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Induced Circuits in Planar Graphs

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Abstract. In [4] we give a polynomial-time algorithm for the problem of finding an induced circuit traversing two given vertices of a planar graph. We give a combinatorial algorithm and a min-max relation for the problem of finding a maximum number of paths connecting two given vertices in a planar graph so that each pair of these paths form an induced circuit.

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Let \( G = (V, E) \) be an undirected graph without loops, and let \( s, t \) be distinct vertices. We call two \( s - t \) paths \( P', P'' \) separate if they form an induced circuit. We consider the problem of finding a maximum number of pairwise separate \( s - t \) paths. For general graphs this is an NP-hard problem: this follows from Bienstock [1] in which it is shown that the problem of deciding if there exists an induced circuit containing \( s \) and \( t \), is NP-complete.

We show that the problem can be solved in polynomial time for planar graphs. Assume that \( G \) is embedded in the 2-sphere \( S_2 \). Moreover, we give a good characterization, based on the following concepts. Let \( C \) be a closed curve in \( S_2 \), not traversing \( s \) or \( t \). The winding number \( w(C) \) of \( C \) is, roughly speaking, the number of times that \( C \) separates \( s \) and \( t \). More precisely, consider any curve \( P \) from \( s \) to \( t \), crossing \( C \) only a finite number of times. Let \( \lambda \) be the number of times \( C \) crosses \( P \) from left to right, and let \( \rho \) be the number of times \( C \) crosses \( P \) from right to left (fixing some orientation of \( C \), and orienting \( P \) from \( s \) to \( t \)). Then \( w(C) = |\lambda - \rho| \). (This number can be seen to be independent of the choice of \( P \).)

We call a closed curve \( C \) alternate if \( C \) does not traverse \( s \) or \( t \), and there exists a sequence

\[ (F_0, w_1, F_1, w_2, F_2, \ldots, w_l, F_l) \]

(where \( l \geq 0 \)) such that

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(i) $F_0, \ldots, F_l$ are faces of $G$, with $F_0 = F_l$;

(ii) $w_i$ is a vertex or edge of $G$ ($i = 1, \ldots, m$);

(iii) $C$ traverses vertices, edges and faces of $G$ in the order (2).

Here, by definition, $C$ traverses an edge $e$ if $C$ follows $e$ from one end vertex to the other.

Let $l(C)$ denote the number $l$ in (2). Now:

**Theorem A.**

(i) There exist $k$ pairwise separate $s-t$ paths, if and only if $l(C) \geq k \cdot w(C)$ for each alternate closed curve $C$.

(ii) A maximum number of pairwise separate $s-t$ paths can be found in polynomial time.

(iii) The curves $C$ in (i) can be restricted to those with $l(C) \leq |V|$.

Before proving the theorem, let us give one small example of a graph where an alternate curve with winding number at least two must be used. Note that a proof of nonexistence of an induced circuit containing $s, t$ (with $s, t$ nonadjacent), by means of an alternate curve is equivalent to there being a vertex cut set which is a clique of size at most two. It is easily verified that the graph of Figure 1 does not contain any induced $s, t$ cycle and yet neither does it have such a clique cut set. Other examples can be constructed for $k > 2$ (see [4]).

**Proof of Theorem A:**

I. **Necessity in (i).** Let $P_1, \ldots, P_k$ be pairwise separate $s-t$ paths, and let $C$ be an alternate closed curve. Then $C$ intersects each $P_i$ at least $w(C)$ times. It is not hard to see that for each $i$, at least $w(C)$ of the $w_j$ in (2) are incident to a vertex in $P_i$ (defining two vertices $v', v''$ to be incident if $v' = v''$). Since distinct $P_i$ and $P_i'$ are separate, there should be at least $k \cdot w(C)$ $w_j$'s, i.e., $l(C) \geq k \cdot w(C)$.

II. **Algorithm.** We next describe an algorithm finding for any $k$, either $k$ pairwise separate $s-t$ paths or an alternate closed curve $C$ with $l(C) < k \cdot w(C)$. We assume, without loss of generality, that there is no edge connecting $s$ and $t$. 

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First we introduce some notation and terminology. We think of \( G \) being embedded on the 2-sphere \( S_2 \). Any \( s-t \) path will be oriented from \( s \) to \( t \). Let \( O \) be an open disk whose boundary contains \( s \) and \( t \). An edge \( e \) (of \( G \)) contained in the closure \( \overline{O} \) of \( O \), connecting two points on the boundary of \( O \), is called a singel relative to \( O \), if any curve from \( s \) to \( t \) contained in \( O \), must cross \( e \). Let \( P', P'' \) be two edge-disjoint \( s-t \) paths, without crossings. Then \( R(P', P'') \) denotes the region encircled by the closed curve \( P' \cdot (P'')^{-1} \) in clockwise orientation. We call the pair \( (P', P'') \) internally separate if \( R(P', P'') \) is an open disk not containing a singel. Note that even if \( (P', P'') \) is internally separate, \( P' \) and \( P'' \) can have a vertex \( v \neq s,t \) in common. Note moreover that \( P' \) and \( P'' \) are separate if and only if both \( (P', P'') \) and \( (P'', P') \) are internally separate.

For \( k = 1 \) the algorithm is trivial: either there exists an \( s-t \) path, or there exists a closed curve \( C \) not intersecting \( G \) with \( w(C) = 1 \) (implying \( l(C) = 0 < 1 \cdot w(C) \)).

Suppose now that \( k > 1 \), and that we have found \( k-1 \) pairwise separate \( s-t \) paths \( P_1, \ldots, P_{k-1} \). In the case that \( k = 2 \) we assume that there exist two internally disjoint \( s-t \) paths \( P_1, P_2 \). If no such pair exists, then it is easy to find an appropriate alternate curve with the help of Menger’s Theorem. Hence for \( k = 2 \) we may choose \( P_1 \) to be \( P \).

We may assume that the first edges of \( P_1, \ldots, P_{k-1} \) occur in this order clockwise at \( s \). Let \( P_k \) be a path ‘parallel’ to the left of \( P_1 \). That is, we add to each edge traversed by \( P_1 \) a parallel edge at the left hand side (with respect to the orientation of \( P_1 \)), and \( P_k \) follows these new edges. (Note that adding parallel edges does not change our problem and in the case \( k = 2 \) we have chosen \( P_2 \) so that \( (P_1, P_2) \) is internally separate.) Then the first edges of \( P_1, \ldots, P_k \) occur in this order clockwise at \( s \), and each pair \( (P_{i-1}, P_i) \) is internally separate \( (i = 2, \ldots, k) \).

Now for \( n = k, k+1, k+2, \ldots \) we do the following. We have pairwise edge-disjoint \( s-t \) paths \( P_{n-k+1}, \ldots, P_n \), without crossings, so that the first edges of \( P_{n-k+1}, \ldots, P_n \) occur in this order clockwise at \( s \), and each pair \( (P_{i-1}, P_i) \) is internally separate \( (i = n-k+2, \ldots, n) \).

If also the pair \( (P_n, P_{n-k+1}) \) is internally separate, then \( P_{n-k+1}, \ldots, P_n \) are pairwise separate, and hence we have \( k \) pairwise separate \( s-t \) paths as required. If \( (P_n, P_{n-k+1}) \) is not internally separate, let \( P_{r+1} \) be the path in \( \overline{R}(P_{n-k+1}, P_{n-k+2}) \) such that \( (P_n, P_{r+1}) \) is internally separate and such that \( R(P_{n+1}, P_{n-k+2}) \) is as large as possible. If \( P_{r+1} \) uses an edge in \( P_{r-k+2} \), then as with \( P_k \), we let \( P_{n+1} \) use a new parallel edge to the left. Then reset \( n := n + 1 \), and repeat.

III. Correctness and running time. Suppose we do \( |V| \) iterations. Let \( m := k + |V| \). Consider the universal covering surface \( U \) of \( S_2 \setminus \{s, t\} \) (see \([5]\)), with projection mapping \( \pi : U \rightarrow S_2 \setminus \{s, t\} \). The inverse \( \pi^{-1}[G \setminus \{s, t\}] \) of \( G \setminus \{s, t\} \) is an infinite graph on \( U \).

(The universal covering surface is obtained from \( S_2 \) by puncturing holes at \( s \) and \( t \) and then cutting between the holes to form a rectangular surface. Copies of this rectangle are glued together to form the covering surface which then contains an infinite number of copies of \( S_2 \setminus \{s, t\} \).

For any lifting \( Q \) of any simple \( s-t \) path \( P \) in \( G \), we denote by \( Q' \) the lifting next to the right of \( Q \). That is, \( Q' \) is to the right of \( Q \) (with respect to the lifted orientation of \( P \) from \( s \) to \( t \)), and there is no other lifting of \( P \) in between of \( Q \) and \( Q' \).

By our construction, there exist liftings \( Q_1, \ldots, Q_m \) of \( P_1, \ldots, P_m \), respectively, so that \( Q_n \) is to the right of \( Q_{n-1} \) (possibly touching) for \( n = 2, \ldots, m \), and such that
$Q_{n-k+2}, \ldots, Q_n$ are contained in the region enclosed by $Q_{n-k+1}$ and $Q'_{n-k+1}$ for $n = k, k+1, \ldots, m$.

For each $n = k+1, \ldots, m$, let $V_n$ denote the set of internal vertices of $Q_n$ which are not vertices of $Q'_{n-k}$. Let $V_k$ be the internal vertices of $Q_k$. Since we did keep shifting, each $V_n \neq \emptyset$. Note that for any $v \in V_n$, there is a $v'$ on $Q_{n-1}$ and a curve $C_v$ from $v'$ to $v$ such that either $C_v$ first traverses an edge and next a face, or only traverses a face (except for $v'$ and $v$). Furthermore, we have $v' \in V_{n-1}$. Otherwise $v'$ is a vertex of $Q'_{n-k}$ which is either a vertex of $Q'_{n-k}$ or adjacent to $Q'_{n-k}$. This contradicts the fact that $(P_{n-k-1}, P_{n-k})$ is internally separate for $n - 1 > k$.

Choose $v_m \in V_m$ and for each $n = m-1, m-2, \ldots, k$, let $v_n$ be the starting vertex of $C_{v_{n+1}}$. Since $m = k + |V|$, there exist $n', n''$ with $m \geq n'' > n' \geq k$ such that $\pi(v_{n''}) = \pi(v_n)$. Let $D$ be the curve

\begin{equation}
C_{v_{n''+1}} \cdot C_{v_{n'+2}} \cdots C_{v_n},
\end{equation}

and let $C$ be the projection $\pi \circ D$ of $D$ to $S$. So $C$ is an alternate closed curve with $l(C) = n'' - n'$. We show that $k \cdot w(C) > n'' - n'$, proving sufficiency in (i).

For any lifting $Q$ of any simple $s-t$ path $P$ and any $i \geq 0$, let $Q^{(i)}$ be the $i$th lifting to the right of $Q$. That is, $Q^{(0)} = Q$ and $Q^{(i+1)} = (Q^{(i)})'$. Let $u := \frac{n'' - n'}{k}$. We must show $w(C) > u$. If $u = 0$, then $w(C) > u = 0$ since $v_{n''} \neq v_{n'}$. If $u > 0$, then $v_{n''}$ is strictly to the right of $Q'_{n-k}$ and $Q'_{n-k}$ is to the right of $Q_{n''}$ (since $Q_{n''-k}$ is to the right of $Q'_{n-k}$, $n'' - k \geq n' + (u - 1)k$). So $v_{n''}$ is strictly to the right of $Q_{n''}^u$. Therefore, $w(C) > u$.

The algorithm given in the proof of the theorem can be extended for any fixed surface $S$ and any fixed $k$, to find $k$ pairwise separate $s-t$ paths in any graph embedded on $S$. It can also be shown ([4]) that the problem of finding a minimum-weight induced circuit traversing two given vertices $s$ and $t$ in a planar graph, is solvable in polynomial time. Moreover, finding a set of $k$ pairwise separate $s-t$ paths of minimum total weight, is solvable in polynomial time for planar graphs.
The proof of the theorem can also be extended to solve a related problem. Let \( G \) be a plane graph whose edges are coloured red and green (we allow multiple edges). We wish to find a maximum number of \( s - t \) paths whose edges are all green and such that there are no red edges between distinct paths. A similar characterization holds for this problem although we have to slightly change the definition of an alternate curve. In (2) we may only use a vertex or a red edge as a \( w_i \).

We also consider the following generalization of Theorem A. For \( d \geq 0 \), a \( d \)-path is a simple dipath of length at most \( d \). A collection of \( s - t \) dipaths is pairwise \( d \)-separate if there is no \( d \)-path connecting internal vertices of distinct dipaths in the collection. We call a closed curve \( C \) (with clockwise orientation relative to \( s \)) \( d \)-alternate if \( C \) does not traverse \( s \) or \( t \), and there exists a sequence

\[
(C_0, p_1, C_1, p_2, C_2, \ldots, p_l, C_l)
\]

such that

(i) \( p_i \) is a \( d \)-path of \( D \setminus \{s, t\} \) with endpoints \( s_i, t_i \) \((i = 1, \ldots, l)\);

(ii) \( C_i \) is a (noncrossing) curve of positive length from \( t_{i-1} \) to \( s_i \) and these are the only vertices of \( D \) that \( C_i \) intersects \((t = 1, \ldots, l) \text{ and } C_0 = C_l\);

(iii) \( C \) traverses the paths and curves given in (2) in the described order;

(iv) \( C_i \) may only cross arcs from right to left (relative to the orientation derived from \( C \)) and none of these arcs corresponds to an arc of any \( p_i \).

Here, by definition, \( C \) traverses a path \( p \) if \( C \) follows \( p \) from one end vertex to the other. (Informally, condition (iv) requires that any arcs crossed by \( C_i \) must be directed towards \( s \).) The following theorem is proved in [3]. In contrast with Theorem A, the proof of necessity is not straightforward.

Theorem B.
For a plane graph \( G \):

(i) There exist \( k \) pairwise \( d \)-separate \( s - t \) paths, if and only if \( l(C) \geq k \cdot w(C) \) for each \( d \)-alternate closed curve \( C \).

(ii) A maximum number of pairwise \( d \)-separate \( s - t \) paths can be found in polynomial time.

(iii) The curves \( C \) in (i) can be restricted to those with \( l(C) \leq |V| \).

We note that in the directed case we do not require the paths in the collection to be induced, i.e., they may have backwards arcs. In fact, Kratochvíl [2] has recently shown that the problem of determining whether there is a single induced \( s - t \) dipath in a planar graph is \( \text{NP} \)-complete.
References


