# 1991

A.L.M. Dekkers, L. de Haan

Optimal choice of sample fraction in extreme-value estimation

Department of Operations Research, Statistics, and System Theory    Report BS-R9107    March

CWI, nationaal instituut voor onderzoek op het gebied van wiskunde en informatica

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).

# Optimal Choice of Sample Fraction in Extreme-Value Estimation

Arnold L. M. Dekkers
*Centre for Mathematics and Computer Science*
*P.O. Box 4079, 1009 AB Amsterdam, The Netherlands*

Laurens de Haan
*Erasmus University*
*P.O. Box 1738, 3000 DR Rotterdam, The Netherlands*

We study the asymptotic bias of the moment estimator $\hat{\gamma}_n$ for the extreme-value index $\gamma \in \mathbb{R}$ under quite natural and general conditions on the underlying distribution function. Furthermore the optimal choice for the sample fraction in estimating $\gamma$ is considered by minimizing the mean squared error of $\hat{\gamma}_n - \gamma$. The results cover all three limiting types of extreme-value theory.
The connection between statistics and regular variation and $\Pi$-variation is handled in a systematic way.

*(1980) Mathematics Subject Classification:* 62E20, 62G30, 26A12.
*Key Words & Phrases:* Extreme-value theory, order statistics, asymptotic normality, mean squared error, regular variation, $\Pi$-variation, inverse complementary function.

## 1 Introduction

Suppose one is given a sequence $X_1, X_2, \ldots$ of i.i.d. observations from some unknown distribution function $F$. Suppose for some constants $a_n > 0$ and $b_n$ and some $\gamma \in \mathbb{R}$

$$\lim_{n\to\infty} P\left\{\frac{\max\{X_1, X_2, \ldots, X_n\} - b_n}{a_n} \le x\right\} = G_\gamma(x) \tag{1}$$

for all $x$, where $G_\gamma(x)$ is one of the extreme-value distributions, given by

$$G_\gamma(x) := \exp -(1+\gamma x)^{-1/\gamma} . \tag{2}$$

Here $\gamma$ is a real parameter, $x$ such that $1+\gamma x > 0$. Interpret $(1+\gamma x)^{-1/\gamma}$ as $e^{-x}$ for $\gamma = 0$. The question is how to estimate $\gamma$, the extreme-value index, from a finite sample $X_1, X_2, \ldots, X_n$. If (1) holds, $F$ is said to be in the domain of attraction of the generalized extreme-value distribution $G_\gamma$ [notation : $F \in \mathcal{D}(G_\gamma)$]. For the extreme-value distributions itself one has $G_\gamma \in \mathcal{D}(G_\gamma)$.

In the last decade much attention has been paid to the estimation of the tail-index of a distribution. This corresponds to estimating $\gamma$ when $\gamma > 0$. Most of the publications are based on the work of Pickands III (1975) and Hill (1975).

Pickands proposed the following estimator for $\gamma \in \mathbb{R}$ and $1 \le k \le [n/4]$

$$\hat{\gamma}_n^{(P)} := (\log 2)^{-1} \log \frac{X_{(n-k,n)} - X_{(n-2k,n)}}{X_{(n-2k,n)} - X_{(n-4k,n)}},$$

Report BS-9107
Centre for Mathematics and Computer Science
P.O.Box 4079, 1009 AB Amsterdam, The Netherlands

where $X_{(1,n)} \leq X_{(2,n)} \leq \ldots \leq X_{(n,n)}$ are the ascending order statistics of $X_1, X_2, \ldots, X_n$. He proved weak consistency of the estimate.

Dekkers and de Haan (1989) gave quite natural and general conditions under which $\sqrt{k}(\hat{\gamma}_n^{(P)} - \gamma)$ is asymptotically normal. Conditions on $k = k(n)$ include $k = k(n) \to \infty$ and $k/n \to 0$ $(n \to \infty)$.

For $\gamma$ positive, Hill introduced the estimator

$$M_n^{(1)} := \frac{1}{k} \sum_{i=0}^{k-1} \log X_{(n-i,n)} - \log X_{(n-k,n)}$$

which involve all $k+1$ upper order statistics instead of only $X_{(n-k,n)}, X_{(n-2k,n)}$ and $X_{(n-4k,n)}$. Mason (1982) proved weak consistency of $M_n^{(1)}$ for any sequence $k = k(n) \to \infty$ , $k(n)/n \to 0$ $(n \to \infty)$ and Deheuvels, Häusler and Mason (1988) proved also strong consistency for sequences $k(n)$, with $k/\log\log n \to \infty$ and $k/n \to 0 (n \to \infty)$. Under certain extra conditions $\sqrt{k}(M_n^{(1)} - \gamma)$ is asymptotically normal with mean zero and variance $\gamma^2$ [see Hall (1982), Davis and Resnick (1984), Csörgö and Mason (1985), Häusler and Teugels (1985), Goldie and Smith (1987) and Dekkers, Einmahl and de Haan (1989)].

Hall (1982) considered distribution functions $F$ which satisfy

$$1 - F(x) = Ax^{-1/\gamma}\{1 + Bx^{-\beta} + o(x^{-\beta})\}, \quad x \to \infty,$$

for $\gamma > 0, A > 0, B \neq 0$ and $\beta > 0$. He proved asymptotic normality for the Hill-estimator and derived an optimal choice for $k$, the number of upper order statistics used in estimating $\gamma$, by minimizing the asymptotic mean squared error of $M_n^{(1)}$. Although he considered an important class of distribution functions, his approach is limited to only $\gamma$ positive.

Using Pickands' well-known key idea [the conditional distribution function of $X - u$, given X exceeds threshold $u$, can be approximated by the generalized Pareto distribution (GPD)] Smith (1987) fits the GPD-distribution by the method of maximum likelihood. The shape-parameter of the fitted GPD-distribution is an estimator of $\gamma$. He obtains asymptotic normality for the MLE-estimators in case $\gamma > -1/2$ and under some extra conditions he obtains also the asymptotic bias of the estimators.

Dekkers, Einmahl and de Haan (1989) considered the problem how to estimate $\gamma$ for general $\gamma \in \mathbb{R}$. They introduced the moment estimator given by

$$\hat{\gamma}_n^{(M)} := M_n^{(1)} + 1 - \tfrac{1}{2}\{1 - (M_n^{(1)})^2/M_n^{(2)}\}^{-1}. \tag{3}$$

where $M_n^{(1)}$ is the Hill estimator and

$$M_n^{(2)} := \frac{1}{k} \sum_{i=0}^{k-1} \{\log X_{(n-i,n)} - \log X_{(n-k,n)}\}^2 \quad ,$$

provided that $x^* = x^*(F) > 0$, which can always be achieved by a simple shift $[x^*(F) := \sup\{x | F(x) < 1\}]$. The moment estimator has some intuitive background [cf. Dekkers, Einmahl and de Haan (1989), section 6] and covers all limiting types of extreme-value theory. Under natural and general conditions the estimator has asymptotically a normal distribution.

All the mentioned estimators for $\gamma$ have one common property. When the number of upper order statistics used in estimating $\gamma$ is small, the variance of the estimator will be large. But on the other hand the use of a large number of upper order statistics will introduce a bias in the estimation in most cases. Balancing the variance and bias components will lead to an optimal choice for $k$. Therefore we want to study the bias of the moment estimator in a systematic way.

So the two main problems which return in all the work and where we like to focus on in this paper are

- how to choose the number of upper order statistics, $k$, involved in estimating $\gamma$,

- are the conditions in some way natural and do they cover all possibilities of tail behaviour?

In section 2 we will give more in detail some conditions and our aim is to show that these conditions are quite natural and general [see appendix A]. In section 3 we will study the moment-estimator for the cases $\gamma > 0$, $\gamma < 0$ and $\gamma$ equals zero. Finally we will give some examples in section 4.

## 2 Regular Variation, Π-variation and extreme-value theory

In this section we want to give some details how the tail behaviour of distribution function $F$ can be translated into terms of the inverse function of $1/(1-F)$. Next we will formulate our "second order" conditions on $F$. Finally we will give a lemma which we need for minimizing the asymptotic mean squared error of $\hat{\gamma}_n$.

Define the function $U : \mathbb{R}^+ \to \mathbb{R}$ by

$$U(x) := \begin{cases} 0 & 0 \le x < 1 \\ \left(\dfrac{1}{1-F}\right)^{\leftarrow}(x) & 1 \le x \end{cases}$$

where the arrow indicates the inverse function defined e.g. by $U(x) := \inf\{y | 1/(1-F(y)) \ge x\}$. Now the domain of attraction condition (1) can be stated in the following way in terms of $U$.

Lemma 2.1

*For a distribution function $F$ holds $F \in \mathcal{D}(G_\gamma)$ if and only if there exists a positive function $a_1$ such that*

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a_1(t)} = \frac{x^\gamma - 1}{\gamma}, \quad (x > 0), \tag{4}$$

*where the right hand side of (4) has to be interpreted as $\log x$ for $\gamma = 0$.*

Proof: Cf. de Haan (1984), lemma 1.

Lemma 2.2

*For $\gamma > 0$, (4) is equivalent to*

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma, \tag{5}$$

*for all $x > 0$, e.g. $U$ is regularly varying with index $\gamma$ [notation $U(t) \in RV_\gamma$], and hence $a_1(t) \sim \gamma U(t)$, $t \to \infty$, i.e. $\lim_{t\to\infty} a_1(t)/(\gamma U(t)) = 1$.*

*For $\gamma < 0$, $F$ has a finite right endpoint, so $U(\infty) = x^* < \infty$, and (4) is equivalent to*

$$U(\infty) - U(t) \in RV_\gamma. \tag{6}$$

*In this case $a_1(t) \sim -\gamma\{U(\infty) - U(t)\}$, $t \to \infty$.*

Proof: Cf. de Haan (1984), corollary 3.

We will call (5) and (6) the first order regular variation conditions on $U$ and for $\gamma = 0$ property (4) the first order Π-variation condition on $U$ [notation $U \in \Pi(a_1)$].

In the following two lemma's the second order conditions are formulated and equivalent conditions are given. In appendix A we will show that these conditions are quite natural and cover all possibilities.

**Lemma 2.3** [Second order regular variation]
*Suppose $\rho > 0$ and $c > 0$.*

1. *For $\gamma < 0$ are equivalent [with either choice of sign] :*

$$(a.) \qquad \pm\{x^{-1/\gamma}[1 - F(U(\infty) - x^{-1})] - c^{1/\gamma}\} \in RV_{-\rho}$$
$$(b.) \qquad \mp\{t^{-\gamma}[U(\infty) - U(t)] - c\} \in RV_{\gamma\rho}.$$

*For $U(\infty) > 0$ these conditions imply the following equivalent conditions*

$$(c.) \qquad \pm\{x^{-1/\gamma}[1 - F(U(\infty)e^{-1/x})] - (c/U(\infty))^{1/\gamma}\} \in RV_{-\rho}$$
$$(d.) \qquad \mp\{t^{-\gamma}[\log U(\infty) - \log U(t)] - c/U(\infty)\} \in RV_{\gamma\rho}\ .$$

2. *For $\gamma > 0$ are equivalent [with either choice of sign] :*

$$(e.) \qquad \pm\{x^{1/\gamma}(1 - F(x)) - c^{1/\gamma}\} \in RV_{-\rho}$$
$$(f.) \qquad \pm\{t^{-\gamma}U(t) - c\} \in RV_{-\gamma\rho}$$
$$(g.) \qquad \pm\{\log U(t) - \gamma\log t - \log c\} \in RV_{-\gamma\rho}.$$

Proof: see appendix B.

**Remark 2.4** Note that the conditions $(d.)$ and $(g.)$ are different, $(g.)$ is equivalent to $(f.)$, but $(d.)$ is not equivalent to $(b.)$. A counter example is the uniform distribution with $U(t) = 1 - 1/t$, which does not satisfy $(b.)$ although it satisfies $(d.)$ with $\gamma = 1$ , $\rho = 1$ and $c = U(\infty) = 1$.

**Lemma 2.5** [Second order $\Pi$-variation]
*Suppose the functions $b_1$, $b_2$, $b_3$, $b_4$, $f$ and $\alpha$ are positive.*

1. *For $\gamma < 0$ are equivalent [with either choice of sign] :*

$$(a.) \qquad \pm\{x^{-1/\gamma}[1 - F(U(\infty) - x^{-1})]\} \in \Pi$$
$$(b.) \qquad \mp\{t^{-\gamma}[U(\infty) - U(t)]\} \in \Pi(b_1)\ .$$

*For $U(\infty) > 0$ these conditions imply the following equivalent conditions*

$$(c.) \qquad \pm\{x^{-1/\gamma}[1 - F(U(\infty)e^{-1/x})]\} \in \Pi$$
$$(d.) \qquad \mp\{t^{-\gamma}[\log U(\infty) - \log U(t)]\} \in \Pi(b_1/U(\infty))\ .$$

2. *For $\gamma = 0$ are equivalent with $\alpha(t) \to 0$ , $t \to x^*$ and $b_2(t) \to 0$ , $t \to \infty$ [with either choice of sign] :*

$$(e.) \qquad \lim_{t\uparrow x^*}\left(\frac{1 - F(\exp(t + xf(t)))}{1 - F(\exp(t))} - e^{-x}\right)/\alpha(t) \ = \ \frac{x^2}{2}e^{-x}$$
$$(f.) \qquad \lim_{t\to\infty}\frac{\log U(tx) - \log U(t) - b_2(t)\log x}{b_3(t)} \ = \ -\ \frac{(\log x)^2}{2}$$

3. *For $\gamma > 0$ are equivalent [with either choice of sign] :*

$$(g.) \qquad \pm\{x^{1/\gamma}(1 - F(x))\} \in \Pi$$
$$(h.) \qquad \pm t^{-\gamma}U(t) \in \Pi(b_4)$$
$$(i.) \qquad \pm\{\log U(t) - \gamma\log t\} \in \Pi(b_4/(t^{-\gamma}U(t))).$$

**PROOF**
For the proof we refer to the appendix of Dekkers and de Haan (1989) and to theorem 3.3 of Dekkers, Einmahl and de Haan (1989).

**REMARK 2.6**
Note that all conditions imply $F \in \mathcal{D}(G_\gamma)$ for appropriate $\gamma$.

**REMARK 2.7**
In the case of the second order regular variation [lemma 2.3] the case $\gamma = 0$ does not exist. See appendix A.

**REMARK 2.8**
In the case of second order $\Pi$-variation with $\gamma = 0$ we have in (*e.*) only the plus sign and in (*f.*) only the minus sign, instead of both choices as for $\gamma \neq 0$. The reason is the following. Condition(*f.*)implies for $x > 1$ and $y > 1$

$$\begin{aligned}\frac{\log U(txy) - \log U(t) - b_2(t)\log xy}{b_3(t)} &= \frac{\log U(txy) - \log U(tx) - b_2(tx)\log x}{b_3(tx)} \cdot \frac{b_3(tx)}{b_3(t)} \\ &+ \frac{\log U(tx) - \log U(t) - b_2(t)\log x}{b_3(t)} \\ &+ \frac{b_2(tx) - b_2(t)}{b_3(t)}\log y \ . \qquad (7)\end{aligned}$$

Now suppose that the left-hand side of (7) tends to $\pm(\log xy)^2/2$ and thus the right hand-side converges also. So $(b_2(tx) - b_2(t))/b_3(t)$ converges to $\pm \log x$ and hence $\pm b_2 \in \Pi(b_3)$. Note that $b_2(t) > 0$ and $b_2(t) \to 0$ , $t \to \infty$ , which is not compatible with $b_2 \in \Pi(b_3)$. This implies $-b_2 \in \Pi(b_3)$ and therefore only the minus sign is possible in condition (*f.*).

In the last part of this section we describe in a general way how to minimize the mean squared error

$$\frac{\sigma^2(\gamma)}{k} + f(\tfrac{n}{k}) ,$$

where $\sigma^2(\gamma)$ denotes the asymptotic variance of the estimator, $n$ the sample size, $k$ the number of used upper order statistics and $f$ is the bias squared, hence $f$ is positive. When the bias is not equal to zero, the mean squared error can be minimized. Let $k_o$ be the value for $k$ for which the minimum is attained. If $f$ is differentiable then $k_o = s^{\leftarrow}(\frac{\sigma^2(\gamma)}{n})$, where $s$ is defined as minus the first derivative of $f$ i.e. $-f'$.

In general $f \in RV_{-2\alpha}$ with $\alpha \geq 0$ and moreover for $\alpha = 0$, $f(t) \to 0$ , $t \to \infty$. The following lemma about the inverse complementary function of $f$, shows that these conditions are already sufficient for obtaining the asymptotical value of $k_o$. For more information concerning the inverse complementary function of a regularly varying function we refer to Geluk and de Haan (1987), section II.1.

**LEMMA 2.9**
*Suppose $\alpha \geq 0$ and $f \in RV_{-\alpha}$. Moreover for $\alpha = 0$ suppose $f(t) \to 0$ , $t \to \infty$ and $f$ is asymptotic to a non-increasing function. There exists a positive decreasing function $s \in RV_{-(\alpha+1)}$, such that*

$$f(t) \sim \int_t^\infty s(u)du \ , \ t \to \infty \ . \qquad (8)$$

*Let $f_c$ denote the inverse complementary function of $f$ defined as*

$$f_c(x) := \inf_{y>0}\{f(y) + xy\} \ , \quad x > 0. \qquad (9)$$

*then $f_c(x)$ exists for sufficiently small $x$ and*

$$f_c(x) \sim \int_0^x s^{\leftarrow}(u)du \ , \ x \to 0 \ ,$$

*where $s^{\leftarrow}$ is the generalized inverse function of $s$ and $s^{\leftarrow} \in RV^0_{-1/(\alpha+1)}$, i.e. $\lim_{x\to 0} s^{\leftarrow}(xy)/s^{\leftarrow}(x) = y^{-1/(\alpha+1)}$ for $y > 0$.*

*The value $y_o(x)$ for which the infimum in (9) is attained, is determined asymptotically by $y_o(x) \sim s^{\leftarrow}(x)$ , $x \to 0$.*

PROOF

For $\alpha = 0$ the conditions imply $-f$ is asymptotic to an element of $\Pi$ [see theorem C.1 of appendix C, due to A.A. Balkema]. For (8) see proposition 1.7.3 [$\alpha > 0$] or proposition 1.19.3 [$\alpha = 0$] of Geluk and de Haan (1987). Let $f_1(t) := \int_t^\infty s(u)du$, $c > 1$ and

$$0 < \varepsilon < \min\left(\sqrt{c - c^{-\alpha} + \{\frac{1+c^{-\alpha}}{2}\}^2} - \frac{1+c^{-\alpha}}{2} \ , \ \frac{c-c^{-\alpha}}{1+c}\right) \ ,$$

then there exists $t_o(c)$ such that for $t > t_o(c)$

$$(1-\varepsilon)f_1(t) \le f(t) \le (1+\varepsilon)f_1(t)$$

and

$$(c^{-\alpha} - \varepsilon)f(t) \le f(ct) \le (c^{-\alpha} + \varepsilon)f(t),$$

hence $f(ct) \le (c^{-\alpha}+\varepsilon)f(t) \le (c^{-\alpha}+\varepsilon)(1+\varepsilon)f_1(t) \le cf_1(t)$, since $(c^{-\alpha}+\varepsilon)(1+\varepsilon) - c < 0$. In a similar way $f_1(ct) \le f(ct)/(1-\varepsilon) \le (c^{-\alpha}+\varepsilon)f(t)/(1-\varepsilon) \le c(f(t)$ and hence

$$\frac{1}{c}\int_{ct}^\infty s(u)du \le f(t) \le c\int_{t/c}^\infty s(u)du \ ,$$

which implies

$$\inf_{y>0}\left\{\frac{1}{c}\int_{cy}^\infty s(u)du + xy\right\} \le f_c(x) \le \inf_{y>0}\left\{c\int_{y/c}^\infty s(u)du + xy\right\}$$

and thus for all $c > 1$

$$\frac{1}{c}\int_0^x s^{\leftarrow}(u)du \le f_c(x) \le c\int_0^x s^{\leftarrow}(u)du \ , (x \to 0).$$

Since $s^{\leftarrow}(x)/c \le y_o(x) \le cs^{\leftarrow}(x)$ for all $c > 1$, we have also proved $y_o(x) \sim s^{\leftarrow}(x)$ , $x \to 0$.

## 3 Optimal choice of sample fraction for the moment estimator

In this section we will state our main results for the optimal choice of $k$ and the corresponding bias for the moment estimator.

Let $X_1, X_2, ..., X_n$ be $n$ i.i.d. random variables of an unknown distribution function $F$, with $F \in \mathcal{D}(G_\gamma)$, and let $Y_1, Y_2, ..., Y_n$ be $n$ i.i.d. random variables of distribution function $1 - x^{-1}$, $(x \ge 1)$. Note that $X_{(n-i,n)} \stackrel{d}{=} U(Y_{(n-i,n)})$ for $0 \le i \le n$. The next lemma gives important properties of $Y_1, Y_2, ..., Y_n$ in relation to the moment estimator $\hat{\gamma}_n$ as defined in (3).

Then we will give the main results for distributions with a second order regularly varying tail [theorem 3.2 for $\gamma < 0$ and theorem 3.4 for $\gamma > 0$]. In theorem 3.5 we will consider distributions functions with a second order $\Pi$-varying tail.

**Lemma 3.1**

*Let $Y_{(1,n)} \leq Y_{(2,n)} \leq \cdots \leq Y_{(n,n)}$ be the ascending order statistics of $Y_1, Y_2, ..., Y_n$. Let $0 < k(n) < n$ and $k(n) \to \infty$, $(n \to \infty)$, then*

1. *for $n \to \infty$, $Y_{(n-k,n)}/(\frac{n}{k}) \to 1$ in probability.*
2. *for $n \to \infty$*

$$\sqrt{k(n)}(P_n^o, Q_n^o) := \sqrt{k(n)} \left( \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \log Y_{(n-i,n)} - \log Y_{(n-k(n),n)} - 1 \, , \right.$$
$$\left. \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \{\log Y_{(n-i,n)} - \log Y_{(n-k(n),n)}\}^2 - 2 \right)$$

*is asymptotically normal with means zero, variances 1 and 20 respectively and covariance 4.*

3. *for $\gamma < 0$ and $n \to \infty$*

$$\sqrt{k(n)}(P_n, Q_n) := \sqrt{k(n)} \left( \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} 1 - \left( \frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}} \right)^\gamma + \frac{\gamma}{1-\gamma} \, , \right.$$
$$\left. \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \left\{ 1 - \left( \frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}} \right)^\gamma \right\}^2 - \frac{2\gamma^2}{(1-\gamma)(1-2\gamma)} \right)$$

*is asymptotically normal with means zero and covariance matrix*

$$\frac{\gamma^2}{(1-\gamma)^2(1-2\gamma)} \begin{pmatrix} 1 & \dfrac{-4\gamma}{1-3\gamma} \\ \dfrac{-4\gamma}{1-3\gamma} & \dfrac{4\gamma^2(5-11\gamma)}{(1-2\gamma)(1-3\gamma)(1-4\gamma)} \end{pmatrix} .$$

**Proof:** Cf. lemma 3.4 Dekkers, Einmahl and de Haan (1989).

**Theorem 3.2**

*Suppose $\gamma < 0$, $U(\infty) > 0$, and condition (d.) of lemma 2.3 holds for $\rho \neq 1$. Define*

$$\begin{aligned} b(t) &:= \frac{c}{U(\infty)} \frac{-\gamma}{1-\gamma} t^\gamma \\ &+ \frac{U(\infty)}{c} \frac{\gamma(1-\gamma)(1-2\gamma)\rho(1+\rho)}{\{1-\gamma(1+\rho)\}\{1-\gamma(2+\rho)\}} \cdot [t^{-\gamma}\{\log U(\infty) - \log U(t)\} - \frac{c}{U(\infty)}] \, . \end{aligned} \tag{10}$$

*Determine $k_o = k_o(n)$ such that the asymptotic second moment of $\hat{\gamma}_n - \gamma$ is minimal and let $\hat{\gamma}_{n,o}$ be the corresponding estimator, then*

$$\sqrt{k_o(n)} \, (\hat{\gamma}_{n,o} - \gamma) \xrightarrow{d} N(b, \sigma^2(\gamma)) \, ,$$

*where the asymptotic bias $b$ and variance $\sigma^2(\gamma)$ are given by*

$$b = \text{sign}(b(t)) \sqrt{\frac{\sigma^2(\gamma)}{-2\gamma \min(1,\rho)}}$$

*and*

$$\sigma^2(\gamma) := (1-\gamma)^2(1-2\gamma)\left(4 - 8\frac{1-2\gamma}{1-3\gamma} + \frac{(5-11\gamma)(1-2\gamma)}{(1-3\gamma)(1-4\gamma)}\right) . \tag{11}$$

*Moreover* $k_o(n) = n/s^{\leftarrow}(1/n)(1+o(1)) \in RV_{\frac{2\gamma\min(1,\rho)}{2\gamma\min(1,\rho)-1}}$, $n \to \infty$, *where* $s^{\leftarrow}$ *is the inverse function of* $s$, *with* $s$ *given by*

$$\frac{\{b(t)\}^2}{\sigma^2(\gamma)} = \int_t^\infty s(u)du\ (1+o(1)) \ , \ t \to \infty .$$

*The existence of such function* $s$ *is guaranteed by the fact that* $b^2(t) \in RV_{2\gamma\min(1,\rho)}$.

PROOF

Assume $\gamma < 0$ and $(d.)$ of lemma 2.3 holds. Define $c_1 := c/U(\infty)$ and $a(t) := t^{-\gamma}\{\log U(\infty) - \log U(t)\} - c_1$ then, since $|a(t)| \in RV_{\gamma\rho}$, for $x > 0$

$$\begin{aligned}
\log U(tx) - \log U(t) &= \log U(\infty) - \log U(t) - \{\log U(\infty) - \log U(tx)\} \\
&= t^\gamma \left[t^{-\gamma}\{\log U(\infty) - \log U(t)\} - x^\gamma (tx)^{-\gamma}\{\log U(\infty) - \log U(tx)\}\right] \\
&= c_1 t^\gamma (1-x^\gamma) + t^\gamma a(t)\{1 - x^\gamma \frac{a(tx)}{a(t)}\} \\
&= c_1 t^\gamma (1-x^\gamma) + t^\gamma a(t)\{1 - x^\gamma x^{\gamma\rho}\} + o(t^\gamma a(t)) \ , \ t \to \infty .
\end{aligned}$$

Also

$$\frac{(Y_{(n-k,n)})^\gamma a(Y_{(n-k,n)})}{\frac{n}{k}^\gamma a(\frac{n}{k})} \to 1 \ , \ n \to \infty ,$$

in probability by lemma 3.1 [notation : $(Y_{(n-k,n)})^\gamma a(Y_{(n-k,n)}) = (\frac{n}{k})^\gamma a(\frac{n}{k})(1+o_p(1)]$. Now one obtains by straightforward calculations using lemma 3.1

$$\begin{aligned}
M_n^{(1)} &= \frac{1}{k}\sum_{i=0}^{k-1} \log X_{(n-i,n)} - \log X_{(n-k,n)} \stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1} \log U(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}} Y_{(n-k,n)}) - \log U(Y_{(n-k,n)}) \\
&= (Y_{(n-k,n)})^\gamma \frac{1}{k}\sum_{i=0}^{k-1}\left[c_1\left\{1 - \left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^\gamma\right\} + a(Y_{(n-k,n)})\left\{1 - \left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{\gamma(1+\rho)}\right\}\right] \\
&\quad + o_p((\tfrac{n}{k})^\gamma a(\tfrac{n}{k})) \\
&\stackrel{d}{=} (Y_{(n-k,n)})^\gamma \left[\frac{-\gamma c_1}{1-\gamma} + c_1 \frac{P_n}{\sqrt{k}} + d_1 a(Y_{(n-k,n)})\right] + o_p((\tfrac{n}{k})^\gamma a(\tfrac{n}{k})) \ ,
\end{aligned} \tag{12}$$

where $d_1 := \int_1^\infty (1 - x^{\gamma(1+\rho)})\frac{dx}{x^2} = \frac{-\gamma(1+\rho)}{1-\gamma(1+\rho)}$ and hence

$$\begin{aligned}
\{M_n^{(1)}\}^2 &\stackrel{d}{=} (Y_{(n-k,n)})^{2\gamma}\left[\frac{\gamma^2 c_1^2}{(1-\gamma)^2} - \frac{2\gamma c_1^2}{1-\gamma}\frac{P_n}{\sqrt{k}} - \frac{2\gamma c_1}{1-\gamma} \cdot d_1 \cdot a(Y_{(n-k,n)})\right] \\
&\quad + o_p((\tfrac{n}{k})^{2\gamma} a(\tfrac{n}{k})) .
\end{aligned} \tag{13}$$

Furthermore, since

$$\begin{aligned}
\{\log U(tx) - \log U(t)\}^2 &= t^{2\gamma}[c_1(1-x^\gamma) + a(t)(1-x^{\gamma(1+\rho)}) + o(a(t))]^2 \\
&= t^{2\gamma}[c_1^2(1-x^\gamma)^2 + 2c_1 a(t)(1-x^\gamma)(1-x^{\gamma(1+\rho)})] + o(t^{2\gamma}a(t)) \ , \ t \to \infty ,
\end{aligned}$$

one obtains

$$M_n^{(2)} \stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1}\left\{\log U(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}} Y_{(n-k,n)}) - \log U(Y_{(n-k,n)})\right\}^2$$

$$
\begin{aligned}
&= (Y_{(n-k,n)})^{2\gamma}\frac{1}{k}\sum_{i=0}^{k-1}\left[c_1^2\left\{1-\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{\gamma}\right\}^2\right.\\
&\quad + \left. 2c_1\left\{1-\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{\gamma}\right\}\left\{1-\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{\gamma(1+\rho)}\right\}a(Y_{(n-k,n)})\right] + o_p((\tfrac{n}{k})^{2\gamma}a(\tfrac{n}{k}))\\
&\stackrel{d}{=} (Y_{(n-k,n)})^{2\gamma}\left[c_1^2\frac{2\gamma^2}{(1-\gamma)(1-2\gamma)}+c_1^2\frac{Q_n}{\sqrt{k}}+d_2a(Y_{(n-k,n)})\right] + o_p((\tfrac{n}{k})^{2\gamma}a(\tfrac{n}{k}))\ , \qquad (14)
\end{aligned}
$$

with $d_2 := 2c_1\int_1^\infty(1-x^\gamma-x^{\gamma(1+\rho)}+x^{\gamma(2+\rho)})\frac{dx}{x^2} = \frac{2c_1\gamma^2(1+\rho)(2-\gamma(2+\rho))}{(1-\gamma)\{1-\gamma(1+\rho)\}\{1-\gamma(2+\rho)\}}$. Combining finally (12), (13) and (14):

$$
\begin{aligned}
\hat{\gamma}_n &= M_n^{(1)}+\frac{1}{2}\frac{M_n^{(2)}-2\{M_n^{(1)}\}^2}{M_n^{(2)}-\{M_n^{(1)}\}^2}\\
&\stackrel{d}{=} \frac{-\gamma c_1}{1-\gamma}(Y_{(n-k,n)})^\gamma + \frac{\gamma+\frac{(1-\gamma)^2(1-2\gamma)}{\gamma^2}\left(\frac{Q_n}{2\sqrt{k}}+\frac{2\gamma}{1-\gamma}\frac{P_n}{\sqrt{k}}+a(Y_{(n-k,n)})\left[\frac{d_2}{2c_1^2}+\frac{2\gamma d_1}{c_1(1-\gamma)}\right]\right)}{1+\frac{(1-\gamma)^2(1-2\gamma)}{\gamma^2}\left(\frac{Q_n}{\sqrt{k}}+\frac{2\gamma}{1-\gamma}\frac{P_n}{\sqrt{k}}+a(Y_{(n-k,n)})\left[\frac{d_2}{c_1^2}+\frac{2\gamma d_1}{c_1(1-\gamma)}\right]\right)}\\
&\quad + o_p(a(\tfrac{n}{k}))\\
&= \frac{-\gamma c_1}{1-\gamma}(Y_{(n-k,n)})^\gamma+\gamma+\frac{(1-\gamma)^2(1-2\gamma)}{\gamma^2}\left[(\tfrac{1}{2}-\gamma)\frac{Q_n}{\sqrt{k}}+(\frac{2\gamma}{1-\gamma}-\frac{2\gamma^2}{1-\gamma})\frac{P_n}{\sqrt{k}}\right.\\
&\quad + \left. a(Y_{(n-k,n)})\{\frac{d_2}{c_1^2}(\tfrac{1}{2}-\gamma)+\frac{2\gamma d_1}{c_1}\}\right]+o_p(a(\tfrac{n}{k}))\\
&= \gamma+\frac{R_n}{\sqrt{k}}+b(\tfrac{n}{k})+o_p(b(\tfrac{n}{k}))\ ,
\end{aligned}
$$

with $b(t)$ as defined in (10) and $R_n := \frac{1}{2}\frac{(1-\gamma)^2(1-2\gamma)^2}{\gamma^2}Q_n+\frac{2(1-\gamma)^2(1-2\gamma)}{\gamma}P_n$, which is asymptotically normal with mean zero and variance $\sigma^2(\gamma)$ as defined in (11). Hence the asymptotic mean squared error of $\hat{\gamma}_n$ equals

$$
\frac{\sigma^2(\gamma)}{k}+\{b(\tfrac{n}{k})+o(b(\tfrac{n}{k}))\}^2\ .
$$

Write $r := n/k$. We are interested in the optimalization problem

$$
\inf_r\left\{\frac{r}{n}+\frac{\{b(r)\}^2}{\sigma^2(\gamma)}+o(\{b(r)\}^2)\right\}\sim\inf_r\left\{\frac{r}{n}+\frac{\{b(r)\}^2}{\sigma^2(\gamma)}\right\}\ . \qquad (15)
$$

The asymptotic equality in (15) follows from lemma 2.9. Define $f(t) := \{b(t)\}^2/\sigma^2(\gamma)$ then $f\in RV_{2\gamma\rho_1}$ with $\rho_1 := \min(1,\rho)$, since $|b(t)|\in RV_{\gamma\min(1,\rho)}$, and so by lemma 2.9 there exists a positive function $s\in RV_{2\gamma\rho_1-1}$ such that

$$
\frac{\{b(t)\}^2}{\sigma^2(\gamma)}=\int_t^\infty s(u)du\ (1+o(1))\ ,\ t\to\infty. \qquad (16)
$$

Let $r_o$ denote the optimal value for $r$ in (15), then [again by lemma 2.9] $r_o(n)=s^{\leftarrow}(1/n)(1+o(1))$ , $n\to\infty$ where $s^{\leftarrow}(1/n)\in RV_{1/(1-2\gamma\rho_1)}$ and hence $k_o(n)=n/s^{\leftarrow}(1/n)(1+o(1))\in RV_{\frac{2\gamma\rho_1}{2\gamma\rho_1-1}}$. Note that $r_o\to\infty$ $(n\to\infty)$ and substitution of $t=n/k_o(n)$ in (16) gives [all the $o$-terms are regularly varying with index $2\gamma\rho_1$]

$$
\frac{\{b(\frac{n}{k_o(n)})\}^2}{\sigma^2(\gamma)} = \int_{r_o}^\infty s(u)du\cdot(1+o(1))
$$

$$
\begin{aligned}
&= \frac{1}{k_o} \cdot \frac{\int_{r_o}^{\infty} s(u)du}{r_o s(r_o)} \cdot (1+o(1)) \\
&= \frac{1}{k_o} \cdot \frac{1}{-2\gamma\rho_1} \cdot (1+o(1)) \ , \ n \to \infty \ ,
\end{aligned}
$$

since $s \in RV_{2\gamma\rho_1 - 1}$ [cf. theorem 1.4 Geluk and de Haan (1987)] and hence

$$
b(\tfrac{n}{k_o}) = \frac{\operatorname{sign}(b(t))}{\sqrt{k_o}} \cdot \sqrt{\frac{\sigma^2(\gamma)}{-2\gamma \min(1,\rho)}} \cdot (1+o(1)) \ , \ n \to \infty \ .
$$

This completes the proof.

Remark 3.3

The above theorem holds also for $\rho = 1$ under the extra condition $|b(t)| \in RV_\gamma$. This condition is not necessarily satisfied because in spite of the fact that both terms of $b(t)$ in (10) are regularly varying with index $\gamma$, they may not have the same sign. In this case the theorem holds also but now with bias $b$ equal to $b = \operatorname{sign}(b(t))\sqrt{\sigma^2(\gamma)/(-2\gamma\rho)}$, where $\rho$ is the index of $b(t)$. The uniform distribution is an example : then $\rho = 1$ and $b(t)$ is regularly varying but with index 2.

Theorem 3.4

*Suppose $\gamma > 0$, condition (g.) of lemma 2.3 holds for $\rho \neq 1/(1-\gamma)$ and define*

$$
b(t) := \frac{\gamma\rho[(1-\gamma)\rho - 1]}{(1+\gamma\rho)^2}\{\log U(t) - \gamma \log t - \log c\}.
$$

*Determine $k_o = k_o(n)$ such that the asymptotic second moment of $\hat{\gamma}_n - \gamma$ is minimal and let $\hat{\gamma}_{n,o}$ be the corresponding estimator, then*

$$
\sqrt{k_o}(\hat{\gamma}_{n,o} - \gamma) \xrightarrow{d} N(b, 1+\gamma^2)
$$

*where $b$ denotes the bias given by*

$$
b = \operatorname{sign}(b(t))\sqrt{\frac{1+\gamma^2}{2\gamma\rho}} \ .
$$

*Moreover $k_o(n) = n/s^{\leftarrow}(1/n)(1+o(1))$, $n \to \infty$, where $s^{\leftarrow}$ is the inverse function of $s$, with $s$ given by*

$$
\frac{\{b(t)\}^2}{1+\gamma^2} = \int_t^{\infty} s(u)du \cdot (1+o(1)) \ , \ t \to \infty
$$

*and furthermore $k_o(n) \in RV_{\frac{2\gamma\rho}{2\gamma\rho+1}}$.*

Proof

Suppose $\gamma > 0$ and suppose that condition (g.) van lemma 2.3 holds. Define $a(t) := \log U(t) - \gamma \log t - \log c$. Since $|a(t)| \in RV_{-\gamma\rho}$, for $x > 0$,

$$
\begin{aligned}
\log U(tx) - \log U(t) &= \log U(tx) - \gamma \log tx - \log c - \{\log U(t) - \gamma \log t - \log c\} + \gamma \log x \\
&= \gamma \log x + \left(x^{-\gamma\rho} - 1\right) \ a(t) \ (1+o(1)) \ , \ t \to \infty \ .
\end{aligned}
$$

One obtains in a similar way as before

$$
\begin{aligned}
M_n^{(1)} &\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1} \log U(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}} Y_{(n-k,n)}) - \log U(Y_{(n-k,n)}) \\
&= \gamma + \frac{1}{k}\sum_{i=1}^{k-1}\left[\gamma\left(\log\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}} - 1\right) + a(Y_{(n-k,n)})\left\{\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{-\gamma\rho} - 1\right\}\right] + o_p(a(\tfrac{n}{k})) \\
&= \gamma \ + \ \gamma\frac{P_n^o}{\sqrt{k}} \ + \ d_1 a(Y_{(n-k,n)}) \ + \ o_p(a(\tfrac{n}{k})) \ , \qquad (17)
\end{aligned}
$$

by lemma 3.1, with $d_1 := \int_1^\infty (x^{-\gamma\rho} - 1)\frac{dx}{x^2} = -\gamma\rho/(1+\gamma\rho)$ [cf. proof of lemma 3.4 Dekkers, Einmahl and de Haan (1989)] and hence

$$\left(M_n^{(1)}\right)^2 = \gamma^2 \ + \ 2\gamma^2\frac{P_n^o}{\sqrt{k}} \ + \ 2\gamma d_1 a(Y_{(n-k,n)}) \ + \ o_p(a(\tfrac{n}{k})). \tag{18}$$

Furthermore $\{\log U(tx) - \log U(t)\}^2 = \{\gamma \log x\}^2 + 2\gamma(x^{-\gamma\rho} - 1)(\log x)a(t) + o(a(t))$ , $t \to \infty$ and hence

$$\begin{aligned} M_n^{(2)} &\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1}\left[\gamma^2\left\{\log\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right\}^2 + 2\gamma\left\{\left(\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}\right)^{-\gamma\rho} - 1\right)\right\}\log\frac{Y_{(n-i,n)}}{Y_{(n-k,n)}}a(Y_{(n-k,n)})\right] \\ &\quad + o_p(a(\tfrac{n}{k})) \\ &\stackrel{d}{=} 2\gamma^2 \ + \ \gamma^2\frac{Q_n^o}{\sqrt{k}} \ + \ d_2 a(Y_{(n-k,n)}) \ + \ o_p(a(\tfrac{n}{k}))\ , \end{aligned} \tag{19}$$

where $d_2 := 2\gamma\int_1^\infty (x^{-\gamma\rho}-1)\log x\frac{dx}{x^2} = -2\gamma^2\rho(2+\gamma\rho)(1+\gamma\rho)^{-2}$ and where $(P_n^o, Q_n^o)$ are asymptotically normal distributed as in lemma 3.1. By combining (17), (18) and (19) one obtains

$$\begin{aligned} \hat{\gamma}_n &= \gamma + \gamma\frac{P_n^o}{\sqrt{k}} + d_1 a(Y_{(n-k,n)}) \\ &\quad + \frac{2\gamma^2 + \gamma^2\frac{Q_n^o}{\sqrt{k}} + d_2 a(Y_{(n-k,n)}) - 2\gamma^2 - 4\gamma^2\frac{P_n^o}{\sqrt{k}} - 4\gamma d_1 a(Y_{(n-k,n)})}{2\gamma^2[2 + \frac{Q_n^o}{\sqrt{k}} + \frac{d_2}{\gamma^2}a(Y_{(n-k,n)}) - 1 - 2\frac{P_n^o}{\sqrt{k}} - \frac{2d_1}{\gamma}a(Y_{(n-k,n)})]} + o_p(a(\tfrac{n}{k})) \\ &= \gamma + \frac{Q_n^o}{2\sqrt{k}} + (\gamma-2)\frac{P_n^o}{\sqrt{k}} + \left(\frac{d_2}{2\gamma^2} + \frac{\gamma-2}{\gamma}d_1\right)a(Y_{(n-k,n)}) + o_p(a(\tfrac{n}{k})) \\ &= \gamma \ + \ \frac{R_n^o}{\sqrt{k}} \ + \ b(\tfrac{n}{k}) \ + \ o_p(b(\tfrac{n}{k}))\ , \end{aligned}$$

with $R_n^o$ asymptotically normal with mean zero and variance $1+\gamma^2$. The rest of the proof is omitted since it follows the same line as the previous one.

REMARK 3.5
In case $\rho = 1/(1-\gamma)$ in theorem 3.4, there may exist a function, say $b^*$, which gives some bias. In this case we have to assume some "third order" condition on $U$.

In the next theorem the case of second order $\Pi$-variation is considered. The conditions and the proofs are slightly different for all the three cases $\gamma < 0$ , $\gamma = 0$ and $\gamma > 0$.

THEOREM 3.6
*Suppose one of the following second order $\Pi$-variation conditions of lemma 2.5 holds: (d.) $[\gamma < 0]$, (f.) $[\gamma = 0]$ or (i.) $[\ \gamma > 0]$. Define the function $a$ as follows*

$$a(t) := \begin{cases} b_1(t)/[t^{-\gamma}\{\log U(\infty) - \log U(t)\}]\ , & \gamma < 0 \\ b_2(t) \ - \ b_3(t)/b_2(t)\ , & \gamma = 0 \\ b_4(t)/\{\log U(t) - \gamma\log t\}\ , & \gamma > 0\ , \end{cases}$$

*and assume that $a^2$ is asymptotic to a non-increasing function and $b_2$ and $b_3/b_2$ are not of the same order. Determine $k_o$ such that the asymptotic second moment of $\hat{\gamma}_n - \gamma$ is minimal and let $\hat{\gamma}_{n,o}$ be the corresponding estimator. Then for $\gamma \in \mathbb{R}$*

$$\sqrt{k_o}(\hat{\gamma}_{n,o} - \gamma) - b_n \xrightarrow{d} N(0, \sigma^2(\gamma))\ , \tag{20}$$

*with variance*

$$\sigma^2(\gamma) := \begin{cases} 1+\gamma^2 \ , & \gamma \geq 0 \\ (1-\gamma)^2(1-2\gamma)\left(4-8\frac{1-2\gamma}{1-3\gamma}+\frac{(5-11\gamma)(1-2\gamma)}{(1-3\gamma)(1-4\gamma)}\right) & \gamma \leq 0 \end{cases}$$

*and where $b_n$ denotes the bias which is a slowly varying sequence and tends to infinity for $n \to \infty$. Moreover $k_o$ is a slowly varying sequence.*

REMARK 3.7
Note that (20) implies $\sqrt{k_o}(\hat{\gamma}_{n,o}-\gamma)/b_n \to 1$ , $n \to \infty$ in probability. Hence the optimal rate of convergence of $\hat{\gamma}_n \to \gamma$ is given by $b_n/\sqrt{k_o}$.

REMARK 3.8 In case $b_3(t) = [b_2(t)]^2\ (1+o(1))$ , $t \to \infty$ we are in the same situation as in theorem 3.2 with $\rho = 1$. An example is the exponential distribution and the Gumbel distribution [cf. section 4]. In this case one has to consider the asymptotic expansion of $a(t)$ and the proof of the theorem to obtain an expression for the bias. order than given in the theorem.

PROOF
For $\gamma < 0$ we give the proof for the plus sign in ($d.$) of lemma 2.5. The condition implies

$$\log U(tx) - \log U(t) = \{\log U(\infty) - \log U(t)\}[1 - x^{\gamma} - (x^{\gamma}\log x)a(t)\ (1+o(1))] \ , \ t \to \infty \ ,$$

where $|a| \in RV_0$ and $a(t) \to 0$ , $t \to \infty$. Now one obtains

$$\begin{aligned} \hat{\gamma}_n &= \gamma + \frac{R_n}{\sqrt{k}} + \frac{-\gamma}{1-\gamma}\{\log U(\infty) - \log U(Y_{(n-k,n)})\} + a(Y_{(n-k,n)}) \ + \ o_p(a(\tfrac{n}{k})) \\ &= \gamma \ + \ \frac{R_n}{\sqrt{k}} \ + \ a(\tfrac{n}{k}) \ + \ o_p(a(\tfrac{n}{k})) \ , \end{aligned}$$

where $R_n$ is asymptotically normal with mean zero and variance $\sigma^2(\gamma)$. The last approximation is valid since $\log U(\infty) - \log U(Y_{(n-k,n)})$ is of lower order than $a$, $|a| \in RV_0$ and $Y_{(n-k,n)}/(\frac{n}{k}) \to 1$ in probability. The mean squared error of $\hat{\gamma}_n - \gamma$ equals

$$\frac{\sigma^2(\gamma)}{k} \ + \{a(\tfrac{n}{k})\}^2\ (1+o(1)) \ , \ n \to \infty \ .$$

Write $r := n/k$. We are interested in the optimalization problem

$$\inf_r \left\{\frac{r}{n} + \frac{\{a(r)\}^2}{\sigma^2(\gamma)}\ (1+o(1))\right\} \sim \inf_r \left\{\frac{r}{n} + \frac{\{a(r)\}^2}{\sigma^2(\gamma)}\right\} \ , \tag{21}$$

with $a^2(t) \to 0$ , $t \to \infty$. Hence the asymptotic equality in (21) follows from lemma 2.9 and by the same lemma there exists a positive function $s \in RV_{-1}$, such that

$$\frac{\{a(t)\}^2}{\sigma^2(\gamma)} = \int_t^{\infty} s(u)d(u)\cdot(1+o(1)) \ , \ t \to \infty \ . \tag{22}$$

Let $r_o$ denote the optimal value for $r$ in (21), then [again by lemma 2.9] $r_o(n) = s^{\leftarrow}(1/n)(1+o(1))$ , $n \to \infty$ where $s^{\leftarrow} \in RV_{-1}$. Note that $r_o \to \infty$ ($n \to \infty$) and $k_o(n) = n/s^{\leftarrow}(1/n)(1+o(1)) \in RV_0$ substitution of $t = n/k_o$ in (22) gives

$$\begin{aligned} \frac{\{a(\frac{n}{k_o})\}^2}{\sigma^2(\gamma)} &= \int_{\frac{n}{k_o}}^{\infty} s(u)du\cdot(1+o(1)) \\ &= \frac{1}{k_o}\cdot\frac{\int_{s^{\leftarrow}(1/n)}^{\infty} s(u)du}{s^{\leftarrow}(1/n)/n}\cdot(1+o(1)) \ , \ n \to \infty \ . \end{aligned} \tag{23}$$

The fraction in (23) tends to infinity [cf. Geluk and de Haan (1987), remark 1 following corollary 1.18]. Hence the asymptotic bias of $\sqrt{k_o}(\hat{\gamma}_{n,o}-\gamma)$ equals

$$b_n \;=\; \operatorname{sign}(a(t)) \left( \frac{\sigma^2(\gamma)\ \int_{s^{\leftarrow}(1/n)}^{\infty} s(u)du}{s^{\leftarrow}(1/n)/n} \right)^{1/2} \quad (1+o(1))\ ,\ n\to\infty\ .$$

where $|b_n|$ is slowly varying and tends to infinity for $n\to\infty$.

For $\gamma=0$ condition $(f.)$ of lemma 2.5 implies for $x>1$

$$\log U(tx)-\log U(t) \;=\; b_2(t)\ [\ \log x - \tfrac{1}{2}(\log x)^2\ [b_3(t)/b_2(t)]\ (1+o(1))\ ]\ ,\ t\to\infty$$

and hence

$$\begin{aligned}\hat{\gamma}_n &= b_2(Y_{(n-k,n)}) - 2\frac{P_n^o}{\sqrt{k}} + \frac{Q_n^o}{2\sqrt{k}} - \ b_3(Y_{(n-k,n)})/b_2(Y_{(n-k,n)}) + o_p(a(\tfrac{n}{k}))\\ &= \frac{R_n}{\sqrt{k}} + a(\tfrac{n}{k}) + o_p(a(\tfrac{n}{k}))\ ,\end{aligned}$$

where $R_n$ is asymptotically standard normal and $b_3(t)=[b_2(t)]^2(1+o(1))\ ,\ t\to\infty$ is excluded. The rest of the proof is as before and is therefore omitted.

For $\gamma>0$ we give the proof with a plus sign in condition $(i.)$ of lemma 2.5 and hence

$$\log U(tx)-\log U(t)=\gamma\log x + a(t)\log x\ (1+o(1))\ ,\ t\to\infty\ .$$

Similar calculations as before give

$$\hat{\gamma}_n=\gamma+\frac{R_n}{\sqrt{k}}+a(\tfrac{n}{k})\ (1+o_p(1))\ .$$

The rest of the proof is omitted since it follows the same line as the part for $\gamma<0$.

# 4 Examples

In this section we discuss the above results applied to some distribution functions.

## *4.1 Uniform distribution*

The uniform distribution does not satisfy condition $(b.)$ of lemma 2.3 since $U(t)=1-1/t\ ,\ t\to\infty$. But the uniform distribution function satisfies condition $(d.)$ of lemma 2.3 with $\gamma=-1\ ,\ \rho=1\ ,\ U(\infty)=c=1$ and hence $t^{-\gamma}\{\log U(\infty)-\log U(t)\}-c/U(\infty)=t\{-\log(1-1/t)\}-1$, which leads to $b_3(t)=1/(2t)-[1/(2t)+1/(3t^2)(1+o(1))]\in RV_{-2}$. So $b(t)=-1/(3t^2)(1+o(1))\ ,\ t\to\infty$. The asymptotic bias of $\hat{\gamma}_{n,o}-\gamma$ is equal to $-\sqrt{6/5}$ and moreover $k_o(n)=(27/10)^{1/5}\cdot n^{4/5}(1+o(1))\ ,\ n\to\infty$.

## *4.2 Cauchy distribution*

Define $F(x):=\frac{1}{2}+\frac{1}{\pi}\arctan x\ ,\ x\in\mathbb{R}$, the Cauchy distribution function. Then $U(t)=\tan(\frac{\pi}{2}-\frac{\pi}{t})=\frac{t}{\pi}\{1-\frac{\pi^2}{3t^2}+o(t^{-3})\}\ ,\ t\to\infty$. The Cauchy distribution satisfies the condition of theorem 3.4 with $\gamma=1\ ,\ c=1/\pi$ and $\rho=2$. The bias $b$ of $\sqrt{k_o}(\hat{\gamma}_{n,o}-\gamma)$ equals $\frac{1}{2}\sqrt{2}$ and $k_o(n)\in RV_{4/5}$, or more precisely

$$b_3(t)=\frac{-2}{9}\log(\pi t^{-1}U(t))=\frac{2\pi^2}{27}t^{-2}+o(t^{-3})\ ,\ t\to\infty$$

and hence $s(t)=2^3\cdot3^{-6}\cdot\pi^4\cdot t^{-5}+o(t^{-5})\ ,\ t\to\infty$. One obtains $s^{\leftarrow}(t)=2^{3/5}\cdot3^{-6/5}\cdot\pi^{4/5}\cdot t^{-1/5}\cdot(1+o(1))\ ,\ t\to\infty$ and finally $k_o(n)=2^{-3/5}\cdot3^{6/5}\cdot(n/\pi)^{4/5}\cdot(1+o(1))\ ,\ n\to\infty$.

### 4.3 *Exponential distribution*

The exponential distribution satisfies condition (*f.*) of lemma 2.5 with $U(t) = \log t$ , $b_2(t) = 1/(\log t)$ and $b_3(t) = 1/(\log t)^2$. Since $b_2$ equals $b_3/b_2$ theorem 3.5 can not be used directly. For the exponential distribution holds

$$\begin{aligned}
\log U(tx) - \log U(t) &= \log(1 + \frac{\log x}{\log t}) \\
&= \frac{\log x}{\log t} - \tfrac{1}{2}(\frac{\log x}{\log t})^2 + \tfrac{1}{3}(\frac{\log x}{\log t})^3 \; (1 + o(1)) \; , \; t \to \infty \; , \qquad (24)
\end{aligned}$$

and hence [cf. proof of theorem 3.5 for $\gamma = 0$] $\hat{\gamma}_n = R_n/\sqrt{k} + a(\frac{n}{k}) + o_p(a_2(\frac{n}{k}))$ with $a(t) = 1/(\log t) - [1 - 2/(3\log t)]/(\log t) + o(1/(\log t)^2) = \frac{2}{3}(\log t)^{-2}\; (1 + o(1))$ , $t \to \infty$.

Moreover $a^2(t) = \int_t^\infty s(u)du$ with $s(u) = 16/(9u(\log u)^5) =: t$. And hence for the inverse function of $s$

$$\frac{16/9}{u(-\log u)^5} < s^{\leftarrow}(u) < \frac{16/9}{u(-\log u - 5\log(-\log u))^5} \; , \; u \to 0 \; ,$$

since $\log t > -\log s - 5 \log\log s$ and $\log t < -\log s - 5\log(-\log s - 5\log(-\log s))$ for $t$ large. For the bias $b_n$ of $\sqrt{k_o}\; (\hat{\gamma}_{n,o} - \gamma)$ holds

$$\begin{aligned}
\{b_n\}^2 &= \frac{\int_{s^{\leftarrow}(1/n)}^{\infty} s(u)du}{s^{\leftarrow}(1/n)/n}(1 + o(1)) \\
&= 1 + \frac{\int_0^{1/n} s^{\leftarrow}(u)du}{s^{\leftarrow}(1/n)/n}(1 + o(1)) \\
&> 1 + (\log n)/8 \; ,
\end{aligned}$$

hence $b_n > \sqrt{1 + (\log n)/8}$ for $n \to \infty$.

### 4.4 *Generalized Extreme-Value distribution*

Let $G_\gamma$ denote the GEV-distribution as defined in (2), then $U(t) = \frac{1}{\gamma}\{[-\log(1 - t^{-1})]^{-\gamma} - 1\}$.

For $\gamma < 0$ holds $U(\infty) = \frac{1}{-\gamma} > 0$ and $t^{-\gamma}[\log U(\infty) - \log U(t)] - c/U(\infty) = \frac{1}{2}[-\gamma t^{-1} + t^{\gamma}](1 + o(1))$ , $t \to \infty$, hence $U$ satisfies the condition of theorem 3.2 with $c = 1/(-\gamma)$ and $\rho = \min(1, 1/(-\gamma)$. The bias $b$ of $\sqrt{k_o}(\hat{\gamma}_{n,o} - \gamma)$ equals $-\sqrt{\sigma^2(\gamma)/2}$ for $\gamma \le -1$, and $\sqrt{\sigma^2(\gamma)/(-2\gamma)}$ for $-1 < \gamma < 0$. The optimal value $k_o(n)$ is for $n \to \infty$,

$$k_o(n) = \begin{cases}
\left[\dfrac{(1-\gamma)^2(1-2\gamma)^2}{8(2-\gamma)^2\sigma^2(\gamma)}\right]^{-1/3} n^{2/3}\; (1 + o(1)) & \gamma < -1 \\
\left[2\sigma^2(-1)\right]^{1/3}\; n^{2/3}\; (1 + o(1)) & \gamma = -1 \\
\left[\dfrac{-2\gamma^5(1+\gamma)^2}{(1-\gamma)^2(1-3\gamma)^2\sigma^2(\gamma)}\right]^{-1/(1-2\gamma)} n^{-2\gamma/(1-2\gamma)}\; (1 + o(1)) & -1 < \gamma < 0 \; .
\end{cases}$$

For $\gamma = 0$ $U(t) = -\log(-\log(1 - 1/t)) = \log t - 1/(2t) + o(1/t)$ , $t \to \infty$, hence $\log U(tx) - \log U(t)$ equals asymptotically the right-hand side of (24). So we are in the same situation as in the example of the exponential distribution.

For $\gamma > 0$, $\log(t^{-\gamma}U(t)/c) = -\gamma t^{-\gamma}/2 - t^{\gamma} + o(t^{-2} + t^{-2\gamma})$ , $t \to \infty$, which satisfies the condition of theorem 3.4 with $c = 1/\gamma$ and $\rho = \min(1, 1/\gamma)$. The bias $b$ of $\sqrt{k_o}(\hat{\gamma}_{n,o} - \gamma)$ equals $\sqrt{(1 + \gamma^2)/(2\gamma)}$ for $0 < \gamma \le 1$ and $\sqrt{(1 + \gamma^2)/2}$ for $\gamma > 1$. Finally one obtains for the optimal value $k_o(n), n \to \infty$

$$k_o(n) = \begin{cases}
\left[\dfrac{(1+\gamma)^4(1+\gamma^2)}{2\gamma^5}\right]^{1/(1+2\gamma)} n^{2\gamma/(1+2\gamma)}\; (1 + o(1)) & 0 < \gamma < 1 \\
[64/9]^{1/3}\; n^{2/3}\; (1 + o(1)) & \gamma = 1 \\
\left[8(1+\gamma^2)(2\gamma - 1)^{-2}\right]^{1/3}\; n^{2/3}\; (1 + o(1)) & \gamma > 1 \; .
\end{cases}$$

## APPENDIX A

In this appendix we want to convince the reader that the conditions in lemma 2.3 and in lemma 2.5 cover all possible second order tail-behaviour of distribution functions. First we formulate relation (4) in term of $\log U$, since the conditions used in section 3 are all formulated in terms of $\log U$.

Define $V(t) := \log U(t)$ and $\gamma_1 := \min(0, \gamma)$. A distribution function $F$ is in the domain of attraction of one of the GEV-distributions if and only if there exists a positive function $a_1$, with $\lim_{t\to\infty} a_1(t) = \gamma_1$, such that

$$\lim_{t\to\infty} \frac{V(tx) - V(t)}{a_1(t)} = \frac{x^{\gamma_1} - 1}{\gamma_1} , \tag{25}$$

where the right hand side of (25) is $\log x$ for $\gamma_1 = 0$.

Now we want to derive second order conditions on $V$. Suppose (25) holds and suppose there exists a function $a$, with $a(t) \to 0$ , $t \to \infty$ such that for all $x > 0$ the limit

$$\lim_{t\to\infty} \frac{\frac{V(tx)-V(t)}{a_1(t)} - \frac{x^{\gamma_1}-1}{\gamma_1}}{a(t)}$$

exists and the limit function is not constant. So we consider for $x > 0$

$$\lim_{t\to\infty} \frac{V(tx) - V(t) - a_1(t)\frac{x^{\gamma_1}-1}{\gamma_1}}{a_2(t)} = H(x) , \tag{26}$$

where $a_2(t) := a_1(t)a(t)$ and the question is what do we know about $H$.

Let $x > 1$ and $y > 1$, then

$$\begin{aligned}
\frac{V(txy) - V(t) - a_1(t)\frac{(xy)^{\gamma_1}-1}{\gamma_1}}{a_2(t)} &= \frac{V(txy) - V(tx) - a_1(tx)\frac{y^{\gamma_1}-1}{\gamma_1}}{a_2(tx)} \frac{a_2(tx)}{a_2(t)} \\
&+ \frac{V(tx) - V(t) - a_1(t)\frac{x^{\gamma_1}-1}{\gamma_1}}{a_2(t)} \\
&+ x^{\gamma_1} \cdot \frac{y^{\gamma_1} - 1}{\gamma_1} \cdot \frac{(tx)^{-\gamma_1}a_1(tx) - t^{-\gamma_1}a_1(t)}{t^{-\gamma_1}a_2(t)} .
\end{aligned}$$

Assuming (26), we know that all parts of both sides of the equation converge for $t \to \infty$ except for $a_2(tx)/a_2(t)$ and

$$\frac{(tx)^{-\gamma_1}a_1(tx) - t^{-\gamma_1}a_1(t)}{t^{-\gamma_1}a_2(t)} . \tag{27}$$

We assume that the function $|a_2|$ is regularly varying. Then (27) converges to $c_1(x^\alpha - 1)/\alpha$ for some $\alpha \le 0$ and $c_1 \ne 0$ [cf. theorem 1.9 of Geluk and de Haan (1987)]. It then follows that $a_2 \in RV_{\gamma_1+\alpha}$ [cf theorem 1.10 of Geluk and de Haan (1987)]. So we obtain the following functional equation

$$H(xy) = H(y)x^{\gamma_1+\alpha} + H(x) + c_1 x^{\gamma_1} \frac{y^{\gamma_1} - 1}{\gamma_1} \frac{x^\alpha - 1}{\alpha} . \tag{28}$$

A solution of (28) is

$$H_1(x) := c_1 \int_1^x s^{\gamma_1 - 1} \frac{s^\alpha - 1}{\alpha} ds = \begin{cases} \frac{c_1}{\alpha}\left[\frac{x^{\gamma_1+\alpha} - 1}{\gamma_1 + \alpha} - \frac{x^{\gamma_1} - 1}{\gamma_1}\right] & \alpha \ne 0 \\ \frac{c_1}{\gamma_1}\left[x^{\gamma_1}\log x - \frac{x^{\gamma_1} - 1}{\gamma_1}\right] & \alpha = 0 \ , \ \gamma_1 < 0 \\ c_1 \frac{(\log x)^2}{2} & \alpha = \gamma_1 = 0 \ . \end{cases}$$

Define $G(x) = H(x) - H_1(x)$, then $G$ satisfies the equation

$$G(xy) - G(x) - G(y)x^{\gamma_1+\alpha} = 0 \ . \tag{29}$$

Since $G$ is measurable, for $\gamma_1 = \alpha = 0$ holds $G(x) = d \cdot \log x$, with $d \neq 0$ [cf. the proof of theorem 1.2 of Geluk and de Haan (1987)]. If $\gamma_1 + \alpha \neq 0$, then also by symmetry

$$G(xy) - G(y) - G(x)y^{\gamma_1+\alpha} = 0$$

and hence

$$G(x)(y^{\gamma_1+\alpha} - 1) - G(y)(x^{\gamma_1+\alpha} - 1) = 0 \ ,$$

which implies $G(x) = d \cdot (x^{\gamma_1+\alpha} - 1)$. So the general solution of (28) is [with $c_2 \neq 0$]

$$H(x) = c_1 \int_1^x s^{\gamma_1 - 1}\frac{s^\alpha - 1}{\alpha} ds + c_2 \frac{x^{\gamma_1+\alpha} - 1}{\gamma_1 + \alpha} \ ,$$

which can be divided into three classes:

$$(a.) \qquad \alpha \neq 0 \qquad H(x) = (\frac{c_1}{\alpha} + c_2)\ \frac{x^{\gamma_1+\alpha} - 1}{\gamma_1 + \alpha} \ - \ \frac{c_1}{\alpha}\ \frac{x^{\gamma_1} - 1}{\gamma_1} \tag{30}$$

$$(b.) \quad \alpha = 0 \wedge \gamma_1 < 0 \quad H(x) = \frac{c_1}{\gamma_1}\ x^{\gamma_1}\ \log x \ + \ (c_2 - \frac{c_1}{\gamma_1})\ \frac{x^{\gamma_1} - 1}{\gamma_1} \tag{31}$$

$$(c.) \qquad \alpha = \gamma_1 = 0 \qquad H(x) = c_1\ \frac{(\log x)^2}{2} \ + \ c_2\ \log x \ . \tag{32}$$

In the solutions $c_2$ can be chosen arbitrarily, since it is the solution of (29).

Note that the class (30) for $H$ is the same as the class obtained under the conditions of theorems 3.2 and 3.4. Further the classes (31) and (32) are the same as the classes obtained under the conditions of theorem 3.5. Hence our theorems cover all possible cases stemming from the equation (26).

## Appendix B

In this appendix we give the proof of Lemma 2.3 [Second order regular variation].

$(b.) \Rightarrow (a.)$ : Suppose $\gamma < 0$ and $t^{-\gamma}\{U(\infty) - U(t)\} - c =: H(t)$ for $t$ sufficiently large, with $H \in RV_{\gamma\rho}$. Replacing now $t$ by $\{1 - F(U(\infty) - x^{-1})\}^{-1}$ one obtains $\{1 - F(U(\infty) - x^{-1})\}^{\gamma} x^{-1} - c = H(\{1 - F(U(\infty) - x^{-1})\}^{-1})$ for $x$ sufficiently large, and $H(\{1 - F(U(\infty) - x^{-1})\}^{-1}) \in RV_{-\rho}$ since $U(\infty) - U(t) \in RV_\gamma$ and $U(\infty) - U(\{1 - F(U(\infty) - x^{-1})\}^{-1}) \in RV_{-1}$.

Now one obtains for $t$ sufficiently large

$$\begin{aligned}
&-\{t^{-1/\gamma}[1 - F(U(\infty) - t^{-1})] - c^{-1/\gamma}\} \\
&= \ - \left[ c^{1/\gamma} \left\{ \frac{t^{-1}[1 - F(U(\infty) - t^{-1})]^{\gamma} - c}{c} + 1 \right\}^{1/\gamma} - c^{1/\gamma} \right] \\
&= \ \frac{c^{-1+1/\gamma}}{-\gamma} H(\frac{1}{1 - F(U(\infty) - t^{-1})})(1 + o(1)) \ , \ t \to \infty \ ,
\end{aligned}$$

where the latter term is positive and $\in RV_{-\rho}$.

$(a.) \Rightarrow (b.)$ : This part of the proof follows the same line.

$(b.) \Rightarrow (d.)$ : Note that $(b.)$ is equivalent with $\mp\{t^{-\gamma}[1 - U(t)/U(\infty)] - c/U(\infty)\} \in RV_{\gamma\rho}$, and use $\log x = (x - 1)(1 + o(1))$ , $x \to 1$.

$(c.) \Leftrightarrow (d.)$ : Use the equivalence of $(a.)$ and $(b.)$.

$(f.) \Rightarrow (e.)$ : Suppose $\gamma > 0$ and $t^{-\gamma}U(t) - c =: H(t)$ , $t \to \infty$ , $H$ positive and $H \in RV_{-\gamma\rho}$. Since $U \in RV_\gamma$ , $1/\{1-F\} \in RV_{1/\gamma}$ and, replacing $t$ by $1/\{1-F(x)\}$,

$$\{1-F(x)\}^\gamma x - c = H(\frac{1}{1-F(x)}) \in RV_{-\rho}.$$

Since $x^{1/\gamma}\{1-F(x)\} - c^{1/\gamma} = [x\{1-F(x)\}^\gamma - c + c]^{1/\gamma} - c^{1/\gamma} =$

$$c^{1/\gamma}\left[1 + \frac{x\{1-F(x)\}^\gamma - c}{\gamma c}(1+o(x^{-\gamma}))\right] - c^{1/\gamma} \ , \ x \to \infty \ ,$$

one obtains for $t$ sufficiently large

$$t^{1/\gamma}\{1-F(t)\} - c^{1/\gamma} = \frac{c^{-1+1/\gamma}}{\gamma} H(\frac{1}{1-F(t)})(1+O(t^{-\rho}))$$

with $c^{-1+1/\gamma}\gamma^{-1}H(1/\{1-F(t)\}) \in RV_{-\rho}$.

$(e.) \Rightarrow (f.)$ : This part of the proof is omitted since it follows the same line as the previous part.

$(f.) \Rightarrow (g.)$ : Suppose $t^{-\gamma}U(t) - c \in RV_{-\gamma\rho}$, then also $t^{-\gamma}U(t)/c - 1 \in RV_{-\gamma\rho}$ and hence $t^{-\gamma}U(t)/c \to 1$ , $t \to \infty$. Now $\log(t^{-\gamma}U(t)/c) = (t^{-\gamma}U(t)/c - 1)(1+o(1)) = c^{-1}(t^{-\gamma}U(t) - c)(1+o(1))$ which is regularly varying with index $-\gamma\rho$.

$(g.) \Rightarrow (f.)$ : Follows the same line as $(f.) \Rightarrow (g.)$.

## Appendix C

The following theorem has been communicated to us by A.A. Balkema.

**Theorem C.1**
*Let $U > 0$ vary slowly and be asymptotic to a non-decreasing function. Then $U$ is asymptotic to an element of $\Pi$.*

**Proof**
Write $g(t) = U(e^t)$. Slow variation of $U$ means that $g(t+x)/g(t) \to 1$ uniformly on bounded $x$-intervals for $t \to \infty$. We shall construct a function $f \sim g$ such that $\log f'$ is continuous and piecewise linear, and $(\log f')' \to 0$. This implies that $V(t) := f(\log t)$ lies in $\Pi$. We may assume that $g(t) \to \infty$ for $t \to \infty$, since else $g$ is asymptotic to a function $f(t) = C - 1/t$ , $C > 0$, which satisfies the condition $(\log f')'(t) = 2/t \to 0$. We may also assume that $g$ is strictly increasing and continuous.

For $t \in \mathbb{R}$ and $c > 1$ define $t_c > t$ by $g(t_c) = cg(t)$. Obviously $t_c - t \to \infty$. This implies that there exists a sequence $y_n = g(x_n)$ such that $y_{n+1} \sim y_n \to \infty$ and $y_{n+1} - y_n =: v_n \sim v_{n-1}$ and such that $x_{n+1} - x_n =: u_n \to \infty$. Indeed choose $x_{n+1}$ so that $g(x_{n+1}) = c_n g(x_n)$ with $c_n > 1$ and $c_n \to 1$ so slowly that $x_{n+1} - x_n \to \infty$. We may assume $c_n$ to be weakly decreasing. In addition we may choose $c_n$ of the form $1 + 1/m$ with $m = m_n$ a positive integer and $m_{n+1} - m_n \in \{0, 1\}$. Increase the value of $c_n$ if necessary. Then

$$\frac{v_{n-1}}{v_n} = \frac{(c_{n-1} - 1)y_{n-1}}{(c_n - 1)y_n} \sim \frac{c_{n-1} - 1}{c_n - 1} = \frac{m_{n-1}}{m_n} \to 1 \ .$$

Let $h$ be piecewise linear such that $h(x_n) = y_n$. The derivative $h'(x) = a_n = v_n/u_n$ is constant on the interval $J_n = (x_n, x_{n+1})$, and $a_n/v_n = 1/u_n \to 0$. The asymptotic relation $v_{n+1} \sim v_n$ implies $a_{n+m}/v_n \to 0$ for any integer $m$. Hence $b_n/v_n \to 0$ where $b_n = a_{n-1} + a_n$ is the sum of the left and right derivative of $h$ in the point $x_n$. Similarly $b_{n+1}/v_n \to 0$.

We shall now give an explicit construction of the function $f$.

Set $f(x_n) = y_n$ so that $f$ agrees with $g$ in the points $x_n$. Since $f$ will be strictly increasing and $y_{n+1} \sim y_n$ this ensures that $f \sim g$. We divide the interval $J_n = (x_n, x_{n+1})$ into two parts by a point $\xi_n$ to be determined later and define

$$f(x+u) = \begin{cases} \varphi_n(x_n + u) = y_n + b_n \int_0^u e^{-\lambda_n t}\, dt & x_n + u \le \xi_n \\ \psi(x_{n+1} - u) = y_{n+1} - b_{n+1} \int_0^u e^{-\lambda_n t}\, dt & x_{n+1} - u > \xi_n \,. \end{cases}$$

We shall choose $\xi_n$ and $\lambda_n > 0$ so that $f$ is $C^1$ on the interval $J_n$.

It is best to look at the derivatives. The function $\varphi_n'$ is decreasing with initial value $b_n > a_n$ in the point $x_n$; the function $\psi_n'$ is increasing with boundary value $b_{n+1} > a_n$ in the point $x_{n+1}$. For $\lambda = 0$ the two derivatives are constant and as $\lambda$ increases, the slopes of the two derivatives increase. Let $\xi(\lambda)$ be the point where they intersect. The function $f'$ agrees with $\max(\varphi_n', \psi_n')$ on the interval $J_n$, and we have to choose $\lambda_n > 0$ so that the average slope over the interval $J_n$ is $a_n$, since this is the derivative of the linear function $h$ on $J_n$. Hence $\xi_n = \xi(\lambda_n) \in J_n$ and $f'(\xi_n) < a_n$.

Now observe that $\varphi_n > \psi_n$ on $J_n$ if $\lambda = 0$ since the slopes exceed $a_n$, and that $\psi_n - \varphi_n > v_n - (b_n + b_{n+1})/\lambda \ge 0$ for $\lambda \ge (b_n + b_{n+1})/v_n \to 0$. This implies $\lambda_n \to 0$, and since $|(\log f')'| = \lambda_n$ on $J_n$ we obtain the desired limit relation $(\log f')'(x) \to 0$ for $x \to \infty$.

## References


S. Csörgö and D.M. Mason (1985). Central limit theorems for sums of extreme values. *Math. Proc. Cambridge Philos. Soc.* 98, 547–558.

R.A. Davis and S.I. Resnick (1984). Tail estimates motivated by extreme-value theory. *Annals of Statistics* 12, 1467–1487.

P. Deheuvels, E. Häusler and D.M. Mason (1988). Almost sure convergence of the Hill estimator. *Math. Proc. Cambridge Philos. Soc.* 104, 371–381.

A.L.M. Dekkers, J.H.J. Einmahl and L. de Haan (1989). A moment estimator for the index of an extreme-value distribution. *Annals of Statistics* 17, 1883–1855.

A.L.M. Dekkers and L. de Haan (1989). On the estimation of the extreme-value index and large quantile estimation. *Annals of Statistics* 17, 1795–1832.

J.L. Geluk and L. de Haan (1987). Regular variation, extensions and Tauberian theorems. *C.W.I. Tract 40. Centrum voor Wiskunde en Informatica / Mathematisch Centrum P.O. Box 4079, 1009 AB Amsterdam.*

C.M. Goldie and R.L. Smith (1987). Slow variation with remainder: Theory and applications. *Quart. J. Math. Oxford Ser. (2)* 38, 45–71.

L. de Haan (1984). Slow variation and characterization of domains of attraction. In *Statistical Extremes and Applications* Ed. J. Tiago de Oliveira, 31–48. *Dordrecht, Reidel.*

P. Hall (1982). On some simple estimates of an exponent of regular variation. *J.R.Statist.Soc. B* 44, 37–42.

E. Häusler and J.L. Teugels (1985). On asymptotic normality of Hill's estimator for the exponent of regular variation. *Annals of Statistics* 13, 743–756.

B.M. Hill (1975). A simple general approach to inference about the tail of a distribution. *Annals of Statistics* 3, 1163–1174.

D.M. Mason (1982). Laws of large numbers for sums of extreme values. *Annals of Probability* 10, 754–764.

J. Pickands III (1975). Statistical inference using extreme order statistics. *Annals of Statistics* 3, 119–131.

R.L. Smith (1987). Estimating tails of probability distributions. *Annals of Statistics* 15, 1174–1207.