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A Pointwise Criterion for Controller Robustness

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We present a pointwise criterion for controller robustness with respect to stability. The term 'point' here refers to complex frequency in the right half plane. The proposed test is based on the concept of the minimal angle between subspaces determined by the plant and the compensator. The test leads to separate balls of uncertainty at each frequency, and may therefore help to reduce conservativeness in the analysis of robustness.

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1. Introduction

Given a plant and a stabilizing controller for it, one defines the robustness of the controller (with respect to stability) as the smallest perturbation of the plant which may cause the closed-loop system to become unstable. Of course, this definition depends on the measure that is chosen for the perturbations. Several distance notions for linear time-invariant systems have been proposed, of which the so-called gap metric [3, 24] has gained much popularity because it is relatively easy to compute [5] and lends itself well to optimization [6]. However, the gap between two systems is a single number, whereas the uncertainty of a model is often seen as a frequency-dependent quantity. A certain amount of frequency dependence can be obtained by introducing suitable weight functions, as for instance in [16]. Here we shall propose a criterion which addresses the dependence on frequency directly by defining a separate ball of allowable uncertainty at every point in the closed right half plane. The proof of the criterion is very simple; nevertheless, it is suggested that the proposed test is a natural and useful tool in frequency-dependent robustness analysis.

2. Robustness of complementarity

The robustness criterion to be presented below will be based on distance notions for subspaces of a finite-dimensional unitary space. In particular, we shall be interested in conditions which will guarantee that two complementary subspaces remain complementary when one of the subspaces is perturbed. To measure the size of the perturbation, we shall use the gap function introduced in [19] and [13]. Let X be a (real or complex) Hilbert space and let Y_1 and Y_2 be closed subspaces of X. Denote the orthogonal projections onto Y_1 and Y_2 by P_1 and P_2 respectively. The *gap* $\delta(Y_1, Y_2)$ between Y_1 and Y_2 is defined by

$$\delta(Y_1, Y_2) = \|P_1 - P_2\| \tag{2.1}$$

or equivalently by

$$\delta(Y_1, Y_2) = \max(\sup\{\|(I-P_2)x\| \mid x \in Y_1,\ \|x\| = 1\},\ \sup\{\|(I-P_1)x\| \mid x \in Y_2,\ \|x\| = 1\}). \tag{2.2}$$

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For a proof that the two expressions are indeed the same, see [1, § 34], or [12, § 15.3]. To express a result on robustness of complementarity, we need a second notion which we take from [8]. The *minimal angle* $\phi(Y, Z)$ between two nonzero subspaces Y and Z of a Banach space X is defined by

$$\sin\phi(Y, Z) = \inf\{\|y - z\| \mid y \in Y,\ z \in Z,\ \max(\|y\|, \|z\|) = 1\}, \quad 0 \leqslant \phi(Y, Z) \leqslant \frac{\pi}{2} \cdot \quad (2.3)$$

In 1963, E. Berkson proved a result on robustness of complementarity of subspaces of Banach spaces which, when specialized to the finite-dimensional case, comes down to the following.

THEOREM 2.1 [2, Thm. 5.2] *Let Y and Z be complementary nontrivial subspaces of a finite-dimensional normed linear space X. Every subspace Y' that satisfies*

$$\delta(Y, Y') < \sin\phi(Y, Z) \quad (2.4)$$

is complementary to Z.

We shall be interested below in the case in which X is a unitary space. For this situation, we want to obtain expressions for the minimal angle which lend themselves for easy computation, and we want to establish whether the bound given by Berkson is sharp.

First of all, let us note that an alternative expression for the minimal angle in the Hilbert space case is given by

$$\sin\phi(Y, Z) = \min(\inf\{\|(I - P_Z)y\| \mid y \in Y,\ \|y\| = 1\}, \inf\{\|(I - P_Y)z\| \mid z \in Z,\ \|z\| = 1\}) \quad (2.5)$$

where P_Y and P_Z denote the orthogonal projections onto Y and Z respectively. Comparing this with the expression given in (2.2) for the gap, we obtain an easy proof of Berkson's theorem for the unitary case. Indeed, since $\sin\phi(Y, Z)$ can at most be equal to one, it follows from the inequality (2.4) that the gap between Y and Y' must be less than one and hence that the dimensions of Y and Y' must be the same. Suppose now that Y' and Z would not be complementary; then there would be a vector z of norm 1 in $Y' \cap Z$. Since we have on the one hand

$$\delta(Y, Y') \geqslant \|(I - P_Y)z\| \quad (2.6)$$

and on the other hand

$$\sin\phi(Y, Z) \leqslant \|(I - P_Y)z\|, \quad (2.7)$$

this would contradict (2.4).

When Y and Z are complementary subspaces, there is a projection onto Z along Y. We now state a formula which uses this projection.

PROPOSITION 2.2 *Let Y and Z be complementary nontrivial subspaces of a unitary space X. Let $N: Z^{\perp} \to Z$ denote the restriction to $Z^{\perp}$ of the projection onto Z along Y. We have*

$$\inf\{\|(I - P_Z)y\| \mid y \in Y,\ \|y\| = 1\} = \frac{1}{\sqrt{1 + \|N\|^2}} \quad (2.8)$$

PROOF Write

$$\tau^2 = \sup\{\|P_Z y\|^2 \mid y \in Y,\ \|y\| = 1\}. \quad (2.9)$$

Since we have

$$\|(I - P_Z)y\|^2 + \|P_Z y\|^2 = 1 \quad (2.10)$$

for all y with $\|y\| = 1$, it is clear that

$$\inf\{\|(I - P_Z)y\| \mid y \in Y,\ \|y\| = 1\} = \sqrt{1 - \tau^2}. \quad (2.11)$$

Let us evaluate τ^2. Note that for $x \in Z^{\perp}$, the vector Nx is defined as the unique element of Z for which we have $x - Nx \in Y$. If we take $y \in Y$ and apply this rule to $x = (I - P_Z)y$, we see that

$$N(I - P_Z)y = -P_Z y. \tag{2.12}$$

Let $y \in Y$ have norm 1. We can then write:

$$\begin{aligned} \|P_Z y\|^2 &= \|N(I - P_Z)y\|^2 \leqslant \|N\|^2 \|(I - P_Z)y\|^2 \leqslant \\ &= \|N\|^2(1 - \|P_Z y\|^2) = \|N\|^2 - \|N\|^2\|P_Z y\|^2 \end{aligned} \tag{2.13}$$

so that

$$\|P_Z y\|^2 \leqslant \frac{\|N\|^2}{1 + \|N\|^2}. \tag{2.14}$$

Since $I - P_Z$ is injective as mapping from Y to $Z^{\perp}$ and since $\dim Y = \dim Z^{\perp}$, the mapping $I - P_Z$ maps Y onto $Z^{\perp}$. Consequently, we may choose $y \in Y$ such that $(I - P_Z)y$ is a maximal vector for N, and then equality will hold in the above. We may therefore conclude that

$$\tau^2 = \frac{\|N\|^2}{1 + \|N\|^2}. \tag{2.15}$$

With (2.11), this produces (2.8).

This formula can be used to show that the two items of which the minimum has to be taken in (2.5) are in fact equal to each other when the subspaces Y and Z are complementary.

COROLLARY 2.3 *Let Y and Z be complementary nontrivial subspaces of a unitary space X. Then we have*

$$\inf\{\|(I - P_Z)y\| \mid y \in Y, \|y\| = 1\} = \inf\{\|(I - P_Y)z\| \mid z \in Z, \|z\| = 1\} \tag{2.16}$$

where P_Y and P_Z denote the orthogonal projections onto Y and Z respectively.

PROOF Let $N: Z^{\perp} \to Z$ denote the restriction to $Z^{\perp}$ of the projection onto Z along Y as in the proposition above, and let $M: Y^{\perp} \to Y$ denote the restriction to $Y^{\perp}$ of the projection onto Y along Z. Note that Z and $Y^{\perp}$ are dual spaces with respect to the duality induced by the inner product on X; the same holds for Y and $Z^{\perp}$. For all $x_1 \in Y^{\perp}$ and $x_2 \in Z^{\perp}$, we have

$$<Mx_1, x_2> = <x_1, x_2> = <x_1, Nx_2>. \tag{2.17}$$

This means that the operators M and N are adjoint operators, and consequently their norms must be equal. The claimed result now follows from the proposition.

From this corollary and from formula (2.5), we immediately obtain four expressions for the minimal angle between two complementary subspaces in terms of various projections. The minimal singular value of an operator T will be denoted by $\sigma_{\min}(T)$.

THEOREM 2.4 *Let Y and Z be complementary nontrivial subspaces of a unitary space X. Let P_Y and P_Z denote the orthogonal projections onto Y and Z respectively, let P_Y^Z denote the projection onto Y along Z, and let P_Z^Y denote the projection onto Z along Y. Then we have*

$$\begin{aligned} \sin\phi(Y, Z) &= \sigma_{\min}((I - P_Z)|_Y) \\ &= \sigma_{\min}((I - P_Y)|_Z) \\ &= \frac{1}{\sqrt{1 + \|P_Z^Y|_{Z^{\perp}}\|^2}} \\ &= \frac{1}{\sqrt{1 + \|P_Y^Z|_{Y^{\perp}}\|^2}}. \end{aligned} \tag{2.18}$$

In terms of matrix representations, one obtains the following formula for the minimal gap between two subspaces.

LEMMA 2.5 *Let Y and Z be complementary nontrivial subspaces of $\mathbb{C}^{m+p}$, with $\dim Y = m$ and $\dim Z = p$. Let A and B be normalized image and kernel representations for Y and Z respectively; that is, we require*

$$A : \mathbb{C}^m \to \mathbb{C}^{m+p}, \quad A^*A = I_m \tag{2.19}$$

and

$$B : \mathbb{C}^{m+p} \to \mathbb{C}^m, \quad BB^* = I_m. \tag{2.20}$$

Under these conditions, we have

$$\sin\phi(Y, Z) = \sigma_{\min}(BA). \tag{2.21}$$

PROOF Note that the elements of norm 1 in Y are exactly those of the form Au where $u \in \mathbb{C}^m$ has norm 1. Moreover, we have $I - P_Z = B^*B$. Therefore, it follows from Thm. 2.4 that

$$\sin\phi(Y, Z) = \sigma_{\min}(B^*BA) = \sqrt{\sigma_{\min}(A^*B^*BA)} = \sigma_{\min}(BA). \tag{2.22}$$

Finally, we show that the bound given by Berkson is sharp, at least in the unitary case.

THEOREM 2.6 *Let Y and Z be complementary nontrivial subspaces of a unitary space X. We have*

$$\inf\{\delta(Y, Y') \mid Y' \cap Z \neq \{0\}\} = \sin\phi(Y, Z). \tag{2.23}$$

PROOF It follows from Thm. 2.1 that the left hand side in the above equation cannot be less than the right hand side; so it suffices to construct a subspace Y' that has a nontrivial intersection with Z and whose distance to Y, as measured by the gap, is equal to $\sin\phi(Y, Z)$. (This will then also show that the infimum in (2.23) is actually a minimum.) Note that the construction is trivial when $\sin\phi(Y, Z) = 1$.

Choose a unitary space U and mappings $A : U \to X$ and $B : X \to U$ such that $Y = \operatorname{im} A$, $Z = \ker B$, $A^*A = I$ and $BB^* = I$. We know from Lemma 2.5 that $\sin\phi(Y, Z)$ is equal to the smallest singular value of BA. Let us simply write σ for this smallest singular value. Take a vector $u \in U$ such that

$$A^*B^*BAu = \sigma^2 u, \quad \|u\| = 1. \tag{2.24}$$

We shall need the norms of $P_Z Au$ and of $P_Y P_Z Au$. One has

$$\begin{aligned}\|P_Z Au\|^2 &= \langle P_Z Au, Au\rangle = \langle (I - B^*B)Au, Au\rangle = \\ &= \langle Au, Au\rangle - \langle A^*B^*BAu, u\rangle = 1 - \sigma^2\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\|P_Y P_Z Au\|^2 &= \langle P_Y P_Z Au, P_Z Au\rangle = \langle AA^*(I - B^*B)Au, (I - B^*B)Au\rangle = \\ &= \langle A^*(I - B^*B)Au, A^*(I - B^*B)Au\rangle = (1 - \sigma^2)^2.\end{aligned} \tag{2.26}$$

Therefore,

$$\|P_Z Au\| = \sqrt{1 - \sigma^2}, \qquad \|P_Y P_Z Au\| = 1 - \sigma^2. \tag{2.27}$$

From this we can also compute the norm of $(I - P_Y)P_Z Au$:

$$\|(I - P_Y)P_Z Au\| = \sqrt{\|P_Z Au\|^2 - \|P_Y P_Z Au\|^2} = \sigma\sqrt{1 - \sigma^2}. \tag{2.28}$$

We see that we have a nontrivial problem ($\sigma \neq 1$) only if $P_Z Au$ neither belongs to Y nor to $Y^\perp$. Under these conditions, the subspace Y' defined by

$$Y' = \operatorname{span} P_Z Au \oplus [(P_Z Au)^\perp \cap Y] \tag{2.29}$$

satisfies $\dim Y' = \dim Y$.

We also claim that $Y' \cap Y^{\perp} = \{0\}$. To see this, suppose that $y' \in Y'$ belongs to $Y^{\perp}$. Write $y' = x - y$ where x is a scalar multiple of $P_Z Au$ and y belongs to $(P_Z Au)^{\perp} \cap Y$. From $y' \in Y^{\perp}$, we get $P_Y x = y$ and we can write

$$\|P_Y x\|^2 = <P_Y x, x> = <y, x> = 0. \tag{2.30}$$

This shows that, indeed, $y' = 0$. From the equalities $\dim Y' = \dim Y$ and $Y' \cap Y^{\perp} = \{0\}$, it follows furthermore that $Y' + Y^{\perp} = X$ and hence that $(Y')^{\perp} \cap Y = \{0\}$. We can therefore conclude that $\delta(Y, Y') < 1$ (see [7, IV.151]).

In this situation, the two items over which the maximum is taken in (2.2) are equal to each other ([12, p. 207]; see also [6, Prop. 3]), and so it will suffice to compute one of the two. It is clear that the maximal value of the quotient $\|(I - P_Y)y'\| / \|y'\|$, for nonzero $y' \in Y'$, is assumed for $y' = P_Z Au$. Using (2.27) and (2.28), we find

$$\delta(Y, Y') = \frac{\|(I - P_Y)P_Z Au\|}{\|P_Z Au\|} = \sigma \tag{2.31}$$

which is what we had to prove.

3. Main result

Let us now consider the problem of stabilization by feedback for linear time-invariant finite-dimensional systems. Following the framework of [22], we shall represent such systems in the form

$$R(\frac{d}{dt})w = 0 \tag{3.1}$$

where $R(s) \in \mathbb{R}^{p \times q}[s]$ is a polynomial matrix of full row rank. The vector w contains input and output variables, but for our main result there is no need to be specific about which components of w are considered as inputs and which are considered as outputs. We will consider the application of a dynamic compensator simply as an operation of adding equations for the external variables:

$$Q(\frac{d}{dt})w = 0 \tag{3.2}$$

where $Q(s) \in \mathbb{R}^{m \times q}[s]$ has full row rank, and $m = q - p$. The closed-loop system is given by

$$\begin{bmatrix} R(s) \\ Q(s) \end{bmatrix} w = 0. \tag{3.3}$$

The feedback loop is said to be *well-posed* if the square polynomial matrix $[R^{\mathrm{T}}(s) \quad Q^{\mathrm{T}}(s)]^{\mathrm{T}}$ is nonsingular, and the closed-loop system is said to be *stable* if this matrix has no zeros in the closed right half plane. Making use of a standard criterion for invertibility of compound matrices, we immediately obtain the following robustness theorem.

Theorem 3.1 *Let a linear system be given by (3.1), and suppose that the system is stabilized by the compensator given by (3.2). The same compensator will also stabilize the system given by*

$$\tilde{R}(s)w = 0 \tag{3.4}$$

($\tilde{R}(s) \in \mathbb{R}^{p \times q}[s]$ of full row rank) provided the following condition is satisfied for all s such that $\mathrm{Re}\, s \geq 0$:

$$\delta(\ker R(s), \ker \tilde{R}(s)) < \sin \phi(\ker R(s), \ker Q(s)). \tag{3.5}$$

Proof For each s, the matrix $[\tilde{R}^{\mathrm{T}}(s) \quad Q^{\mathrm{T}}(s)]^{\mathrm{T}}$ is nonsingular if and only if the subspaces $\ker \tilde{R}(s)$ and $\ker Q(s)$ are complementary. The result therefore follows from Thm. 2.1.

It follows from Thm. 2.6 that the above bound is the best pointwise bound that can be given in terms of the gap metric. Lemma 2.5 provides a method to compute the minimal angle. For ways of computing the gap, see [9, §§2.4, 12.4].

4. Remarks

1. The system given by (3.1) is said to be *stabilizable* if $R(s)$ has full row rank as a matrix over $\mathbb{C}$ at every point s with $\operatorname{Re} s \geqslant 0$. It is easily seen that this is indeed a necessary and sufficient condition for a stabilizing compensator of the form (3.2) to exist. The condition of stabilizability can also be formulated as the requirement that $\ker R(s)$ is of constant dimension m in the right half plane.

2. We see from Lemma 2.5 that, to compute the minimal angle, it is convenient to have a kernel representation for one of the two subspaces $\ker R(s)$ and $\ker Q(s)$, and an image representation for the other. Such representations are easily obtained if an 'MA' representation in the sense of [23] is used; that is, we represent for instance $\ker Q(s)$ as $\operatorname{im} M(s)$. In the more common input/output terminology, this means that a left fractional representation is used for the plant, and a right fractional representation for the controller.

3. Note that $\ker R(s)$, for s with $\operatorname{Re} s \geqslant 0$, will not be changed if $R(s)$ is premultiplied by a rational H_∞-matrix. This means that also RH_∞-matrices (as for instance in [21]) can be used to define the mapping $s \mapsto \ker R(s)$ on $\{s \mid \operatorname{Re} s \geqslant 0\}$. The coprimeness condition from the input/output framework is equivalent to the stabilizability condition mentioned under 1.

4. Let us denote the set $\{s \in \mathbb{C} \mid \operatorname{Re} s \geqslant 0\} \cup \{\infty\}$ by $\mathbb{H}$ (for 'hemisphere'). It can be seen as in [14] that the mapping $s \mapsto \ker R(s)$, for a stabilizable system given by $R(s)$, can be extended to a continuous mapping from $\mathbb{H}$ to the Grassmannian manifold $G^m(\mathbb{C}^q)$ of m-dimensional subspaces of the q-dimensional complex space. It may also be verified that (for given $R(s)$, $\tilde{R}(s)$, and $Q(s)$) both sides of the inequality (3.5) represent continuous functions on $\mathbb{H}$. Since $\mathbb{H}$ is compact, the two functions must have a maximum and a minimum on $\mathbb{H}$. Define

$$d(R, \tilde{R}) = \max_{s \in \mathbb{H}} \delta(R(s), \tilde{R}(s)) \tag{4.1}$$

and

$$s(R, Q) = \min_{s \in \mathbb{H}} \sin\phi(R(s), Q(s)). \tag{4.2}$$

An obvious corollary of our main result is then the following.

Corollary 4.1 *Let a linear system be given by (3.1), and suppose that the system is stabilized by the compensator given by (3.2). The same compensator will also stabilize the system given by (3.4) if*

$$d(R, \tilde{R}) < s(R, Q). \tag{4.3}$$

Of course, this result will in general be conservative with respect to the pointwise criterion. It was recently shown in [18] (see also [17] for the scalar case) that the metric on plants given by the distance measure (4.1) is topologically equivalent to the graph metric introduced by Vidyasagar [20], which in its turn is topologically equivalent to the gap metric of Zames and El-Sakkary, as shown in [25].

5. Suppose now that we define a regular input/output structure for (3.1), that is, we write

$$R(s) = [D(s) \quad -N(s)] \tag{4.4}$$

where $D(s)$ is square and invertible. The system is said to be *stable* if $D(s)$ is invertible as a matrix over $\mathbb{C}$ for each s with $\operatorname{Re} s \geqslant 0$. In this case, we have the representation (for $\operatorname{Re} s \geqslant 0$)

$$\ker R(s) = \operatorname{im} \begin{bmatrix} G(s) \\ I \end{bmatrix} \tag{4.5}$$

where $G(s)$ is the transfer matrix defined by $G(s) = D^{-1}(s)N(s)$. If $G(s)$ is proper rational, then the abobe representation can also be used at $s = \infty$. It is a special feature of the scalar case that the same representation may even be used for unstable systems. This is due to the fact that $G^1(\mathbb{C}^2)$ is homeomorphic to the Riemann sphere via the identifications

$$\text{span}\begin{bmatrix}\alpha\\1\end{bmatrix} \mapsto \alpha, \quad \text{span}\begin{bmatrix}1\\0\end{bmatrix} \mapsto \infty. \tag{4.6}$$

This makes it possible to develop a theory for the scalar case based on transfer functions (cf. for instance [4, 17]). The metric on the Riemann sphere corresponding to the gap on $G^1(\mathbb{C}^2)$ is known as the *chordal metric* (see for instance [10]). A similar approach, however, does not seem feasible for the general (unstable, multivariable) case.

6. Obviously, an input/output system represented by (4.4) is stable if and only if it is stabilized by the compensator given by $Q(s) = [0 \quad I]$. The criterion of Thm. 3.1 can therefore be used to obtain results on robust stability. In this connection it is useful to note the following consequence of Lemma 2.5: if $D(s)$ is stable, then for all s with $\mathrm{Re}\, s \geq 0$ one has

$$\sin\phi(\ker[D(s) \quad -N(s)], \ker[0 \quad I]) = \|I + (G(s))^*G(s)\|^{-\frac{1}{2}}. \tag{4.7}$$

In the scalar case we write $g(s)$ rather than $G(s)$, and we obtain the following corollary of Thm. 3.1 in which we use χ to denote the chordal metric on the Riemann sphere.

COROLLARY 4.2 *Let the rational functions $g(s)$ and $h(s)$ represent scalar stabilizable systems. If $g(s)$ is stable and*

$$\chi(g(s), h(s)) < \frac{1}{\sqrt{1 + |g(s)|^2}} \tag{4.8}$$

for all s with $\mathrm{Re}\, s \geq 0$, then $h(s)$ is also stable.

This result appears in [4].

7. We have seen that (3.5) gives the sharpest possible pointwise bound in terms of the gap metric on $G^m(\mathbb{C}^q)$. In specific applications, however, there may be good reasons to use a different metric. In that case one would of course also be interested in obtaining a sharpest bound as in Thm. 2.6. At the present, little seems to be known in this direction. Modifications of the theorem could be made in at least the following respects.

(i) Use of the gap with a different vector norm. If the norm does not correspond to an inner product, the Banach space version of the definition of the gap has to be used (see for instance [11, Ch. IV, §2]).

(ii) Use of a different distance notion on $G^m(\mathbb{C}^q)$. An example is the distance notion proposed in [15]: for subspaces Y and Y' of equal dimension, define

$$r_0(Y, Y') = \inf\{\|I - A\| \mid A : \mathbb{C}^q \mapsto \mathbb{C}^q \text{ invertible, and } AY' = Y\}. \tag{4.9}$$

Note that this definition uses an operator norm, for which again various choices are possible.

(iii) In some applications it may be reasonable to take the infimum in (2.23) over a restricted set of subspaces. This happens for instance when we know that the perturbed system is dissipative.

Note that all modifications may be made dependent on s. This allows considerable flexibility in dealing with specific features that may be present in particular applications. It remains to be seen, though, to what extent it will be possible to obtain sharp and computationally tractable bounds in various possible situations.

8. The extension of the material from this paper to the discrete-time case is straightforward. Extension to infinite-dimensional systems is perhaps less straightforward, but seems certainly possible. A much more open question is whether it is possible to do some sort of optimization with respect to the right hand side of the inequality in (3.5). Another interesting subject of study would be to describe the effects of performance constraints on the pointwise minimal angle between plant and controller.

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