from s to t, crossing C only a finite number of times. Let λ be the number of times C crosses P from left to right, and let ρ be the number of times C crosses P from right to left (fixing some orientation of C, and orienting P from s to t). Then $w(C) = |\lambda - \rho|$. (This number can be seen to be independent of the choice of P.)

We call a closed curve C *alternate* if C does not traverse s or t, and there exists a sequence

$$(F_0, w_1, F_1, w_2, F_2, \ldots, w_l, F_l)$$

(where $l \geq 0$) such that

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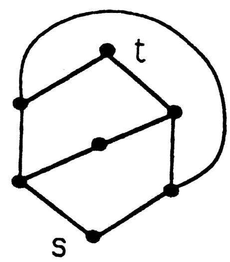

Figure 1:

(i) $F_0, \ldots, F_l$ are faces of G, with $F_0 = F_l$;

(ii) w_i is a vertex or edge of G $(i = 1, \ldots, m)$;

(iii) C traverses vertices, edges and faces of G in the order (2).

Here, by definition, C *traverses* an edge e if C follows e from one end vertex to the other.
Let $l(C)$ denote the number l in (2). Now:

Theorem A.

(i) *There exist k pairwise separate $s - t$ paths, if and only if $l(C) \geq k \cdot w(C)$ for each alternate closed curve C.*

(ii) *A maximum number of pairwise separate $s - t$ paths can be found in polynomial time.*

(iii) *The curves C in* (i) *can be restricted to those with $l(C) \leq |V|$.*

Before proving the theorem, let us give one small example of a graph where an alternate curve with winding number at least two must be used. Note that a proof of nonexistence of an induced circuit containing s, t (with s, t nonadjacent), by means of an alternate curve is equivalent to there being a vertex cut set which is a clique of size at most two. It is easily verified that the graph of Figure 1 does not contain any induced s, t cycle and yet neither does it have such a clique cut set. Other examples can be constructed for $k > 2$ (see [4]).

Proof of Theorem A:
I.Necessity in (i). Let $P_1, \ldots, P_k$ be pairwise separate $s-t$ paths, and let C be an alternate closed curve. Then C intersects each P_i at least $w(C)$ times. It is not hard to see that for each i, at least $w(C)$ of the w_j in (2) are incident to a vertex in P_i (defining two vertices v', v'' to be *incident* if $v' = v''$). Since distinct P_i and $P_{i'}$ are separate, there should be at least $k \cdot w(C)$ w_j's, i.e., $l(C) \geq k \cdot w(C)$.

II.Algorithm. We next describe an algorithm finding for any k,either k pairwise separate $s - t$ paths or an alternate closed curve C with $l(C) < k \cdot w(C)$. We assume, without loss of generality, that there is no edge connecting s and t.

First we introduce some notation and terminology. We think of G being embedded on the 2-sphere S_2. Any $s-t$ path will be oriented from s to t. Let O be an open disk whose boundary contains s and t. An edge e (of G) contained in the closure $\overline{O}$ of O, connecting two points on the boundary of O, is called a *singel relative to* O, if any curve from s to t contained in O, must cross e. Let P', P'' be two edge-disjoint $s-t$ paths, without crossings. Then $R(P', P'')$ denotes the region encircled by the closed curve $P' \cdot (P'')^{-1}$ in clockwise orientation. We call the pair (P', P'') *internally separate* if $R(P', P'')$ is an open disk not containing a singel. Note that even if (P', P'') is internally separate, P' and P'' can have a vertex $v \neq s, t$ in common. Note moreover that P' and P'' are separate if and only if both (P', P'') and (P'', P') are internally separate.

For $k = 1$ the algorithm is trivial: either there exists an $s-t$ path, or there exists a closed curve C not intersecting G with $w(C) = 1$ (implying $l(C) = 0 < 1 \cdot w(C)$).

Suppose now that $k > 1$, and that we have found $k-1$ pairwise separate $s-t$ paths $P_1, \ldots, P_{k-1}$. In the case that $k = 2$ we assume that there exist two internally disjoint $s-t$ paths P, Q. If no such pair exists, then it is easy to find an appropriate alternate curve with the help of Menger's Theorem. Hence for $k = 2$ we may choose P_1 to be P.

We may assume that the first edges of $P_1, \ldots, P_{k-1}$ occur in this order clockwise at s. Let P_k be a path 'parallel' to the left of P_1. That is, we add to each edge traversed by P_1 a parallel edge at the left hand side (with respect to the orientation of P_1), and P_k follows these new edges. (Note that adding parallel edges does not change our problem and in the case $k = 2$ we have chosen P_1 so that (P_1, P_2) is internally separate.) Then the first edges of $P_1, \ldots, P_k$ occur in this order clockwise at s, and each pair (P_{i-1}, P_i) is internally separate $(i = 2, \ldots, k)$.

Now for $n = k, k+1, k+2, \ldots$ we do the following. We have pairwise edge-disjoint $s-t$ paths $P_{n-k+1}, \ldots, P_n$, without crossings, so that the first edges of $P_{n-k+1}, \ldots, P_n$ occur in this order clockwise at s, and each pair (P_{i-1}, P_i) is internally separate $(i = n-k+2, \ldots, n)$.

If also the pair (P_n, P_{n-k+1}) is internally separate, then $P_{n-k+1}, \ldots, P_n$ are pairwise separate, and hence we have k pairwise separate $s-t$ paths as required. If (P_n, P_{n-k+1}) is not internally separate, let P_{n+1} be the path in $\overline{R}(P_{n-k+1}, P_{n-k+2})$ such that (P_n, P_{n+1}) is internally separate and such that $R(P_{n+1}, P_{n-k+2})$ is as large as possible. If P_{n+1} uses an edge in P_{n-k+2}, then as with P_k, we let P_{n+1} use a new parallel edge to the left. Then reset $n := n+1$, and repeat.

III. **Correctness and running time.** Suppose we do $|V|$ iterations. Let $m := k + |V|$. Consider the universal covering surface U of $S_2 \setminus \{s, t\}$ (see [5]), with projection mapping $\pi : U \longrightarrow S_2 \setminus \{s, t\}$. The inverse $\pi^{-1}[G \setminus \{s, t\}]$ of $G \setminus \{s, t\}$ is an infinite graph on U. (The universal covering surface is obtained from S_2 by puncturing holes at s and t and then cutting between the holes to form a rectangular surface. Copies of this rectangle are glued together to form the covering surface which then contains an infinite number of copies of $S_2 \setminus \{s, t\}$.)

For any lifting Q of any simple $s-t$ path P in G, we denote by Q' the lifting next to the right of Q. That is, Q' is to the right of Q (with respect to the lifted orientation of P from s to t), and there is no other lifting of P in between of Q and Q'.

By our construction, there exist liftings $Q_1, \ldots, Q_m$ of $P_1, \ldots, P_m$, respectively, so that Q_n is to the right of Q_{n-1} (possibly touching) for $n = 2, \ldots, m$, and such that

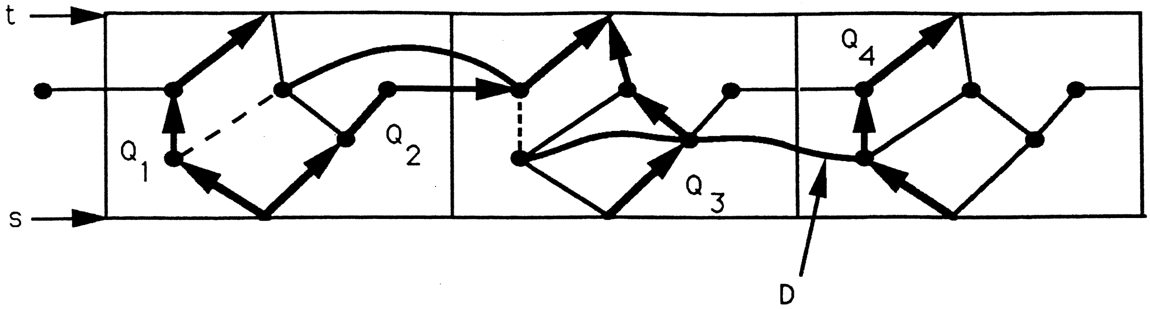

Figure 2: Example of the algorithm applied to the graph of Figure 1.

$Q_{n-k+2},\ldots,Q_n$ are contained in the region enclosed by Q_{n-k+1} and Q'_{n-k+1} for $n = k, k+1, \ldots, m$.

For each $n = k+1, \ldots, m$, let V_n denote the set of internal vertices of Q_n which are not vertices of Q'_{n-k}. Let V_k be the internal vertices of Q_k. Since we did keep shifting, each $V_n \neq \phi$. Note that for any $v \in V_n$, there is a v' on Q_{n-1} and a curve C_v from v' to v such that either C_v first traverses an edge and next a face, or only traverses a face (except for v' and v). Furthermore, we have $v' \in V_{n-1}$. Otherwise v' is a vertex of Q'_{n-k-1} which is either a vertex of Q'_{n-k} or adjacent to Q'_{n-k}. This contradicts the fact that (P_{n-k-1}, P_{n-k}) is internally separate for $n-1 > k$.

Choose $v_m \in V_m$ and for each $n = m-1, m-2, \ldots k$, let v_n be the starting vertex of $C_{v_{n+1}}$. Since $m = k+|V|$, there exist n', n'' with $m \geq n'' > n' \geq k$ such that $\pi(v_{n''}) = \pi(v_{n'})$. Let D be the curve

$$(1) \qquad C_{v_{n'+1}} \cdot C_{v_{n'+2}} \cdot \ldots \cdot C_{v_{n''}},$$

and let C be the projection $\pi \circ D$ of D to S. So C is an alternate closed curve with $l(C) = n'' - n'$. We show that $k \cdot w(C) > n'' - n'$, proving sufficiency in (i).

For any lifting Q of any simple $s-t$ path P and any $i \geq 0$, let $Q^{(i)}$ be the ith lifting to the right of Q. That is, $Q^{(0)} = Q$ and $Q^{(i+1)} = (Q^{(i)})'$.

Let $u := \lfloor \frac{n''-n'}{k} \rfloor$. We must show $w(C) > u$. If $u = 0$, then $w(C) > u = 0$ since $v_{n''} \neq v_{n'}$. If $u > 0$, then $v_{n''}$ is strictly to the right of $Q'_{n''-k}$ and $Q'_{n''-k}$ is to the right of $Q^{(u)}_{n'}$ (since $Q_{n''-k}$ is to the right of $Q^{(u-1)}_{n'}$, as $n'' - k \geq n' + (u-1)k$). So $v_{n''}$ is strictly to the right of $Q^{(u)}_{n'}$. Therefore, $w(C) > u$.

□

The algorithm given in the proof of the theorem can be extended for any fixed surface S and any fixed k, to find k pairwise separate $s-t$ paths in any graph embedded on S. It can also be shown ([4]) that the problem of finding a minimum-weight induced circuit traversing two given vertices s and t in a planar graph, is solvable in polynomial time. Moreover, finding a set of k pairwise separate $s-t$ paths of minimum total weight, is solvable in polynomial time for planar graphs.

The proof of the theorem can also be extended to solve a related problem. Let G be a plane graph whose edges are coloured red and green (we allow multiple edges). We wish to find a maximum number of $s-t$ paths whose edges are all green and such that there are no red edges between distinct paths. A similar characterization holds for this problem although we have to slightly change the definition of an alternate curve. In (2) we may only use a vertex or a red edge as a w_i.

We also consider the following generalization of Theorem A. For $d \geq 0$, a *d-path* is a simple dipath of length at most d. A collection of $s-t$ dipaths is pairwise d-separate if there is is no d-path connecting internal vertices of distinct dipaths in the collection. We call a closed curve C (with clockwise orientation relative to s) *d-alternate* if C does not traverse s or t, and there exists a sequence

(2) $$(C_0, p_1, C_1, p_2, C_2, \ldots, p_l, C_l)$$

such that

(3)

(i) p_i is a d-path of $D \setminus \{s,t\}$ with endpoints s_i, t_i ($i = 1, \ldots, l$);

(ii) C_i is a (noncrossing) curve of positive length from t_{i-1} to s_i and these are the only vertices of D that C_i intersects ($i = 1, \ldots, l$ and $C_0 = C_l$);

(iii) C traverses the paths and curves given in (2) in the described order;

(iv) C_i may only cross arcs from right to left (relative to the orientation derived from C) and none of these arcs corresponds to an arc of any p_i.

Here, by definition, C *traverses* a path p if C follows p from one end vertex to the other. (Informally, condition (iv) requires that any arcs crossed by C_i must be directed towards s.) The following theorem is proved in [3]. In contrast with Theorem A, the proof of necessity is not straightforward.

Theorem B.
For a plane graph G:

(i) *There exist k pairwise d-separate $s-t$ paths, if and only if $l(C) \geq k \cdot w(C)$ for each d-alternate closed curve C.*

(ii) *A maximum number of pairwise d-separate $s-t$ paths can be found in polynomial time.*

(iii) *The curves C in* (i) *can be restricted to those with $l(C) \leq |V|$.*

We note that in the directed case we do not require the paths in the collection to be induced, i.e., they may have backwards arcs. In fact, Kratochvil [2] has recently shown that the problem of determining whether there is a single induced $s-t$ dipath in a planar graph is NP-complete.

References

[1] D. Bienstock (1989), *Private Communication.*

[2] J. Kratochvil, Personal communication, (1991).

[3] C. McDiarmid, B. Reed, A. Schrijver and B. Shepherd, Non-interfering dipaths in planar digraphs, *Centrum voor Wiskunde en Informatica Technical Report* (1991).

[4] C. McDiarmid, B. Reed, A. Schrijver and B. Shepherd, Packing induced paths in planar graphs, *University of Waterloo Technical Report* (1990).

[5] W.S. Massey, Algebraic Topology: An Introduction, Graduate texts in mathematics, 56, *Springer-Verlag*, (1967).