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State Space Formulas for Transfer Poles at Infinity

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Formulas for the transfer pole/zero structure at infinity of linear time-invariant systems are given in terms of subspaces of the input and output spaces. A characterization of the existence of a transfer function as well as its left/right invertibility is given by using the same subspaces. For systems that are represented in descriptor form the formulas are given in terms of the matrices E, A, B, C and D. No specific assumptions are made on the descriptor representation; in particular, the matrices E and A may be nonsquare. As easy corollaries, known formulas are recovered for the zeros at infinity of standard state space representations and for the infinite elementary divisors of matrix pencils $sE - A$. Furthermore, we give a geometric proof of the fact that the transfer zeros at infinity of descriptor representations that are controllable and observable at infinity coincide with the invariant zeros at infinity, as defined from the system pencil, whereas the transfer poles at infinity can be calculated as the zeros at infinity of $sE - A$. Again, the invertibility of $sE - A$ is not required.

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1. Introduction and preliminaries

In this paper we consider transfer poles and zeros at infinity of a linear time-invariant system. In the literature these are also called transmission poles and zeros at infinity. Examples of situations in which transfer poles and zeros at infinity play an important role are non-interacting control (Vardulakis, 1980; Descusse and Dion, 1982), disturbance decoupling (Verghese, 1978; Emre and Hautus, 1980), root locus theory (Owens, 1978) and singular optimal control (Francis, 1979). We will give state space expressions for the pole/zero structure at infinity of systems, represented in descriptor form:

$$\begin{aligned} \sigma E\xi &= A\xi + Bu \\ y &= C\xi + Du. \end{aligned} \tag{1.1}$$

No restrictions will be placed upon the descriptor representation. In particular, it is not required that $sE - A$ is invertible; the matrices E and A may be even nonsquare. It should be noted that, when $sE - A$ is not invertible, a transfer function may still exist. Indeed, a system, given by (1.1), has a transfer function $T(s)$ if the condition $y(s) = T(s)u(s)$, for rational vectors $y(s)$ and $u(s)$, is equivalent to the existence of a rational vector $x(s)$ such that $(sE - A)x(s) = Bu(s)$ and $y(s) = Cx(s) + Du(s)$. In this paper we will characterize the existence of a transfer function in terms of the matrices E, A, B, C and D.

In recent papers on H^∞ control it is shown that the descriptor form is well-suited for representing the H^∞-controller formulae (Safonov *et al.*, 1989). More generally, descriptor representations arise when systems are described by a combination of differential (or difference) equations and algebraic equations. This situation typically arises when the system is an interconnection of subsystems. Thus, descriptor representations have proved useful in circuit modeling (Rosenbrock, 1974) and econometric modeling (Luenberger and Arbel, 1978). In this context, examples with E and A nonsquare can be easily found. For an example of a descriptor representation for which no transfer function exists the reader is referred to (Dziurla and Newcomb, 1987).

At finite points the transfer poles and zeros of a system are defined through the Smith-McMillan

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canonical form of the transfer function (McMillan, 1952; Rosenbrock, 1970). It was not until (Hautus, 1975) that an analogous form at infinity was introduced which was named the "Smith-McMillan form at infinity" in (Verghese, 1978; Verghese and Kailath, 1979a). This made it possible to define transfer poles/zeros at infinity and at finite points of the complex plane in a completely analogous way. For the sake of clarity we recall the Smith-McMillan form at infinity in the following theorem which states that any rational matrix can be brought into this form by left and right multiplication by "biproper" matrices (i. e. proper matrices that have a proper inverse).

THEOREM 1.1 (Hautus, 1975; Verghese, 1978) *For every rational matrix $T(s)$ there exist biproper matrices $M(s)$ and $N(s)$ such that*

$$M(s)T(s)N(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{1.2}$$

with $D(s) = \operatorname{diag}(s^{n_1}, \cdots, s^{n_l})$. Moreover, the indices $n_1 \in \mathbb{Z}, \cdots, n_l \in \mathbb{Z}$ are unique up to order.

When the indices are ordered such that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_l$ the right-hand side of (1.2) is called the Smith-McMillan form at infinity of $T(s)$. Transfer poles/zeros at infinity are then defined through this canonical form:

DEFINITION 1.2 A system with transfer function $T(s)$ is said to have *l poles at infinity* of orders $n_1, \cdots, n_l$ (zeros of order k are counted as poles of order $-k$) if the Smith-McMillan form at infinity of $T(s)$ is identical to

$$\begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{1.3}$$

where $D(s) = \operatorname{diag}(s^{n_1}, \cdots, s^{n_l})$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_l$.

Of course, the above notion coincides with more classical definitions in which the point at infinity is transformed to a finite point in the complex plane, see (McMillan, 1952; Rosenbrock, 1970). In this way, poles/zeros at infinity of $T(s)$ can (for example) be defined from the standard Smith-McMillan form of $T(s^{-1})$.

Let us quickly review some results for standard state space representations of the form

$$\begin{aligned} \sigma x &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{1.4}$$

Note that, because of the strict properness of the transfer function, there are only poles at infinity of negative order. In (Morse, 1973) a canonical form is derived under the transformation group $(C,A,B) \to (NCT^{-1}, T(A + BF + KC)T^{-1}, TBM)$. Here N, T and M are non-singular matrices, corresponding to a change of basis in the output space, state space and input space, respectively. The matrix F corresponds to state feedback while K stands for output injection. Morse (1975, p.68) noted that certain integers which turn up in the Morse canonical form coincide with the orders of the transfer zeros at infinity. Referring to Owens (1978), Verghese and Kailath (1979a) worked this out in more detail: for the strictly proper transfer function $T(s) = C(sI - A)^{-1}B$ Morse's transformations lead to biproper matrices $M(s) = N[I - C(sI - A)^{-1}K]^{-1}$ and $N(s) = [I - F(sI - A)^{-1}B]^{-1}M$ for which

$$M(s)T(s)N(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{1.5}$$

with $D(s) = \operatorname{diag}(s^{n_1}, \cdots, s^{n_l})$ and $0 > n_1 \geqslant n_2 \geqslant \cdots \geqslant n_l$. In (Thorp, 1973) it is shown that two matrix pencils of the form

$$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \tag{1.6}$$

are related by Morse's transformation group if and only if they are strictly equivalent in the sense of Kronecker (1868, 1874; Gantmacher, 1959). As a consequence, there is a 1-1 correspondence between the orders of the zeros at infinity n_i and the degrees of the infinite elementary divisors: they differ by one.

The above implies that in the strictly proper case there exists a state space procedure to derive the integers $n_1, \cdots, n_l$: compute the Morse canonical form. However, the Morse canonical form displays more structure than just the zero structure at infinity and it would therefore be more efficient to have explicit state space expressions for the orders of the zeros at infinity. These were given by Commault and Dion (1982) by exploiting Morse's canonical form. A different approach was taken in (Nijmeijer and Schumacher, 1985) where state space expressions are derived directly from Definition 1.2 above, so without recourse to a state space canonical form. The expressions describe the number of n_i's which are less than or equal to a certain integer j. From these data the n_i's can of course be reconstructed in a trivial way.

In this context, Verghese *et al.* (1979) considered a descriptor representation (1.1), given by (E, A, B, C, D). They defined the zero structure at infinity of the matrix

$$P(s) = \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} \tag{1.7}$$

from the Smith-McMillan form at infinity of $P(s)$ (Definition 1.2 above). These zeros are called the *invariant zeros at infinity* of the system. Again, a connection with the Kronecker theory can be made, see (Verghese *et al.*, 1979): the orders of the invariant zeros at infinity are equal to the degrees of the infinite elementary divisors of $P(s)$ plus one. In (Malabre, 1989) state space formulas for an (E, A, B, C) representation were given from which the orders of the invariant zeros at infinity can be derived. It is important to note that, when the matrix E is not invertible, the invariant zeros at infinity do not necessarily coincide with the transfer zeros at infinity that we consider in this paper: in contrast with the standard state space case, "cancellations at infinity" might occur in the transfer function. This will be illustrated by an example of an electrical network in section 4. In (Verghese, 1978; Verghese and Kailath, 1979b) it is shown that only under certain conditions the two notions coincide. A weaker version of this result was used in (Lebret and Malabre, 1990) to derive state space formulas for transfer zeros at infinity of an (E, A, B, C) representation. Our methods here are entirely different, and we obtain the result of Verghese *et al.* as a corollary. In fact, we obtain a stronger version since we do not require $sE - A$ to be invertible.

For the sake of brevity we will write "poles" instead of "transfer poles" throughout this paper. The paper is organized as follows. In the next section we introduce certain subspaces of Y and U for systems that are represented by a set of higher-order differential or difference equations. It will be shown that these subspaces completely determine the pole/zero structure at infinity of the system. In fact, they determine more than that: the existence of a transfer function as well as the existence of its inverse can also be characterized in terms of the subspaces. In Section 3 we consider a first order representation which we call the pencil representation, see also (Kuijper and Schumacher, 1990a):

$$\begin{aligned} \sigma G z &= F z \\ y &= H_y z \\ u &= H_u z. \end{aligned} \tag{1.8}$$

We derive expressions for the pole/zero structure at infinity in terms of the matrices G, F, H_y and H_u. We then proceed with descriptor representations in Section 4. By using the results on the pencil representation, state space expressions in terms of the matrices E, A, B, C and D are obtained. The existence of the transfer function as well as the existence of its inverse are then characterized.

2. Polynomial characterization

As a starting point, we consider a linear time-invariant system that is given by equations of the form

$$R_1(\sigma)y - R_2(\sigma)u = 0. \tag{2.1}$$

Here σ denotes differentiation or shift, depending on whether one works in continuous or discrete time, $R_1(s)$ is a polynomial matrix of size $r \times p$ and $R_2(s)$ is a polynomial matrix of size $r \times m$. Inputs and outputs are jointly referred to as *external variables*, and (2.1) may then be rewritten as

$$R(\sigma)w = 0 \tag{2.2}$$

where $R(s) = [R_1(s) \quad -R_2(s)]$ and $w = [y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}}$ is the vector of external variables, which takes its values in $W (= Y \oplus U)$. Following the terminology of J. C. Willems (1986), we shall refer to (2.1) (and also to (2.2)) as an AR (*autoregressive*) representation. We will assume that the decomposition of W into Y and U is fixed. This means that we do not have any freedom in relabeling certain variables as outputs and others as inputs.

In this paper we are only interested in the interactions between inputs and outputs: trajectories of the output variables that are not influenced by the input variables (the so-called "uncontrollable behaviour" in (Kuijper and Schumacher, 1990b)) are not important for the pole/zero structure of the transfer function. For that reason the concept of "external equivalence", which we used in (Kuijper and Schumacher, 1990a, 1990b), can be replaced by the weaker notion of equivalence that was introduced in (Aplevich, 1981). We will term this type of equivalence "input-output equivalence". Note that for any system, represented by an AR matrix $R(s)$, one can associate in a unique way the rational vector space $\ker R(s)$.

Definition 2.1 The AR representations

$$R(\sigma)w = 0 \tag{2.3}$$

and

$$\tilde{R}(\sigma)w = 0 \tag{2.4}$$

are *input-output equivalent* if

$$\ker R(s) = \ker \tilde{R}(s). \tag{2.5}$$

It should be noted that, in case the transfer function of the system exists, input-output equivalence coincides with transfer equivalence (by which systems are defined to be equivalent if their transfer functions coincide).

In the following the space of proper rational W-valued functions is denoted by $W[[s^{-1}]]$. The formal Laurent series at infinity of an element $w(s)$ of $W[[s^{-1}]]$ is written as

$$w(s) = w_0 + w_1 s^{-1} + w_2 s^{-2} + \cdots \tag{2.6}$$

In the sequel we will write $w_0 = w(\infty)$. As in (Kuijper and Schumacher, 1990a) we define the following subspace $W^0 \subset W$:

$$W^0 = \{w \in W \mid \exists\, w(s) \in W[[s^{-1}]] \text{ such that } w(s) \in \ker R(s) \text{ and } w = w(\infty)\}. \tag{2.7}$$

Note that W^0 does not depend on the choice of inputs and outputs. In (Kuijper and Schumacher, 1990a) it was shown that $\dim W^0 = \dim \ker R(s)$ corresponds to the minimum number of driving variables of the system.

We will now proceed by defining subspaces $W^k \subset W$ ($k \in \mathbb{Z}$). This time, our definition will be dependent on the choice of inputs and outputs. Later, it will be shown that these subspaces determine the pole/zero structure at infinity of the system. Corresponding to the decomposition $W = Y \oplus U$, we will partition $w(s)$ as $w(s) = [y(s)^{\mathrm{T}} \quad u(s)^{\mathrm{T}}]^{\mathrm{T}}$.

DEFINITION 2.2

$$W^k = \{[y^{\mathrm T} \quad u^{\mathrm T}]^{\mathrm T} \in W \mid \exists\, [y(s)^{\mathrm T} \quad u(s)^{\mathrm T}]^{\mathrm T} \in W[[s^{-1}]] \text{ such that}$$
$$[s^k R_1(s) \quad -R_2(s)][y(s)^{\mathrm T} \quad u(s)^{\mathrm T}]^{\mathrm T} = 0 \text{ and } [y^{\mathrm T} \quad u^{\mathrm T}]^{\mathrm T} = [y(\infty)^{\mathrm T} \quad u(\infty)^{\mathrm T}]^{\mathrm T}\}.$$

LEMMA 2.3 *For all $k \in \mathbb{Z}$ we have*
(i) $\dim W^k = \dim W^0$
(ii) $\pi_Y W^k \subset \pi_Y W^{k-1}$
(iii) $\pi_U W^k \subset \pi_U W^{k+1}$
(iv) $\dim \pi_Y W^k + \dim \pi_U W^{k-1} = \dim W^k = \dim W^0$.

PROOF Equality (i) follows from

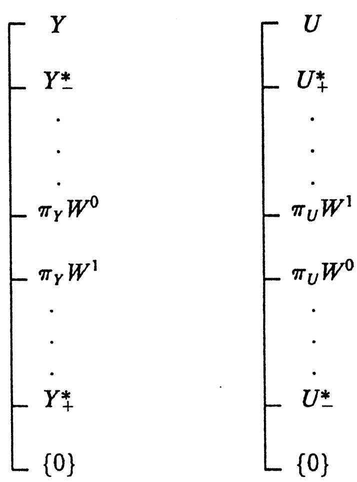
$$\dim\ker[s^k R_1(s) \quad -R_2(s)] = \dim\ker[R_1(s) \quad -R_2(s)] =$$
$$= \dim W^0 \tag{2.8}$$

where the last equality was proven in (Kuijper and Schumacher, 1990a). Next, the inclusions (ii) and (iii) are immediate from the definition while equality (iv) follows from

$$\dim W^k = \dim \pi_Y W^k + \dim\{[y^{\mathrm T} \quad u^{\mathrm T}]^{\mathrm T} \in W^k \mid y = 0\} =$$
$$= \dim \pi_y W^k + \dim\{u \in U \mid \exists\, [y(s)^{\mathrm T} \quad u(s)^{\mathrm T}]^{\mathrm T} \in W[[s^{-1}]] \text{ such that}$$
$$sy(s) \in W[[s^{-1}]],\ [s^{k-1}R_1(s) \quad -R_2(s)][sy(s)^{\mathrm T} \quad u(s)^{\mathrm T}]^{\mathrm T} = 0 \text{ and } u = u(\infty)\} =$$
$$= \dim \pi_Y W^k + \dim \pi_U W^{k-1}. \tag{2.9}$$

This completes the proof of the lemma.

As a result of the above lemma, for $k \geqslant 0$, the sequences $\{\pi_Y W^k\}$ and $\{\pi_U W^{-k}\}$ are non-increasing while the sequences $\{\pi_Y W^{-k}\}$ and $\{\pi_U W^k\}$ are non-decreasing. Since we are dealing with finite dimensional spaces, the four sequences have limits that are reached in finitely many steps. We will denote the limit spaces by Y^*_+, U^*_-, Y^*_- and U^*_+, respectively. We then have the following situation:



Our definition of W^k stems from the approach in (Nijmeijer and Schumacher, 1985) where the subject of "zeros at infinity" is also handled by using subspaces of Y and U. However, these are defined from the transfer function whereas our definition of W^k is more general: it is also applicable for systems without a transfer function. We proceed with the following theorems in which the AR matrix $[R_1(s) \quad -R_2(s)]$ is assumed to have full row rank, which is of course no restriction under input-output equivalence.

THEOREM 2.4 *Let a system be given by*

$$R_1(\sigma)y - R_2(\sigma)u = 0$$

where $[R_1(s) \quad -R_2(s)]$ *is assumed to have full row rank. Then the transfer function* $T(s)$ *of the system exists if and only if the following conditions are satisfied:*

(i) $Y^*_+ = \{0\}$

(ii) $U^*_+ = U$.

If these conditions are satisfied, we have

(iii) $\dim\ker T(s) = \dim U^*_-$

(iv) $\operatorname{rank} T(s) = \dim Y^*_-$.

PROOF From Definition 2.2 it follows immediately that

$$\dim Y^*_+ = \dim \ker R_1(s) \tag{2.10}$$

and

$$\dim U^*_- = \dim \ker R_2(s). \tag{2.11}$$

We also have that $\dim W^0 = \dim\ker[R_1(s) \quad -R_2(s)] = p+m-r$, so from Lemma 2.3 (iv) it follows that

$$\dim U^*_+ = p+m-r-\dim Y^*_+ \tag{2.12}$$

and

$$\dim Y^*_- = p+m-r-\dim U^*_-. \tag{2.13}$$

The invertibility of $R_1(s)$, i.e. the existence of $T(s) = R_1^{-1}(s)R_2(s)$, is now clearly equivalent to conditions (i) and (ii). Next, assume that $T(s)$ exists. Then $p = r$ and

$$\dim\ker T(s) = \dim\ker R_2(s) = \dim U^*_- \tag{2.14}$$

$$\operatorname{rank} T(s) = m - \dim\ker R_2(s) = m - \dim U^*_- = \dim Y^*_- \tag{2.15}$$

where the last equality follows from (2.13).

COROLLARY 2.5 *Let a system be given as in Theorem 2.4. Assume that the transfer function* $T(s) = R_1^{-1}(s)R_2(s)$ *exists. Then*

a) $T(s)$ *is left invertible if and only if* $U^*_- = \{0\}$

b) $T(s)$ *is right invertible if and only if* $Y^*_- = Y$.

THEOREM 2.6 *Let a system be given as in Theorem 2.4. Assume that the transfer function* $T(s) = R_1^{-1}(s)R_2(s)$ *exists. Denote the number of poles at infinity of* $T(s)$ *of order* $\geqslant k$ *by* p_k *and the number of poles at infinity of order* $\leqslant k$ *by* s_k $(k \in \mathbb{Z})$. *Then*

$$p_k = \dim \pi_Y W^k \tag{2.16}$$

and

$$s_k = \dim \pi_U W^k - \dim U^*_-. \tag{2.17}$$

PROOF We can write W^k in the following way:

$$W^k = \{[y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}} \in W \mid \exists\, [y(s)^{\mathrm{T}} \quad u(s)^{\mathrm{T}}]^{\mathrm{T}} \in W[[s^{-1}]] \text{ such that } y(s) = s^{-k}T(s)u(s) \text{ and } [y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}} = [y(\infty)^{\mathrm{T}} \quad u(\infty)^{\mathrm{T}}]^{\mathrm{T}}\}. \tag{2.18}$$

It is clear that dim $\pi_Y W^k$ and dim $\pi_U W^k$ are invariant under left and right multiplication of $T(s)$ by biproper matrices. Therefore we may assume that $T(s)$ is in the form (1.3) i. e. the Smith-McMillan form at infinity. It is then not difficult to see that (2.16) holds. In order to prove (2.17), it should be noted that we have

$$s_k + p_{k+1} = \operatorname{rank} T(s) = \dim Y^*_- \tag{2.19}$$

by Theorem 2.4. So we get

$$s_k = \dim Y^*_- - \dim \pi_Y W^{k+1}. \tag{2.20}$$

Using Lemma 2.3 (iv), we get

$$s_k = \dim Y^*_- - (\dim W^0 - \dim \pi_U W^k) = \dim \pi_U W^k - \dim U^*_-. \tag{2.21}$$

REMARK 2.7 Note that the orders of the poles and zeros at infinity of $T(s)$ can be derived from either the p_k's or the s_k's. One therefore has a choice to consider either subspaces of Y or subspaces of U.

REMARK 2.8 Consider the following transformation group for the matrix $[R_1(s) \quad -R_2(s)]$:

$$[R_1(s) \quad -R_2(s)] \rightarrow F(s)[R_1(s) \quad -R_2(s)] \begin{bmatrix} B_1(s) & 0 \\ 0 & B_2(s) \end{bmatrix} \tag{2.22}$$

where $F(s)$ is a non-singular rational matrix of size $r \times r$ and $B_1(s)$ and $B_2(s)$ are biproper matrices of sizes $p \times p$ and $m \times m$, respectively. The dimensions of the spaces W^k ($k \in \mathbb{Z}$) are obviously invariant under transformations of this type. For the case that $R_1(s)$ is non-singular we will now show that the Smith-McMillan form at infinity of $T(s) = R_1^{-1}(s)R_2(s)$ induces a canonical form for $[R_1(s) \quad -R_2(s)]$ under (2.22) of the type

$$[D_1(s) \quad -D_2(s)] \tag{2.23}$$

where

$$D_1(s) = \begin{bmatrix} \Delta_1(s) & 0 \\ 0 & I \end{bmatrix}, \tag{2.24}$$

$$\Delta_1(s) = \operatorname{diag}(s^{i_1}, \cdots, s^{i_l}), \qquad 0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_l \tag{2.25}$$

$$D_2(s) = \begin{bmatrix} \Delta_2(s) & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.26}$$

$$\Delta_2(s) = \operatorname{diag}(s^{j_1}, \cdots, s^{j_l}), \qquad j_1 \geqslant j_2 \geqslant \cdots \geqslant j_l \geqslant 0 \tag{2.27}$$

and for which the degrees i_k and j_k ($k = 1, \ldots, l$) are as small as possible. Indeed, writing the Smith-McMillan form at infinity as

$$M(s)T(s)N(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{2.28}$$

with $M(s)$ and $N(s)$ biproper and

$$D(s) = \operatorname{diag}(s^{n_1}, \cdots, s^{n_l}), \qquad n_1 \geqslant n_2 \geqslant \cdots \geqslant n_j \geqslant 0 > n_{j+1} \geqslant \cdots \geqslant n_l \tag{2.29}$$

we can decompose $D(s)$ as $D(s) = A_1^{-1}(s)A_2(s)$ where

$$A_1(s) = \operatorname{diag}(1, \cdots, 1, s^{-n_{j+1}}, \cdots, s^{-n_l}), \tag{2.30}$$

$$A_2(s) = \operatorname{diag}(s^{n_1}, \cdots, s^{n_j}, 1, \cdots, 1). \tag{2.31}$$

Now defining $B_1(s) = M^{-1}(s)$, $B_2(s) = N(s)$ and

$$F(s) = \begin{bmatrix} A_1(s) & 0 \\ 0 & I \end{bmatrix} M(s) R_1^{-1}(s) \tag{2.32}$$

we get the canonical form

$$F(s)[R_1(s) \quad -R_2(s)] \begin{bmatrix} B_1(s) & 0 \\ 0 & B_2(s) \end{bmatrix} = \left[\begin{array}{cc|cc} A_1(s) & 0 & -A_2(s) & 0 \\ 0 & I & 0 & 0 \end{array}\right]. \tag{2.33}$$

Note that the canonical form displays the orders of the poles/zeros at infinity of $T(s)$ explicitly. It remains an open question whether a similar canonical form can be given in the situation where $R_1(s)$ is not invertible.

REMARK 2.9 In discrete time the subspaces W^k can also be expressed in terms of a behaviour (Willems, 1986) as follows. Let $\mathscr{B}$ be a behaviour, i. e. a linear shift-invariant subspace of $W^{\mathbb{Z}_+}$. Denoting an element w of $W^{\mathbb{Z}_+}$ by

$$w = (w_0, w_1, \cdots) \tag{2.34}$$

the shift σ is given by

$$\sigma: (w_0, w_1, \cdots) \mapsto (w_1, w_2, \cdots). \tag{2.35}$$

Further, the forward shift is given by

$$\sigma^*: (w_0, w_1, \cdots) \mapsto (0, w_0, w_1, \cdots), \tag{2.36}$$

and the evaluation mapping at time 0 is the mapping

$$\chi: (w_0, w_1, \cdots) \mapsto w_0. \tag{2.37}$$

Following (Willems, 1986) we define the subspace

$$\mathscr{B}^0 = \{w \in \mathscr{B} \mid (\sigma^*)^k w \in \mathscr{B} \ \ \forall k \geqslant 0\}. \tag{2.38}$$

The space $\mathscr{B}^0$ corresponds to the rational vector space $\ker [R_1(s) \quad -R_2(s)]$ that we considered before, see (Kuijper and Schumacher, 1990a). It is now easily seen that we have

$$W^0 = \chi \mathscr{B}^0, \tag{2.39}$$

$$W^k = \chi\{[y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}} \in \mathscr{B}^0 \mid u_0 = u_1 = \cdots = u_{k-1} = 0\} \qquad \text{for } k > 0 \tag{2.40}$$

$$W^k = \chi\{[y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}} \in \mathscr{B}^0 \mid y_0 = y_1 = \cdots = y_{-k-1} = 0\} \qquad \text{for } k < 0. \tag{2.41}$$

3. CHARACTERIZATION IN TERMS OF A PENCIL REPRESENTATION

In the previous section we considered AR representations for which we defined spaces $W^k \subset W \ (k \in \mathbb{Z})$ that are invariant under input-output equivalence. Formulated differently, the spaces W^k were defined for a rational vector space of the form $\ker[R_1(s) \quad -R_2(s)]$ where $R_1(s)$ and $R_2(s)$ are polynomial matrices. It is not difficult to see that a rational vector space of the form $\ker R(s)$, where $R(s)$ is a polynomial matrix, can also be represented in the form

$$\ker R(s) = H[\ker(sG - F)] \tag{3.1}$$

where F, G and H are constant matrices, see (Kuijper and Schumacher, 1990c). The right-hand side of (3.1) corresponds to a pencil representation

$$\begin{aligned} \sigma G z &= F z \\ w &= H z. \end{aligned} \tag{3.2}$$

Here, $F, G : Z \rightarrow X$ and $H : Z \rightarrow W$; Z is the space of internal variables and X is the equation space. We will call the representation (3.2) *input-output equivalent* with the AR representation

$$R(\sigma)w = 0 \tag{3.3}$$

if (3.1) holds. Note that there always exists an AR representation that is input-output equivalent with a given pencil representation. Therefore the spaces W^k can also be defined for a system that is represented in pencil form (F, G, H): choose $R(s)$ such that (3.1) holds and apply Definition 2.2. In the next two lemmas we express the spaces W^k, $\pi_Y W^k$ and $\pi_U W^k$ in terms of the pencil matrices themselves, so without recourse to an equivalent AR representation. The lemmas will immediately lead to a characterization of the existence, invertibility and pole/zero structure at infinity of the transfer function of the system in terms of the matrices F, G and H.

LEMMA 3.1 *Let a system be given by a pencil representation*

$$\begin{aligned} \sigma G z &= F z \\ y &= H_y z \\ u &= H_u z. \end{aligned} \tag{3.4}$$

Let $k \in \mathbb{Z}$, $y \in Y$ and $u \in U$. Then $[y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}} \in W^k$ if and only if there exists a rational vector $z(s)$ with Laurent expansion

$$z(s) = z_{-l}s^l + z_{-l+1}s^{l-1} + \cdots + z_0 + z_1 s^{-1} + \cdots$$

such that the following conditions hold:
(i) $(sG - F)z(s) = 0$
(ii) *(for $k \geqslant 0$) $H_y z(s)$ and $s^k H_u z(s)$ are proper, $y = H_y z_0$, $u = H_u z_k$*
(iii) *(for $k \leqslant 0$) $s^{-k} H_y z(s)$ and $H_u z(s)$ are proper, $y = H_y z_{-k}$, $u = H_u z_0$.*

PROOF Let $[R_1(s) \quad -R_2(s)]$ be a polynomial matrix such that

$$\ker[R_1(s) \quad -R_2(s)] = [H_y^{\mathrm{T}} \quad H_u^{\mathrm{T}}]^{\mathrm{T}}[\ker(sG - F)]. \tag{3.5}$$

We then have

$$[y(s)^{\mathrm{T}} \quad u(s)^{\mathrm{T}}]^{\mathrm{T}} \in \ker[s^k R_1(s) \quad -R_2(s)] \tag{3.6}$$

if and only if, for $k \geqslant 0$,

$$[y(s)^{\mathrm{T}} \quad s^{-k}u(s)^{\mathrm{T}}]^{\mathrm{T}} \in \ker[R_1(s) \quad -R_2(s)] = [H_y^{\mathrm{T}} \quad H_u^{\mathrm{T}}]^{\mathrm{T}}[\ker(sG - F)] \tag{3.7}$$

or, for $k \leqslant 0$,

$$[s^k y(s)^{\mathrm{T}} \quad u(s)^{\mathrm{T}}]^{\mathrm{T}} \in \ker[R_1(s) \quad -R_2(s)] = [H_y^{\mathrm{T}} \quad H_u^{\mathrm{T}}]^{\mathrm{T}}[\ker(sG - F)]. \tag{3.8}$$

From this the statements in the lemma follow immediately.

In the above lemma, we had to make a distinction between the case "$k > 0$" and the case "$k < 0$". In the sequel, this will also be necessary. We therefore introduce a new notation. For $k > 0$ we define

$$Y_+^k = \pi_Y W^k, \qquad U_+^k = \pi_U W^k. \tag{3.9}$$

For $k < 0$ we define

$$Y_-^{-k} = \pi_Y W^k, \qquad U_-^{-k} = \pi_U W^k. \tag{3.10}$$

For $k = 0$ we define

$$Y_+^0 = Y_-^0 = \pi_Y W^0, \qquad U_+^0 = U_-^0 = \pi_U W^0. \tag{3.11}$$

Before presenting the next lemma, we introduce the following subspaces of Z. Consider the iteration

$$\mathcal{N}^0 = \{0\}, \quad \mathcal{N}^{m+1} = G^{-1}F[\mathcal{N}^m \cap \ker H]. \tag{3.12}$$

The sequence $\{\mathcal{N}^k\}$ is non-decreasing and has a limit space $\mathcal{N}^*$. Next, consider the iteration

$$\mathcal{Z}^0 = Z, \quad \mathcal{Z}^{m+1} = F^{-1}G\mathcal{Z}^m. \tag{3.13}$$

The sequence $\{\mathcal{Z}^k\}$ is non-increasing and has a limit space $\mathcal{Z}^*$. To some extent the spaces $\mathcal{N}^*$ and $\mathcal{Z}^*$ express the redundancy of the representation: $\mathcal{N}^* = \ker G$ when $[G^T \quad H^T]^T$ is injective, while surjectivity of G leads to $\mathcal{Z}^* = Z$. In the following lemma the spaces Y_-^k, Y_+^k, U_-^k and U_+^k are expressed in terms of the matrices G, F, H_y and H_u of a pencil representation. We remark here that results for non-minimal pencil representations are needed in the next section. Therefore, minimality is not assumed in the lemma.

LEMMA 3.2 *Let a system be given by a pencil representation (F,G,H_y,H_u). Let the spaces $\mathcal{N}^*$ and $\mathcal{Z}^*$ be defined as before, i. e. as limit spaces of the iterations (3.12) and (3.13), respectively. Then for $k \geqslant 0$ we have*
a)

$$Y_-^k = H_y[\mathcal{T}^k \cap \mathcal{Z}^*] \tag{3.14}$$

where $\mathcal{T}^k$ is given by the iteration

$$\mathcal{T}^0 = \mathcal{N}^*, \quad \mathcal{T}^{m+1} = G^{-1}F[\mathcal{T}^m \cap \ker H_y] \tag{3.15}$$

b)

$$Y_+^k = H_y[\bar{\mathcal{T}}^k \cap \mathcal{N}^*] \tag{3.16}$$

where $\bar{\mathcal{T}}^k$ is given by the iteration

$$\bar{\mathcal{T}}^0 = \mathcal{Z}^*, \quad \bar{\mathcal{T}}^{m+1} = F^{-1}G\bar{\mathcal{T}}^m \cap \ker H_u \tag{3.17}$$

c)

$$U_-^k = H_u[\mathcal{V}^k \cap \mathcal{N}^*] \tag{3.18}$$

where $\mathcal{V}^k$ is given by the iteration

$$\mathcal{V}^0 = \mathcal{Z}^*, \quad \mathcal{V}^{m+1} = F^{-1}G\mathcal{V}^m \cap \ker H_y \tag{3.19}$$

d)

$$U_+^k = H_u[\bar{\mathcal{V}}^k \cap \mathcal{Z}^*] \tag{3.20}$$

where $\bar{\mathcal{V}}^k$ is given by the iteration

$$\bar{\mathcal{V}}^0 = \mathcal{N}^*, \quad \bar{\mathcal{V}}^{m+1} = G^{-1}F[\bar{\mathcal{V}}^m \cap \ker H_u]. \tag{3.21}$$

PROOF See Appendix.

With obvious notation we also get

$$Y_-^* = H_y[\mathcal{T}^* \cap \mathcal{Z}^*] \tag{3.22}$$

$$Y_+^* = H_y[\bar{\mathcal{T}}^* \cap \mathcal{N}^*] \tag{3.23}$$

$$U_-^* = H_u[\mathcal{V}^* \cap \mathcal{N}^*] \tag{3.24}$$

$$U_+^* = H_u[\bar{\mathcal{V}}^* \cap \mathcal{Z}^*]. \tag{3.25}$$

We now present the main theorem of this section:

THEOREM 3.3 *Let a system be given by a pencil representation (F,G,H_y,H_u). Let the spaces $\mathcal{N}^*$ and $\mathcal{Z}^*$ be defined as before i. e. as limit spaces of the iterations (3.12) and (3.13) respectively. Let $\mathcal{T}^k, \bar{\mathcal{T}}^k, \mathcal{V}^k, \bar{\mathcal{V}}^k$ be* *defined by the iterations (3.15), (3.17), (3.19) and (3.21), respectively and let* $\mathscr{T}^*$, $\bar{\mathscr{T}}^*$, $\mathscr{V}^*$ *and* $\bar{\mathscr{V}}^*$ *be the corresponding limit spaces. Then we have*

a) *the system has a transfer function if and only if the following conditions are satisfied:*

(i) $\bar{\mathscr{T}}^* \cap \mathscr{N}^* \subset \ker H_y$

(ii) $H_u[\bar{\mathscr{V}}^* \cap \mathscr{Z}^*] = U$.

Assume that the transfer function $T(s)$ *exists. Then we have*

b) $\dim \ker T(s) = \dim H_u[\mathscr{V}^* \cap \mathscr{N}^*]$

c) $\operatorname{rank} T(s) = \dim H_y[\mathscr{T}^* \cap \mathscr{Z}^*]$.

Denote the number of poles at infinity of $T(s)$ *of order* $\geqslant k$ *by* p_k *and the number of poles at infinity of order* $\leqslant k$ *by* s_k $(k \in \mathbb{Z})$. *Then we have*

d)
$$p_k = \begin{cases} \dim H_y[\bar{\mathscr{T}}^k \cap \mathscr{N}^*] & \text{for } k \geqslant 0 \\ \dim H_y[\mathscr{T}^{-k} \cap \mathscr{Z}^*] & \text{for } k \leqslant 0 \end{cases}$$

$$s_k = \begin{cases} \dim H_u[\bar{\mathscr{V}}^k \cap \mathscr{Z}^*] - \dim H_u[\mathscr{V}^* \cap \mathscr{N}^*] & \text{for } k \geqslant 0 \\ \dim H_u[\mathscr{V}^{-k} \cap \mathscr{N}^*] - \dim H_u[\mathscr{V}^* \cap \mathscr{N}^*] & \text{for } k \leqslant 0 . \end{cases}$$

PROOF Combining the previous lemma with Theorem 2.4, we get a), b) and c). Statement d) follows from Theorem 2.6.

COROLLARY 3.4 *Let a system be given as in Theorem 3.3. Let the spaces* $\mathscr{N}^*$, $\mathscr{Z}^*$, $\mathscr{T}^*$ *and* $\mathscr{V}^*$ *be defined as before, i. e. as limit spaces of the iterations (3.12), (3.13), (3.15) and (3.19), respectively. Assume that the transfer function* $T(s)$ *exists. Then*

a) $T(s)$ *is left invertible if and only if* $\mathscr{V}^* \cap \mathscr{N}^* \subset \ker H_u$

b) $T(s)$ *is right invertible if and only if* $H_y[\mathscr{T}^* \cap \mathscr{Z}^*] = Y$.

4. CHARACTERIZATION IN TERMS OF A DESCRIPTOR REPRESENTATION

In this section we start from a descriptor representation

$$\begin{aligned} \sigma E\xi &= A\xi + Bu \\ y &= C\xi + Du. \end{aligned} \tag{4.1}$$

Here the matrices E and A are not necessarily square; the domain of the mappings E and A will be denoted by X_d (descriptor space) while the codomain will be denoted by X_e (equation space). In order to get expressions in terms of the matrices E, A, B, C and D for the subspaces that are of interest in this paper, we merely need to rewrite the descriptor representation (E, A, B, C, D) as a pencil representation (F, G, H_y, H_u) by defining

$$\begin{aligned} G &= [E \quad 0], \qquad F = [A \quad B], \\ H_y &= [C \quad D], \qquad H_u = [0 \quad I]. \end{aligned} \tag{4.2}$$

It should be noted that we now have $Z = X_d \oplus U$. Since the matrix G has the form $[\,*\quad 0]$, it turns out that the expressions can be given in terms of subspaces of X_d rather than subspaces of $X_d \oplus U$. For example, we have $\mathscr{N}^* = N^* \oplus U$ where N^* is defined as the limit space of the iteration

$$N^0 = \{0\}, \qquad N^{m+1} = E^{-1}A[N^m \cap \ker C]. \tag{4.3}$$

It turns out that the expressions for Y_-^k, Y_+^k, U_-^k and U_+^k can be formulated in such a way that only N^* plays a role. In a similar way there is a correspondence between $\mathscr{Z}^*$ and the subspace $X^* \subset X_d$ that is defined as the limit space of the iteration

$$X^0 = X_d, \qquad X^{m+1} = A^{-1}[EX^m + \operatorname{im} B]. \tag{4.4}$$

The next lemma follows from Lemma 3.2 in a straightforward way and will therefore be stated without proof.

LEMMA 4.1 *Let a system be given by a descriptor representation (E, A, B, C, D). Then we have*

a) for $k \geqslant 0$

$$Y_-^k = \{y \in Y \mid \exists \xi \in T^k, u \in U \text{ such that } A\xi + Bu \in EX^* \text{ and } y = C\xi + Du\} \tag{4.5}$$

where T^k is given by the iteration

$$T^0 = N^*,$$
$$T^{m+1} = \{\xi \in X_d \mid \exists \overline{\xi} \in T^m, u \in U \text{ such that } E\xi = A\overline{\xi} + Bu \text{ and } C\overline{\xi} + Du = 0\} \tag{4.6}$$

b) for $k \geqslant 1$

$$Y_+^k = C[\overline{T}^k \cap N^*] \tag{4.7}$$

where $\overline{T}^k$ is given by the iteration

$$\overline{T}^0 = X^*,$$
$$\overline{T}^{m+1} = A^{-1}E\overline{T}^m \tag{4.8}$$

c) for $k \geqslant 1$

$$U_-^k = \{u \in U \mid \exists \xi \in N^* \text{ such that } A\xi + Bu \in EV^{k-1} \text{ and } C\xi + Du = 0\} \tag{4.9}$$

where V^k is given by the iteration

$$V^0 = X^*,$$
$$V^{m+1} = \{\xi \in X_d \mid \exists u \in U \text{ such that } A\xi + Bu \in EV^m \text{ and } C\xi + Du = 0\} \tag{4.10}$$

d) for $k \geqslant 0$

$$U_+^k = B^{-1}[A\overline{V}^k + EX^*] \tag{4.11}$$

where $\overline{V}^k$ is given by the iteration

$$\overline{V}^0 = N^*,$$
$$\overline{V}^{m+1} = E^{-1}A\overline{V}^m. \tag{4.12}$$

EXAMPLE 4.2 Consider the descriptor representation $(E, A, I, I, 0)$ where $X_d = Y$ and $X_e = U$. For this representation the above formulas become:

$$Y_+^k = \tilde{T}^k \cap \ker E \qquad \text{for } k \geqslant 0 \tag{4.13}$$
$$U_+^k = A\tilde{V}^k + \operatorname{im} E \qquad \text{for } k \geqslant 0 \tag{4.14}$$
$$Y_-^k = X_d \qquad \text{for } k \geqslant 1 \tag{4.15}$$
$$U_-^k = \{0\} \qquad \text{for } k \geqslant 2 \tag{4.16}$$
$$U_-^1 = \operatorname{im} E \tag{4.17}$$

where $\tilde{T}^k$ is given by the iteration

$$\tilde{T}^0 = X_d,$$
$$\tilde{T}^{m+1} = A^{-1}E\tilde{T}^m \tag{4.18}$$

and $\tilde{V}^k$ is given by the iteration

$$\tilde{V}^0 = \ker E,$$
$$\tilde{V}^{m+1} = E^{-1}A\tilde{V}^m. \tag{4.19}$$

The formulas in this example will be needed in later results on matrix pencils.

Combining Lemma 4.1 with Theorem 2.4 and Theorem 2.6, we immediately get the main theorem of this section.

THEOREM 4.3 *Let a system be given by a descriptor representation (E, A, B, C, D). Let the spaces N^* and X^* be defined as before, i. e. as limit spaces of the iterations (4.3) and (4.4), respectively. Let T^k, $\overline{T}^k$, V^k and $\overline{V}^k$ ($k \geqslant 0$) be defined by the iterations (4.6), (4.8), (4.10) and (4.12), respectively and let T^*, $\overline{T}^*$, V^* and $\overline{V}^*$ be the corresponding limit spaces. Then we have*

a) *the system has a transfer function if and only if the following conditions are satisfied:*
   (i) $\overline{T}^* \cap N^* \subset \ker C$
   (ii) $B^{-1}[A\overline{V}^* + EX^*] = U.$

Assume that the transfer function $T(s)$ exists. Then we have

b) $\dim\ker T(s) = \{u \in U \mid \exists \xi \in N^* \text{ such that } A\xi + Bu \in EV^* \text{ and } C\xi + Du = 0\}$
c) $\operatorname{rank} T(s) = \{y \in Y \mid \exists \xi \in T^*, u \in U \text{ such that } A\xi + Bu \in EX^* \text{ and } y = C\xi + Du\}.$

Denote the number of poles at infinity of $T(s)$ of order $\geqslant k$ by p_k and the number of poles at infinity of order $\leqslant k$ by s_k ($k \in \mathbb{Z}$). Then we have

d)
$$p_k = \begin{cases} \dim C[\overline{T}^k \cap N^*] & \text{for } k \geqslant 1 \\ \dim\{y \in Y \mid \exists \xi \in T^{-k}, u \in U \text{ such that } A\xi + Bu \in EX^* \text{ and } y = C\xi + Du\} & \text{for } k \leqslant 0 \end{cases}$$

$$s_k = \begin{cases} \dim B^{-1}[A\overline{V}^k + EX^*] - \dim U^*_- & \text{for } k \geqslant 0 \\ \dim\{u \in U \mid \exists \xi \in N^* \text{ such that } A\xi + Bu \in EV^{-k-1} \text{ and } C\xi + Du = 0\} - \dim U^*_- & \text{for } k \leqslant -1 \end{cases}$$

where $U^*_- = \{u \in U \mid \exists \xi \in N^* \text{ such that } A\xi + Bu \in EV^* \text{ and } C\xi + Du = 0\}.$

COROLLARY 4.4 *Let a system be given as in Theorem 4.3. Let the spaces N^*, X^*, T^* and V^* be defined as before, i. e. as limit spaces of the iterations (4.3), (4.4), (4.6) and (4.10), respectively. Assume that the transfer function $T(s)$ exists. Then*

a) *$T(s)$ is left invertible if and only if*

$$\{u \in U \mid \exists \xi \in N^* \text{ such that } A\xi + Bu \in EV^* \text{ and } C\xi + Du = 0\} = \{0\}$$

b) *$T(s)$ is right invertible if and only if*

$$\{y \in Y \mid \exists \xi \in T^*, u \in U \text{ such that } A\xi + Bu \in EX^* \text{ and } y = C\xi + Du\} = Y.$$

REMARK 4.5 Let us consider a standard state space representation, i. e. a descriptor representation with $E = I$:

$$\sigma x = Ax + Bu$$
$$y = Cx + Du. \tag{4.20}$$

Here $A : X \to X$ where X is the state space. In this case it is easily checked that we have $X^* = X$ and $N^* = \{0\}$. Further, $Y^k_+ = \{0\}$ for all $k \geqslant 1$ and $U^k_+ = U$ for all $k \geqslant 0$. This is not surprising (see

Theorem 2.6) since the transfer function of the system is proper: there are no poles at infinity of positive order. Next, from Lemma 4.1 it follows that we have for $k \geqslant 0$

$$Y_-^k = CT^k + \operatorname{im} D \tag{4.21}$$

where

$$T^0 = \{0\},$$

$$T^{m+1} = \{x \in X \mid \exists \bar{x} \in T^m, u \in U \text{ such that } x = A\bar{x} + Bu \text{ and } C\bar{x} + Du = 0\}. \tag{4.22}$$

For $k \geqslant 1$ we have

$$U_-^k = B^{-1}V^{k-1} \cap \ker D \tag{4.23}$$

where

$$V^0 = X,$$

$$V^{m+1} = \{x \in X \mid \exists u \in U \text{ such that } Ax + Bu \in V^m \text{ and } Cx + Du = 0\}. \tag{4.24}$$

Denote the number of zeros at infinity of $T(s) = C(sI - A)^{-1}B + D$ of order $\leqslant k$ by t_k and the number of zeros at infinity of order $\geqslant k$ by j_k ($k \geqslant 0$). With some linear algebra it now follows from Theorem 4.3 that

$$t_k = \dim \operatorname{im} [C \quad D] - \operatorname{codim} (T^k + V^1) \qquad \text{for } k \geqslant 0 \tag{4.25}$$

$$j_0 = \dim \operatorname{im} [C \quad D] - \operatorname{codim} (T^* + V^1) \tag{4.26}$$

$$j_k = \dim (V^{k-1} \cap T^1) - \dim (V^* \cap T^1) \qquad \text{for } k \geqslant 1. \tag{4.27}$$

Formula (4.27) is the generalization of a result in (Malabre, 1982) where the matrix D is assumed to be zero (see also (Nijmeijer and Schumacher, 1985)).

In the following we shall give some more results that arise from Theorem 4.3. The next corollary characterizes the properness of the transfer function in terms of the matrices E, A, B, C and D.

COROLLARY 4.6 *Let a system be given by a descriptor representation* (E, A, B, C, D) *for which the transfer function* $T(s)$ *exists. Let the spaces* N^* *and* X^* *be defined as before, i. e. as limit spaces of the iterations (4.3) and (4.4), respectively. Then* $T(s)$ *is proper if and only if*

$$N^* \cap A^{-1}EX^* \subset \ker C. \tag{4.28}$$

PROOF In the notation of Theorem 4.3, there are no poles at infinity if and only if $p_1 = 0$. By result d) of Theorem 4.3 this is equivalent to (4.28).

REMARK 4.7 The above corollary can be made more specific for instance when we assume that the descriptor representation (E, A, B, C, D) satisfies the following conditions:

(i) E and A are square

(ii) $[E \quad B]$ has full row rank

(iii) $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank.

We then have $X^* = X_d$ and $N^* = \ker E$ because of conditions (ii) and (iii). According to the above corollary the transfer function is proper if and only if

$$\ker E \cap A^{-1}[\operatorname{im} E] = \{0\}. \tag{4.29}$$

By a suitable choice of coordinates we can rewrite the descriptor representation in the form

$$\sigma\xi_1 = A_{11}\xi_1 + A_{12}\xi_2 + B_1u \tag{4.30}$$

$$0 = A_{21}\xi_1 + A_{22}\xi_2 + B_2u \tag{4.31}$$

$$y = C_1\xi_1 + C_2\xi_2 + Du. \tag{4.32}$$

Clearly, (4.29) holds if and only if A_{22} is invertible. When A_{22} is invertible, we can rewrite (4.31) as

$$\xi_2 = -A_{22}^{-1}(A_{21}\xi_1 + B_2u). \tag{4.33}$$

Substitution of this expression into the equations (4.30) and (4.32) leads to an equivalent standard state space representation. The above corollary tells us that, under the assumptions (i)-(iii), this is the only circumstance under which the representation can be rewritten in standard state space form.

Next, our approach also yields geometric conditions for the zero structure at infinity of the pencil $sE - A$. Left and right invertibility are characterized along the same lines.

THEOREM 4.8 *Let $E, A: X_d \to X_e$ be linear mappings. Let $\tilde{T}^k$ and $\tilde{V}^k$ $(k \geq 0)$ be defined by the iterations (4.18) and (4.19), respectively and let $\tilde{T}^*$ and $\tilde{V}^*$ be the corresponding limit spaces. Denote the number of zeros at infinity of $sE - A$ of order $\leq k$ by t_k and the number of zeros at infinity of order $\geq k$ by j_k $(k \geq 0)$. Then we have (only counting zeros of non-negative order)*

a)

$$t_k = \dim(A\tilde{V}^k + \operatorname{im} E) - \dim \operatorname{im} E \tag{4.34}$$

$$j_k = \dim(\tilde{T}^k \cap \ker E) - \dim(\tilde{T}^* \cap \ker E) \tag{4.35}$$

b)

$$\dim \ker(sE - A) = \dim(\tilde{T}^* \cap \ker E) \tag{4.36}$$

c)

$$\operatorname{rank}(sE - A) = \dim(A\tilde{V}^* + \operatorname{im} E). \tag{4.37}$$

In particular, $sE - A$ has no zeros at infinity of positive order if and only if

$$\dim(A[\ker E] + \operatorname{im} E) = \dim(A\tilde{V}^* + \operatorname{im} E) \tag{4.38}$$

or, equivalently,

$$\dim(A^{-1}[\operatorname{im} E] \cap \ker E) = \dim(\tilde{T}^* \cap \ker E), \tag{4.39}$$

$sE - A$ has full column rank if and only if

$$\tilde{T}^* \cap \ker E = \{0\} \tag{4.40}$$

and $sE - A$ has full row rank if and only if

$$A\tilde{V}^* + \operatorname{im} E = X_e. \tag{4.41}$$

PROOF Consider the system that is represented by the AR representation

$$y - (\sigma E - A)u = 0 \tag{4.42}$$

where $Y = X_e$ and $U = X_d$. The above system has the pencil $sE - A$ as its transfer function. According to Theorems 2.4 and 2.6 left/right invertibility as well as the zero structure at infinity of $sE - A$ are completely determined by the spaces Y^*_-, U^*_-, Y^k_- and U^k_- $(k \geq 0)$. For the purpose of calculating these spaces in a recursive way, we note that (4.42) is input-output equivalent with the descriptor representation

$$\begin{aligned} \sigma E\xi &= A\xi + y \\ u &= \xi. \end{aligned} \tag{4.43}$$

Using the results of Example 4.2, we get for $k \geq 0$:

$$Y_-^k = A\tilde{V}^k + \operatorname{im} E \tag{4.44}$$

$$U_-^k = \tilde{T}^k \cap \ker E \tag{4.45}$$

We also have $Y_+^1 = \operatorname{im} E$. As a result of Theorem 2.6 we now have (only counting zeros of non-negative order):

$$\begin{aligned} t_k &= \dim Y_-^k - \dim Y_+^1 = \\ &= \dim (A\tilde{V}^k + \operatorname{im} E) - \dim \operatorname{im} E \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} j_k &= \dim U_-^k - \dim U_-^* = \\ &= \dim (\tilde{T}^k \cap \ker E) - \dim (\tilde{T}^* \cap \ker E). \end{aligned} \tag{4.47}$$

Next, we have from Theorem 2.4 that

$$\dim \ker (sE - A) = \dim U_-^* = \dim (\tilde{T}^* \cap \ker E) \tag{4.48}$$

and

$$\operatorname{rank} (sE - A) = \dim Y_-^* = \dim (A\tilde{V}^* + \operatorname{im} E). \tag{4.49}$$

This completes the proof of the theorem.

REMARK 4.9 A k-th order zero at infinity of the pencil $sE - A$ corresponds to an infinite elementary divisor of degree $k+1$, as defined in the Kronecker theory, see (Verghese *et al.*, 1979). In (Malabre, 1989) recursive formulas are given for the infinite elementary divisor structure of $sE - A$; on page 598 of that paper formula (4.47) can be found. In the same paper right invertibility is characterized by a formula that is different from (4.41), namely by:

$$A\tilde{V}^* + E\tilde{T}^* = X_e. \tag{4.50}$$

The equivalence of (4.41) and (4.50) can be established by considering the Kronecker canonical form of $sE - A$ as in (Schumacher, 1988). Since (4.41) does not require the computation of the space $\tilde{T}^*$, it is easier to check than formula (4.50). Equation (4.36) appears also in (Bernhard, 1982; Armentano, 1986).

Application of the last result of Theorem 4.8 immediately leads to more general versions of results in (Armentano, 1986).

COROLLARY 4.10 *Let E, $A : X_d \to X_e$ and $B : U \to X_d$ be linear mappings. Let $J^* \subset X_d$ be the limit space of the iteration*

$$\begin{aligned} J^0 &= \{0\}, \\ J^{m+1} &= E^{-1}[AJ^m + \operatorname{im} B]. \end{aligned} \tag{4.51}$$

Then $[sE - A \quad B]$ has no zeros at infinity of positive order if and only if

$$\dim (A[\ker E] + \operatorname{im} E + \operatorname{im} B) = \dim (AJ^* + \operatorname{im} E + \operatorname{im} B). \tag{4.52}$$

In particular, when $[sE - A \quad B]$ has full row rank, $[sE - A \quad B]$ has no zeros at infinity of positive order if and only if

$$A[\ker E] + \operatorname{im} E + \operatorname{im} B = X_e. \tag{4.53}$$

COROLLARY 4.11 *Let E, $A : X_d \to X_e$ and $C : X_d \to Y$ be linear mappings. Let $L^* \subset X_d$ be the limit space of the iteration*

$$L^0 = X_d,$$

$$L^{m+1} = A^{-1}EL^{m} \cap \ker C. \tag{4.54}$$

Then $[sE^{T} - A^{T} \quad C^{T}]^{T}$ *has no zeros at infinity of positive order if and only if*

$$\dim(A^{-1}[\operatorname{im} E] \cap \ker E \cap \ker C) = \dim(L^{*} \cap \ker E). \tag{4.55}$$

In particular, when $[sE^{T} - A^{T} \quad C^{T}]^{T}$ *has full column rank,* $[sE^{T} - A^{T} \quad C^{T}]^{T}$ *has no zeros at infinity of positive order if and only if*

$$A^{-1}[\operatorname{im} E] \cap \ker E \cap \ker C = \{0\}. \tag{4.56}$$

Following the terminology of (Verghese *et al.*, 1981; Lewis, 1986) we shall call a descriptor representation *controllable at infinity* if $[sE - A \quad B]$ has no zeros at infinity of positive order whereas the representation is called *observable at infinity* if $[sE^{T} - A^{T} \quad C^{T}]^{T}$ has no zeros at infinity of positive order. The following theorem presents the analogon of a well-known property of standard state space representations that are controllable and observable, see (Rosenbrock, 1970): the pole structure at $\alpha \in \mathbb{C}$ of the transfer function $C(sI - A)^{-1}B + D$ is equal to the zero structure at $\alpha \in \mathbb{C}$ of $sI - A$ while the zero structure at $\alpha \in \mathbb{C}$ equals the zero structure at $\alpha \in \mathbb{C}$ of the matrix

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}.$$

THEOREM 4.12 *Let a system be given by a descriptor representation (E, A, B, C, D) for which a transfer function T(s) exists and for which the matrices* $[sE - A \quad B]$ *and* $[sE^{T} - A^{T} \quad C^{T}]^{T}$ *have full row rank and full column rank, respectively. Next, assume that the following conditions are satisfied:*

(i) $[sE - A \quad B]$ *has no zeros at infinity of positive order*

(ii) $[sE^{T} - A^{T} \quad C^{T}]^{T}$ *has no zeros at infinity of positive order.*

Then for all $k \geq 1$

a) the number of poles at infinity of order k of T(s) equals the number of zeros at infinity of order k of $sE - A$

b) the number of zeros at infinity of order k of T(s) equals the number of zeros at infinity of order k of the matrix

$$\begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix}.$$

PROOF Let $k \geq 1$. According to Corollaries 4.10 and 4.11 it follows from (i) and (ii) that

$$A[\ker E] + \operatorname{im} E + \operatorname{im} B = X_{e} \tag{4.57}$$

and

$$A^{-1}[\operatorname{im} E] \cap \ker E \cap \ker C = \{0\}. \tag{4.58}$$

Equation (4.58) immediately gives $N^{*} = \ker E$ (where N^{*} is the limit space of iteration (4.3)). Next, from (4.57) it follows that

$$\ker E + A^{-1}(\operatorname{im}[E \quad B]) = X_{d}. \tag{4.59}$$

It is easily seen that this implies that $EX^{1} = EX^{*} = \operatorname{im} E$ (where X^{1} and X^{*} are defined from iteration (4.4)). With p_k defined as the number of poles at infinity of order $\geq k$ of $T(s)$, Theorem 4.3 then yields

$$\begin{aligned} p_k &= \dim C[\tilde{T}^{k} \cap \ker E] = \\ &= \dim[\tilde{T}^{k} \cap \ker E] \end{aligned} \tag{4.60}$$

where $\tilde{T}^{k}$ is given by iteration (4.18) and the last equality holds because of (4.58). The right-hand side of

(4.60) now equals the formula that is obtained in Theorem 4.8 for the number of zeros of order $\geqslant k$ of $sE - A$ up to a constant. This proves a). Next, with j_k defined as the number of zeros at infinity of order $\geqslant k$ of $T(s)$, Theorem 4.3 yields

$$j_k = \dim\{u \in U \mid \exists \xi \in \ker E \text{ such that } A\xi + Bu \in EV^{k-1} \text{ and } C\xi + Du = 0\} - K \quad (4.61)$$

where K is a constant and V^k is given by the iteration

$$V^0 = X_d,$$
$$V^{m+1} = \{\xi \in X_d \mid \exists u \in U \text{ such that } A\xi + Bu \in EV^m \text{ and } C\xi + Du = 0\}. \quad (4.62)$$

Here we made use of the fact that $EX^* = \operatorname{im} E$. On the other hand, application of Theorem 4.8 on the pencil $s\bar{E} - \bar{A}$ where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \quad (4.63)$$

yields

$$\bar{j}_k = \dim\{[\xi^T \quad u^T]^T \mid \xi \in \ker E, A\xi + Bu \in EV^{k-1} \text{ and } C\xi + Du = 0\} - \bar{K}. \quad (4.64)$$

Here $\bar{j}_k$ denotes the number of zeros at infinity of order $\geqslant k$ of the matrix

$$\begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} \quad (4.65)$$

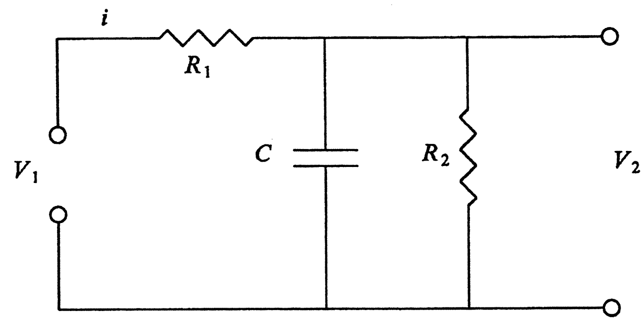
and $\bar{K}$ denotes another constant. Because of (4.58) it now follows easily that j_k equals $\bar{j}_k$ up to a constant. From this b) can be concluded.

REMARK 4.13 For the case that $sE - A$ is invertible the results of the above theorem were proven in (Verghese, 1978; Verghese and Kailath, 1979b) in a completely different way.

REMARK 4.14 In the above theorem no conclusions were drawn concerning the zeros at infinity of order zero. However, for a specific class of descriptor representations we have the following result. Let us consider an $(E, A, B, C, 0)$ representation for which $[E \quad B]$ has full row rank and $[E^T \quad C^T]^T$ has full column rank (these are necessary conditions for minimality w.r.t. the size of E, see (Grimm, 1988)). Denote the transfer function of the system that is represented by $(E, A, B, C, 0)$ by $T(s)$. It can then be easily shown that the number of zeros at infinity of order zero of $T(s)$ equals the number of zeros at infinity of order zero of $sE - A$.

When no assumptions are made on the descriptor representation, the number of invariant zeros at infinity (= zeros at infinity of (4.65)) is larger than or equal to the number of transfer zeros at infinity as can be easily seen from the proof of Theorem 4.12. The question arises what the system-theoretic interpretation of the "redundant" invariant zeros at infinity is. In the following an example of such "redundancy" is given.

EXAMPLE 4.15 Consider the electrical network consisting of a capacitor C and two resistors R_1 and R_2 as shown in the following figure:



We set the numerical values of C and R_2 equal to 1. Let us consider the voltage V_1 as an input and the voltage V_2 as an output. Taking V_2 and the current i as the descriptor state variables ξ_1 and ξ_2, respectively, we can represent the system by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & R_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{4.66}$$

From Cor. 4.11 it can be immediately concluded that the representation is observable at infinity for all values of R_1 except zero. As can be calculated by using Theorem 4.8 for the system pencil (1.7), there is an invariant zero at infinity of order 1. For $R_1 \neq 0$ this invariant zero corresponds to the zero at infinity of the transfer function

$$T(s) = C(sE - A)^{-1}B = \frac{-1}{R_1 s + R_1 + 1}.$$

For $R_1 = 0$, however, there are no transfer zeros at infinity: the invariant zero of the system is "redundant".

5. Conclusions

Through the definition of subspaces of Y and U we gave formulas for the transfer pole/zero structure at infinity of linear time-invariant systems. For descriptor representations the formulas were given in terms of the matrices E, A, B, C and D. Requirements of for instance squareness or invertibility of $sE - A$ were not imposed. An important tool for deriving the descriptor results was the so-called pencil representation. The above-mentioned subspaces of Y and U were also employed to characterize the *existence* of a transfer function as well as its left/right invertibility. The approach enabled a purely geometric proof of the fact that the transfer zeros at infinity coincide with the invariant zeros at infinity if the descriptor representation is controllable and observable at infinity. As an issue for further research we mention the development of an analog for descriptor representations of the nine-fold decomposition in (Aling and Schumacher, 1984).

6. Appendix

For the proof of Lemma 3.2 we need the following lemma which can be considered as a generalization of a result in (Hautus, 1980).

Lemma A.1

Let $F, G: Z \to X$ be linear mappings. Let $V \subset Z$ be a subspace for which $FV \subset GV$. Let $z \in V$. Then there exists a strictly proper rational function $\xi(s)$ for which

$$(sG - F)\xi(s) = Gz. \tag{6.1}$$

PROOF Decompose Z as $Z_1 \oplus Z_2$ where $Z_2 =$ ker G. Decompose X as $X_1 \oplus X_2$ where $X_1 =$ im G. Accordingly, write

$$G = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}, \quad Gz = \begin{bmatrix} x \\ 0 \end{bmatrix} \tag{6.2}$$

Let $\pi_{Z_1}: Z \to Z_1$ be the projection onto Z_1 along Z_2. Denote $\bar{V} = \pi_{Z_1} V$. Then $\bar{V}$ is a subspace of Z_1 for which

$$\begin{bmatrix} A \\ C \end{bmatrix} \bar{V} \subset (\bar{V} \times \{0\}) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix}. \tag{6.3}$$

Since we are dealing with finite dimensional spaces we can then find a mapping $K: Z_1 \to Z_2$ such that

$$(A + BK)\, \bar{V} \subset \bar{V}, \qquad (C + DK)\, \bar{V} = \{0\}. \tag{6.4}$$

Now define the strictly proper rational function

$$\bar{\xi}(s) = (sI - A - BK)^{-1} x. \tag{6.5}$$

Then $\bar{\xi}(s) \in \bar{V}$ for all $s \in \mathbb{C}$, so $(C + DK)\bar{\xi}(s) = 0$. Defining

$$\xi(s) = \begin{bmatrix} \bar{\xi}(s) \\ K\bar{\xi}(s) \end{bmatrix} \tag{6.6}$$

we now have

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \xi(s) = \begin{bmatrix} x \\ 0 \end{bmatrix} \tag{6.7}$$

and this completes the proof of the lemma.

PROOF OF LEMMA 3.2

Let $k \geqslant 0$.

a) First we prove that $Y^k_- \subset H_y[\mathscr{T}^k \cap \mathscr{Z}^*]$. Let $y \in Y^k_-$. Then it follows from Lemma 3.1 that there exists a rational vector $z(s)$ with Laurent expansion

$$z(s) = z_{-l}s^l + z_{-l+1}s^{l-1} + \cdots + z_0 + z_1 s^{-1} + \cdots + z_k s^{-k} + \cdots$$

such that

(i) $(sG - F)z(s) = 0$

(ii) $s^k H_y z(s)$ and $H_u z(s)$ are proper

(iii) $y = H_y z_k$.

From (i) it follows that $z_i \in \mathscr{Z}^*$ for all $i \geqslant -l$. In particular, we have $z_k \in \mathscr{Z}^*$. From (i) and (ii) it follows that $z_{-l} \in \mathscr{N}^1$, $z_{-l+1} \in \mathscr{N}^2$, $\cdots$, $z_0 \in \mathscr{N}^{l+1}$, so we certainly have $z_0 \in \mathscr{N}^*$. So, by definition, $z_0 \in \mathscr{T}^0$. Now (ii) implies that $z_1 \in \mathscr{T}^1$, $z_2 \in \mathscr{T}^2$, $\cdots$, $z_k \in \mathscr{T}^k$. Finally, it follows from (iii) that $y \in H_y[\mathscr{T}^k \cap \mathscr{Z}^*]$. Next, we prove that $H_y[\mathscr{T}^k \cap \mathscr{Z}^*] \subset Y^k_-$. Let $y \in H_y[\mathscr{T}^k \cap \mathscr{Z}^*]$. Then there exists $z_k \in \mathscr{T}^k \cap \mathscr{Z}^*$ such that $y = H_y z_k$. Since $z_k \in \mathscr{T}^k$ we can find $z_0, z_1, \cdots, z_{k-1}$ such that $Fz_{k-1} = Gz_k$, $Fz_{k-2} = Gz_{k-1}$, $\cdots$, $Fz_0 = Gz_1$ while $H_y z_{k-1} = 0$, $H_y z_{k-2} = 0$, $\cdots$, $H_y z_0 = 0$ and $z_0 \in \mathscr{N}^*$. Since $z_0 \in \mathscr{N}^*$ there exist $z_{-1}, z_{-2}, \cdots, z_{-l}$ such that $Gz_0 = Fz_{-1}$, $Gz_{-1} = Fz_{-2}$, $\cdots$, $Gz_{-l+1} = Fz_{-l}$, $Gz_{-l} = 0$ while $Hz_{-1} = Hz_{-2} = \cdots = Hz_{-l} = 0$. Furthermore, $z_k \in \mathscr{Z}^*$ implies that we can find $z_{k+1} \in \mathscr{Z}^*$ such that $Fz_k = Gz_{k+1}$. According to Lemma A.1 there exists a strictly proper rational function $\xi(s)$ such that $(sG - F)\xi(s) = Gz_{k+1}$. Now define

$$z(s) = z_{-l}s^l + z_{-l+1}s^{l-1} + \cdots + z_0 + z_1 s^{-1} + \cdots + z_k s^{-k} + \xi(s)s^{-k}.$$

Then it is easily seen that conditions (i)-(iii) hold. From Lemma 3.1 it now follows that $y \in Y^k_-$.

b) First we prove that $Y_+^k \subset H_y[\bar{\mathcal{T}}^k \cap \mathcal{N}^*]$. Let $y \in Y_+^k$. Then it follows from Lemma 3.1 that there exists a rational vector $z(s)$ with Laurent expansion

$$z(s) = z_{-l}s^l + z_{-l+1}s^{l-1} + \cdots + z_0 + z_1 s^{-1} + \cdots + z_k s^{-k} + \cdots$$

such that
(i) $(sG - F)z(s) = 0$
(ii) $H_y z(s)$ and $s^k H_u z(s)$ are proper
(iii) $y = H_y z_0$.
As in part a) of this proof, it follows that $z_0 \in \mathcal{N}^*$ and $z_k \in \mathcal{Z}^*$. So, by definition, $z_k \in \bar{\mathcal{T}}^0$. It now follows from (i) and (ii) that $z_{k-1} \in \bar{\mathcal{T}}^1$, $z_{k-2} \in \bar{\mathcal{T}}^2$, $\cdots$, $z_0 \in \bar{\mathcal{T}}^k$. Together with (iii) this implies that $y \in H_y[\bar{\mathcal{T}}^k \cap \mathcal{N}^*]$. Next, we prove that $H_y[\bar{\mathcal{T}}^k \cap \mathcal{N}^*] \subset Y_+^k$. Let $y \in H_y[\bar{\mathcal{T}}^k \cap \mathcal{N}^*]$. Then there exists $z_0 \in \bar{\mathcal{T}}^k \cap \mathcal{N}^*$ such that $y = H_y z_0$. Since $z_0 \in \mathcal{N}^*$ there exist $z_{-1}, z_{-2}, \cdots, z_{-l}$ such that $Gz_0 = Fz_{-1}$, $Gz_{-1} = Fz_{-2}$, $\cdots$, $Gz_{-l+1} = Fz_{-l}$, $Gz_{-l} = 0$ while $Hz_{-1} = 0$, $Hz_{-2} = 0$, $\cdots Hz_{-l} = 0$. Furthermore, $z_0 \in \bar{\mathcal{T}}^k$ implies that we can find $z_1, z_2, \cdots, z_k$ such that $Fz_0 = Gz_1$, $Fz_1 = Gz_2$, $\cdots$, $Fz_{k-1} = Gz_k$ while $H_u z_0 = 0$, $H_u z_1 = 0$, $\cdots$, $H_u z_{k-1} = 0$ and $z_k \in \mathcal{Z}^*$. Now $z_k \in \mathcal{Z}^*$ implies, as in the proof of part a), that there exists a strictly proper rational function $\xi(s)$ for which $(sG - F)\xi(s) = Fz_k$. Now define

$$z(s) = z_{-l}s^l + z_{-l+1}s^{l-1} + \cdots + z_0 + z_1 s^{-1} + \cdots + z_k s^{-k} + \xi(s)s^{-k}.$$

It is easily seen that conditions (i)-(iii) hold. From Lemma 3.1 it now follows that $y \in Y_+^k$.

By interchanging y and u, part c) and d) can be proven completely analogously to part b) and part a), respectively. This completes the proof of the lemma.

7. References

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