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Right Angle Free Subsets in the Plane

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Abstract

In this paper we investigate the complexity of finding maximum right angle free subsets in the plane. For a set of rational points P in the plane, the *right angle number* $\rho(P)$ (respectively *rectilinear right angle number* $\rho_R(P)$) of P is the cardinality of a maximum subset of P , no three members of which form a right angle triangle (respectively a right angle triangle with its side or base parallel to the x -axis). It is shown that both parameters are NP-hard to compute. The latter problem is also shown to be equivalent to finding a minimum dominating set in a bipartite graph. This is used to show that there is a polynomial algorithm for computing $\rho_R(P)$ when P is a horizontally-convex subset of the lattice $\mathbb{Z} \times \mathbb{Z}$ (P is *horizontally-convex* if for any pair of points in P which lie on a horizontal line, every lattice point between them is also in P). We then show that this algorithm yields a $\frac{1}{2}$ -approximate algorithm for the right angle number of a convex subregion of the lattice.

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1 Introduction

Let P be a set of n points in the plane. A subset, S , of P is *right angle free (raf)* if no three points of S form the vertices of a right angle triangle. The *right angle number* of P , denoted by $\rho(P)$, is the cardinality of a maximum raf subset of P .

Let $f(n)$ be the largest integer such that for any set of n points in the plane, one has $\rho(P) \geq f(n)$. Erdős and Silverman [3] noted that a $k \times k$ rectangular subset of the integer lattice has right angle number $2k - 2$. Hence $f(k^2) \leq 2k - 2$. Abbott [1] proved that $f(n) \geq \sqrt{n}$. It is still unknown, however, whether $\lim_{n \rightarrow \infty} f(n)/n^{1/2}$ exists.

For points $x = (x_1, x_2)$, $y = (y_1, y_2)$ the *taxi-cab distance* between x and y is $|x_1 - y_1| + |x_2 - y_2|$. The *k-diamond*, D_k , is the set of all points on the integer lattice which are within taxi-cab distance k from the origin (D_3 is shown in Figure 1).

The set of points on either axis form an raf subset of D_k of size $2k + 1$. Paul Lemke [4] has recently shown that raf subsets of size greater than $2k + 1$ exist in D_k . It has been conjectured that $\lim_{k \rightarrow \infty} \rho(D_k)/(2k + 1) = 1$. This would imply that if it exists, $\lim_{n \rightarrow \infty} f(n)/n^{1/2} \leq \sqrt{2}$.

In this paper we examine the associated computational problem. Specifically, we refer to RAF as the decision problem: "Given an integer k and subset P of rational points, determine whether $\rho(P) \geq k$."

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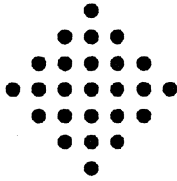


Figure 1:

We also consider a related problem where we only consider a finite number of orientations of the right angle triangles. A set of points S in the plane is *rectilinear right angle free* (**rraf**) if no three points of S form a right angle triangle with a base side which is parallel to the vector $(1,0)$, i.e., a base which is horizontal. For a set of points P , the rectilinear right angle number of P , denoted by $\rho_R(P)$, is the cardinality of a maximum **rraf** subset of P . (Note that $\rho_R(P) \geq \rho(P)$.) We let **RRAF** denote the decision problem: “Given an integer k and set of points P , determine whether $\rho_R(P) \geq k$ ”.

An algorithm to solve **RAF** is *polynomial* if its running time is bounded by a polynomial in the number of binary bits needed to encode the given set of points as input. A *pseudo-polynomial* algorithm is defined similarly but the points in the plane may be encoded in unary. A problem is *strongly NP-complete* if the existence of a pseudo-polynomial algorithm for that problem implies the existence of a polynomial algorithm for some NP-complete problem.

In the next section we show that both **RAF** and **RRAF** are strongly NP-complete. In Section 3 we show that **RRAF** is polynomially solvable for sets P satisfying a certain convexity condition. This is used to give a $1/2$ -approximate algorithm for **RAF** restricted to convex subregions of the lattice.

2 Complexity Results

2.1 RAF

We show that **RAF** is strongly NP-complete. (Clearly it is in NP.) This is achieved by showing that the existence of a pseudo-polynomial algorithm for **RAF** implies a polynomial algorithm to solve the stable set problem for a general graph. As usual, $\alpha(G)$ denotes the size of a largest stable set in G . We begin with a few definitions.

We say z *centres* a right angle with *endpoints* x and y if (x, z, y) forms a right angle triangle with hypotenuse \overline{xy} . For points x and y , denote by $C(\{x, y\})$, the circle whose diameter is determined by \overline{xy} . Then z centres a right angle with endpoints x, y if and only if $z \in C(\{x, y\})$. Thus a set of points S , is **raf** if and only if $C(\{x, y\}) \cap S = \{x, y\}$ for all $\{x, y\} \subseteq S$.

Theorem 2.1 *RAF is strongly NP-complete.*

Proof: Let $G = (\{1, \dots, n\}, E)$ be a graph with a maximum independent set of size α . We create an instance P of **RAF** of size $n + |E|$ for which $\alpha = |E| - \rho(P)$. Set

$V' = \{\bar{i} = (i, 0) : 1 \leq i \leq n\}$. Construct a set E' of points iteratively by initializing $E' = \phi$ and then repeating the following steps for each edge $e \in E$.

- Let $ij \in E$ be an edge not yet considered. Pick a point z in the first quadrant of $C(\{\bar{i}, \bar{j}\})$, that is belonging to $C^1(\{\bar{i}, \bar{j}\}) \equiv \{(x_1, x_2) \in C(\{\bar{i}, \bar{j}\}) : x_2 > 0, x_1 > (j - i)/2\}$, such that:

(*) z is only in the right triangle (\bar{i}, z, \bar{j})
(we call \bar{i}, \bar{j} the endpoints of z).

- Set $E' = E' \cup \{z\}$.

Let $P = V' \cup E'$.

Claim. $\rho(P) = |E| + \alpha(G)$.

Proof. By construction each z in P is in exactly one right angle triangle of P - the one associated with its endpoints (i.e. an edge of G). Thus for any independent subset I of G , $E' \cup \{\bar{i} : i \in I\}$ is raf. Hence $\rho(P) \geq |E| + \alpha(G)$.

Now let S be a maximum raf subset of P which maximizes $|S \cap E'|$. If there exists $z \in E' - S$, then by maximality both endpoints of z , say x and y are in S . But then $(S - \{x\}) \cup \{z\}$ is raf, a contradiction. Hence $E' \subseteq S$ and so for any $\bar{i}, \bar{j} \in S \cap V'$ we must have $ij \notin E$. Thus $|S| \leq |E'| + \alpha(G) = |E| + \alpha(G)$.

End of Claim.

Some details still remain. In particular, how do we construct E' in time polynomially bounded in n ? It suffices to show that the following problems can be solved in polynomial time:

- determining whether x and y are endpoints of a right angle with centre z ;
- augmenting E' .

We first note that if x, y and z are rational points, then the problem (a) can be answered by evaluating an equation of the form

$$(1) \quad (z_1 - ((y_1 - x_1)/2))^2 + (z_2 - ((y_2 - x_2)/2))^2 = (x_1 - y_1)^2 + (y_2 - x_2)^2.$$

Now consider problem (b), i.e., finding a point z satisfying (*). First we show that there is a polynomial $p(\cdot)$ such that at most $p(n)$ points on $C^1(\{\bar{i}, \bar{j}\})$ fail condition (*). Consider any candidate $z^{(1)}$, for augmenting E'_1 . Let the endpoints of $z^{(1)}$ be \bar{i} and \bar{j} , i.e., $z^{(1)} \in C^1(\{\bar{i}, \bar{j}\}) \cap Q$. The condition (*) is satisfied by $z^{(1)}$ if

$$(2) \quad z \notin C(\{u, v\}) \text{ for all } \{u, v\} \subseteq (E' \cup V') - \{\bar{i}, \bar{j}\}$$

and

$$(3) \quad |C(\{z, \bar{i}\}) \cap (C' \cup E')| = |C(\{z, \bar{j}\}) \cap (V' \cup E')| = 1.$$

(To check whether (2) and (3) hold requires at most $\binom{|V' \cup E'|}{2}$ evaluations of the form given in (1).) If one of the conditions fails, then pick another candidate $z^{(2)} \in C^1(\{\bar{i}, \bar{j}\}) - z^{(1)}$. Since any circle (line) intersects quadrant one of $C(\{\bar{i}, \bar{j}\}) - \{\bar{j}\}$ in at most one point we know that condition (2) (respectively (3)) can fail for at most $\binom{|V' \cup E'| - 2}{2}$ (respectively $2|V' \cup E'| - 2$) distinct points. Thus after at most $\binom{|V' \cup E'|}{2} = p(n)$ candidates we are able to augment E' .

Now we know that we need only generate a polynomial number of points on a circle for each new edge considered. We must show how to find these points and how to make sure that their sizes do not grow too quickly. Given any Pythagorean Triple (a, b, c) , where we assume $a \leq b$, the point

$$z = \left(j - (j - i)b_2/c^2, (j - i)ab/c^2 \right)$$

is rational and lies on $C(\{\bar{i}, \bar{j}\})$. Since the equations

$$a = 2m(m + 1), \quad b = (m + 1)^2 - m^2, \quad c = m^2 + (m + 1)^2$$

yield distinct Pythagorean Triples, we can always generate a z with unary representation bounded by a polynomial in n .

This completes the proof of Theorem 2.1. \square

A set P of points in $\mathbb{Z} \times \mathbb{Z}$ is *horizontally-convex* if for any two points of P which lie on a horizontal line, each lattice point between them is also in P . We define *vertically-convex* in a similar fashion and say that the subset P is *rectilinear-convex* if it is both horizontally and vertically-convex. We can modify the reduction in Theorem 2.1 so that each member in P is alone in its row and column. This is done by “tilting” the configuration slightly. We can also scale the resulting set of points so that it lies on the integer lattice. This shows that RAF remains NP-hard for rectilinear-convex regions.

2.2 RRAF

We show that RRAF is also NP-complete. We show in fact that this problem is equivalent to the domination problem for bipartite graphs. This latter problem is NP-complete. For $G = (V, E)$, a subset D of V is a *dominating set* of G if each vertex of $V - D$ is adjacent to some vertex of D . The *domination number* of G , denoted by $\gamma(G)$, is the size of a smallest dominating set of G .

We start by defining a bipartite graph $B(P)$, associated with a set of points P in the plane. The graph B has bipartition $(H(P), V(P))$. Roughly speaking, the members of $H(P)$ correspond to the horizontal lines in the plane which contain a point of P . Similarly $V(P)$ corresponds to the vertical lines which contain points of P . More precisely, we let $H(P) = \{b_{y_0} : \text{there exists } x \text{ such that } (x, y_0) \in P\}$, and $V(P) = \{a_{x_0} : \text{there exists } y \text{ such that } (x_0, y) \in P\}$ (where a and b are indeterminates). Also $E(P) = \{a_x b_y : (x, y) \in P\}$ and $B(P) = (H(P) \cup V(P), E(P))$. Clearly $B(P)$ is a bipartite graph. The next theorem shows how $\gamma(B)$ is related to $\rho_R(P)$.

Theorem 2.2 *For any set of points P , $\rho_R(P) + \gamma(B(P)) = |H(P) \cup V(P)|$.*

Proof: A subset S of P , is *rraf* if and only if the subset of $E(P)$ corresponding to S induces a subgraph of $B(P)$ without a path of length three. It is easy to see there exists a maximum such subset E' of $E(P)$ for which $B[E']$ (the graph induced by the edges E') has no isolated vertices. Hence if we take from each component of $B[E']$, a vertex of maximum degree we obtain a dominating set for the bipartite graph $B(P)$. Thus $\gamma(B(P)) \leq |H \cup V| - \rho_R(P)$. Conversely let D be a minimum dominating set of $B(P)$ and consider a minimal subset E' of $E(P)$ such that D is a dominating set of $B[E']$. Evidently $B[E']$ has no path of length 3, and so $\rho_R(P) \geq |H \cup V| - \gamma(B(P))$. Hence $\rho_R(P) + \gamma(B) = |H \cup V|$. \square

Corollary 2.3 *RRAF is strongly NP-complete.*

Proof: It is immediate that a pseudo-polynomial algorithm for RRAF implies the existence of a polynomial algorithm to solve the domination problem for bipartite graphs. Since this problem is NP-complete, the result follows. \square

By defining a certain type of weighted RAF problem, we obtain a problem equivalent to the minimum weight domination problem. Given weights on the x and y coordinates, the *induced weight* of an *rraf* subset S corresponds to summing the weights of the vertices in S . However, these weights depend on S as follows. If $(x, y) \in S$ is not alone in its row, then it assumes the weight of its y coordinate. Similarly, if there is another vertex in S with same x coordinate, then it assumes the weight of its x coordinate. If (x, y) is alone in its row and column, then its weight becomes the maximum of the corresponding coordinate weights. The induced weight RRAF problem is find a maximum induced weight *rraf* subset in a given set of points. We leave it as an exercise to show that Theorem 2.2 can be extended to give a similar relation between weighted dominating set problem and the induced weight RRAF problem.

3 RRAF in Horizontally Convex Sets

We now show that there is a polynomial time algorithm to solve RRAF if the input consists of a horizontally-convex subset of the lattice. In fact if P is horizontally convex, then (in $B(P)$) each node of $H(P)$ is adjacent to a “contiguous” block of vertices in $V(P)$. Such bipartite graphs are called *convex*. We give an extension of the cardinality algorithm outlined in [2] which finds a minimum weight dominating set in convex graphs in polynomial time.

Theorem 3.1 *The weighted domination problem for convex graphs can be solved in polynomial time.*

Corollary 3.2 *The problem RRAF restricted to horizontally-convex sets is solvable in polynomial time.*

We now give an algorithm for finding a minimum weight dominating set in a convex graph $B = (V = (I \cup J), E)$ (where J has some ordering which shows B to be convex). Let J_1, J_2, \dots, J_m be a partition of J such that each J_i contains vertices whose neighbourhoods are identical and is maximal subject to its forming a contiguous block relative to the ordering on J . We let $I_i = N(J_i)$, $J(i) = J_1 \cup \dots \cup J_i$ and $F_i = \{x \in I : N(x) \subseteq J(i)\}$. Note that for

a node in F_i , either it is in a dominating set or a neighbour in $J(i)$ must be in a dominating set. We assume without loss of generality, that positive weights $w \in \mathbb{Z}^V$ are assigned to the vertices.

We say that a set $D \subseteq V$ is *type* (k, t_k) if

- (4) $D \subseteq I \cup J(k)$ and
- (5) D dominates all vertices $J(t_k)$,
- (6) D dominates all *forced* vertices F_k .

It is easy to see that any minimum weight dominating set will contain either 0, 1 or $|J_k|$ vertices from J_k . A type (k, t_k) set is said to *obey* (c_k, s_k) if

- (7) $J_i \cap D = \phi$ for $i > s_k$,
- (8) $|J_{s_k} \cap D| = 1$ if $c_k = 1$ and $s_k > 0$,
- (9) $|J_{s_k} \cap D| = 0$ if $c_k = 0$ or $s_k = 0$,
- (10) $J_{s_k} \subseteq D$ if $c_k = 2$ and $s_k > 0$.

We construct a collection of sets $D_k(c_k, s_k, t_k)$, $k = 1, \dots, m$, $c_k = 0, 1, 2$, $s_k = 0, \dots, k$ and $t_k \leq k$ such that:

- (11) $D_k(c_k, s_k, t_k)$ is a minimum weight (k, t_k) set obeying (c_k, s_k)
- (12) subject to minimality in (11), $D_k(c_k, s_k, t_k)$ dominates $J(i)$
for maximum $i \geq t_k$.

Note that from such a collection we implicitly know a collection of sets $D_k(t_k)$ which are the minimum weight type (k, t_k) sets which also satisfy (12). We now show how to construct this collection in polynomial time. In the following we let z_i be any minimum weight vertex of I_i which maximizes $r(z_i)$. It is easy to see that the initial sets are obtained as follows: $D_1(2, 1, 0) = D_1(2, 1, 1) = J_1$; $D_1(1, 1, 0) = \{z\}$ for any $z \in J_1$ of minimum weight; $D_1(1, 1, 1) = D_1(1, 1, 0) \cup \{z_1\}$; $D_1(0, 0, 0) = F_1$ and finally

$$D_1(0, 0, 1) = \begin{cases} F_1 & \text{if } F_1 \neq \phi \\ z_1 & \text{otherwise} \end{cases}$$

We proceed to show how to construct a set $D_{k+1}(c_{k+1}, s_{k+1}, t_{k+1})$ for $k \geq 1$.

Case 1 $k + 1 > s_{k+1}$.

First let F be the subset of F_{k+1} consisting of vertices adjacent to J_{k+1} but not to $J_{s_{k+1}}$. If $F \neq \phi$, then let j be the smallest integer such that some vertex of F is adjacent to J_{j+1} . We then have:

$$(I) \quad D_{k+1}(c_{k+1}, s_{k+1}, t_{k+1}) = D_l(c_{k+1}, s_{k+1}, l)$$

where $l = \min\{j, t_{k+1}\}$. When $F = \phi$, if either $D_k(c_{k+1}, s_{k+1}, t_{k+1})$ dominates J_{k+1} or $t_{k+1} \leq k$, we have:

$$(II) \quad D_{k+1}(c_{k+1}, s_{k+1}, t_{k+1}) = D_k(c_{k+1}, s_{k+1}, t_{k+1}).$$

Otherwise $t_{k+1} = k+1$ and $D_k(c_{k+1}, s_{k+1}, t_{k+1})$ does not dominate J_{k+1} and $D_{k+1}(c_{k+1}, s_{k+1}, t_{k+1})$ becomes an appropriate minimum weight set in:

(III) $\{D_k(c_{k+1}, s_{k+1}, j) \cup \{z\} : z \in I_{k+1} \text{ is a minimum weight vertex with } l(z) = j + 1 \text{ and subject to this maximizes } r(z)\}$.

It is easy to see that in Cases (I) and (II), the sets satisfy (11) and (12). Now consider the third case and let D be any set satisfying (11) and (12) and let $z \in D \cap I_{k+1}$. It follows that $D - \{z\}$ is a type $(k, l(z) - 1)$ set obeying (c_{k+1}, s_{k+1}) . Hence there is a set of at most the weight of D which is considered in (III). Since each of the sets in (III) is *feasible*, we easily deduce that we can find a new set satisfying (11) and so in fact also (12).

Case 2 $s_{k+1} = k + 1, c_{k+1} = 1$.

If $D_k(t_{k+1})$ dominates J_{k+1} or if $t_{k+1} \leq k$, then we have:

(IV) $D_{k+1}(1, s_{k+1}, t_{k+1}) = D_k(t_{k+1}) \cup \{z\}$

for any $z \in J_{k+1}$ of minimum weight. So suppose $t_{k+1} = k + 1$ and $D_k(t_{k+1})$ does not dominate J_{k+1} . We can argue as in (III) to show that a suitable set for our collection can be found in:

(V) $\{D_k(c, s, t) \cup \{z\} \cup \{z'\} : \text{for all minimum weight } z' \in J_{k+1} \text{ and } z \in I_{k+1} \text{ with } l(z) = t + 1 \text{ and all possible choices of } c, s\}$.

Case 3 $s_{k+1} = k + 1, c_{k+1} = 2$.

In this case it is easy to see that

$$D_{k+1}(2, k + 1, t_{k+1}) = D_k(t_{k+1}) \cup J_{k+1}$$

gives a suitable set.

The procedure described above can be performed in polynomial time. Hence we can find a minimum weight dominating set in a convex bipartite graph in polynomial time and so we can also compute ρ_R for horizontally-convex sets. In general, an algorithm which computes ρ_R does not give a constant bound for the parameter ρ .

Theorem 3.3 *There does not exist a positive constant C such that $\rho(P) \geq C\rho_R(P)$ for every rectilinear-convex subset P of $\mathbb{Z} \times \mathbb{Z}$.*

Proof: For any nonnegative integer n define S_n as follows. Let L_n consist of points $\{(i, i) : i = 1, \dots, n\}$; define $L_n^k = \{x + (k, -k) : x \in L_n\}$. Set $S_n = \bigcup_{k=0}^{n-1} L_n^k$. Each S_n is rectilinear-convex, indeed each point is alone in its row and column. Hence $\rho_R(S_n) = |S_n| = n^2$, but S_n is just a "tilted" $n \times n$ grid and so $\rho(S_n) = 2n - 2$. \square

If P is forced to be convex however, such a bound for $\rho(P)$ does exist. (A subset of the integer lattice is *convex* if it is the set of integer points contained in some convex polygon.)

Theorem 3.4 *For any convex subset P of $\mathbb{Z} \times \mathbb{Z}$, $\rho(P) \geq (1/2)\rho_R(P)$.*

Proof: Consider S , a maximum rraf subset of P . Define S_h to be the points of S for which there is another vertex of S with the same y coordinate. Similarly define S_v . Without loss of generality $|S_h| \geq |S_v|$. Set $S' = S - S_v$. Note that no vertical line contains more than one point of S' . Order the y coordinates of the points in order of increasing x coordinate, say y_1, y_2, \dots, y_k and suppose S has been chosen to maximize the length of the initial monotone subsequence of $\{y_i\}_{i=1}^k$. We claim that $\{y_i\}_{i=1}^k$ itself is monotone. If not,

then without loss of generality there is some i such that $y_1 \leq y_2 \dots \leq y_i, y_{i+1} < y_i$ and either $y_{i-1} \leq y_{i+1}$ or $y_{i+1} \leq y_{i-1}$. We consider only the case where $y_{i+1} \geq y_{i-1}$, the other case being similar.

For each i , let x_i be the x coordinate of the i^{th} point in S' . By convexity, every lattice point contained in the triangle determined by $(x_{i-1}, y_{i-1}), (x_i, y_i)$ and (x_{i+1}, y_{i+1}) is in P . In particular $(x_i, y_{i+1}) \in P$. Thus $(S - \{(x_i, y_i)\}) \cup \{(x_i, y_{i+1})\}$ contradicts the choice of S . Hence $\{y_i\}_{i=1}^k$ is monotone and so S' is rraf and the theorem follows. \square

Corollary 3.5 *There is a $\frac{1}{2}$ -approximate algorithm for RAF restricted to convex regions of the integer lattice.*

Proof: Given S a maximum rraf subset, we can polynomially construct S' as in Theorem 3.4. \square

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