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Zero Drift Random Walk on $\mathbb{N}\times\mathbb{N}$ with Reflection, Ergodicity Conditions

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Summary
The present study is an addition to author's study [1]. It concerns the explicit formulation of the conditions for the random walk to be positive, null or nonrecurrent in terms of the first and second moments of the various one-step displacement vectors.

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1. Introduction

The present study is a continuation of author's previous study: *The two-dimensional random walk, its hitting process and its classification,* [1]. This continuation concerns an explicit formulation of the necessary and sufficient conditions for the two-dimensional random walk with zero drifts on the lattice in $\mathbb{R}_2^+$ with reflecting boundary, cf. [1] section 6. In section 6 of [1] such conditions have been formulated in theorem (6.2) [1]; unfortunately, a condition used in its proof has not been mentioned in its formulation, see errata list below. In section 2 of this study we present the theorem which formulates in terms of the first and second moments of the various one-step displacement vectors at points on the boundary and at those of the interior of the state space the conditions for the random to be positive or null recurrent; those for nonrecurrence have been formulated in theorem 6.1 of [1]. Next to this result explicit expressions are given for the first moments of the distribution of the *hitting point* in the case of positive recurrence.

We conclude this introduction with several remarks on the existant litterature, notation and errata to be made in [1].

Remark 1.1. In [2], Fayolle, Malyshev and Menshikov investigate the same problem, By using Lyapounov functions they derive conditions for the classification of the random walk. Their study made the present author to reconsider his results in [1], the outcome is the present study of which the results differ essentially with those of [2]. □

Remark 1.2. In author's study [1] theorem 2.3 of [1] concerning the finiteness of the first entrance time into the boundary out from a point of the interior plays an essential role. This theorem formulated and proved in [3], contains a result derived also in [4], the present author was and unfortunately could not be aware of it. □

Remark 1.3. Notation. In the present study we shall use the same notations, definitions of symbols and problem formulation as in [1]. References to formulas and theorems in [1] are indicated by adding the symbol [1]. □

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REMARK 1.4. Errata in text of [1].

page 27. in (6.32): replace second + by -,

in (6.33): "-" signs in front of a_1 and of a_2 should all be replaced by "+" signs,

In theorem 6.2:
"three" to be replaced by "four"

in theorem 6.2:
condition iv. should read:

iv. the linear equations (6.32) have a finite, unique solution with
$E\{\mathbf{h}_1\} > K_1(1)$, $E\{\mathbf{h}_2\} > K_2(1)$.

page 28. 11th line from above: (7.43) $\rightarrow$ (6.35).

in last line of proof: iii. $\rightarrow$ iii. or iv.

third line above (6.38) should read:

So if (6.34) has a solution with $E\{\mathbf{h}_1\} > K_1(1)$ then necessarily ...

2. CONDITIONS FOR POSITIVE AND NULL RECURRENCE.

For the problem formulation and meaning of symbols see [1].

In theorem 6.1 of [1] the classification of the $\mathbf{z}_n$-process has been described apart from the case with $\alpha_{12} < 0$ and the $\mathbf{k}_m$-process being positive recurrent; in that case the $\mathbf{z}_n$-process can be null recurrent of positive recurrent. In this section we shall formulate the condition which distinguish these possibilities. So it will be *assumed* here that

$$(1-\mu_1)(1-\nu_2) > \mu_2\,\nu_1, \quad \mu_1 < 1, \quad \nu_2 < 1. \tag{2.1}$$

From (2.1) and theorem 6.1 it follows that the $\mathbf{k}_m$-process possesses a stationary distribution and for this distribution the probabilities K_0, $K_1(1)$ and $K_2(1)$ are given by [1] (6.17).

We introduce two stochastic variables $\mathbf{u}$ and $\mathbf{v}$ defined as follows:

$$\begin{array}{llll} \mathbf{u} := \boldsymbol{\xi}_0 & \text{and } \mathbf{v} := \boldsymbol{\eta}_0 & \text{with probability} & K_0, \\ := \boldsymbol{\xi}_1 - 1 & \quad ,, \quad := \boldsymbol{\eta}_1 & \quad ,, \quad ,, & K_1 \equiv K_1(1), \\ := \boldsymbol{\xi}_2 & \quad ,, \quad := \boldsymbol{\eta}_2 - 1 & \quad ,, \quad ,, & K_2 \equiv K_2(1). \end{array} \tag{2.2}$$

From (2.2) and [1] (6.33) it follows readily by using [1] (6.11) that (note remark 1.4, errata, above)

$$\begin{aligned} E\{\mathbf{u}\} &= E\{\mathbf{v}\} = 0, \\ B_1 &= E\{\mathbf{uv}\} + a_1\, E\{\mathbf{v}^2\}, \\ B_2 &= E\{\mathbf{uv}\} + a_2\, E\{\mathbf{u}^2\}, \end{aligned} \tag{2.3}$$

with a_1 and a_2 being given by [1] (6.27). With

$$\epsilon_1 := \frac{\nu_1}{1-\mu_1} > 0, \quad \epsilon_2 := \frac{\mu_2}{1-\nu_2} > 0, \tag{2.4}$$

and assuming

$$E\{\boldsymbol{\xi}_j^2\} < \infty, \quad E\{\boldsymbol{\eta}_j^2\} < \infty, \quad j = 0,1,2, \tag{2.5}$$

put, cf. (2.2) and [1] (6.27),

$$D_b := \epsilon_1 \,\mathrm{E}\{\mathbf{u}^2\} + 2\,\mathrm{E}\{\mathbf{uv}\} + \epsilon_2 \,\mathrm{E}\{\mathbf{v}^2\}, \tag{2.6}$$

$$D_i := \epsilon_1 \alpha_1 + 2\alpha_{12} + \epsilon_2 \alpha_2 .$$

Consider the two linear equations for H_1 and H_2

$$\nu_1 H_1 + [1 - 2\frac{a_1}{\epsilon_2}] H_2 = -\mathrm{E}\{\mathbf{uv}\} - a_1 \,\mathrm{E}\{\mathbf{v}^2\}, \tag{2.7}$$

$$[1 - 2\frac{a_2}{\epsilon_1}]\nu_1 H_1 + \mu_2 H_2 = -\mathrm{E}\{\mathbf{uv}\} - a_2 \,\mathrm{E}\{\mathbf{u}^2\},$$

of which the solution reads: for $D_i \neq 0$.

$$H_1 = \frac{1}{2}\frac{1}{1-\mu_1}[\mathrm{E}\{\mathbf{u}^2\} - \alpha_1 \frac{D_b}{D_i}], \tag{2.8}$$

$$H_2 = \frac{1}{2}\frac{1}{1-\nu_2}[\mathrm{E}\{\mathbf{v}^2\} - \alpha_2 \frac{D_b}{D_i}].$$

If the first moments $\mathrm{E}\{\mathbf{h}_1\}$ and $\mathrm{E}\{\mathbf{h}_2\}$ of the stationary distribution of the $\mathbf{k}_m$-process are finite then they satisfy (2.7), cf. [1] (6.32), they are the unique solution, and

$$\mathrm{E}\{\mathbf{h}_1 | \mathbf{h}_1 > 0, \mathbf{h}_2 = 0\} > 1, \quad \mathrm{E}\{\mathbf{h}_2 | \mathbf{h}_2 > 0, \mathbf{h}_1 = 0\} > 1. \tag{2.9}$$

Denote by M_B the first moment of the entrance time into the boundary B out from a point of B; if we start at B (if the first transition leads directly to B then $M_B = 1$). From [1] (6.39) and (2.2) it is easily derived that

$$M_B = 1 + \frac{\Delta}{-\alpha_2}, \tag{2.10}$$

with, cf. [1] (6.39) and (2.2),

$$\Delta = \nu_1 \,\mathrm{E}\{\mathbf{h}_1\} + \mu_2 \,\mathrm{E}\{\mathbf{h}_2\} + \mathrm{E}\{\mathbf{uv}\}. \tag{2.11}$$

THEOREM 2.1. *For* $\mu_3 = 1$, $\nu_3 = 1$, $\mathrm{E}\{\xi_3^2\} < \infty$, $\mathrm{E}\{\eta_3^2\} < \infty$ *the* $\mathbf{z}_n$*-process is positive recurrent if the following four conditions apply:*

i. $(1-\mu_1)(1-\nu_2) > \mu_2 \nu_1, \quad \mu_1 < 1, \quad \nu_2 < 1,$ (2.12)

ii. $\alpha_{12} < 0,$

iii. $\mathrm{E}\{\xi_j^2\} < \infty, \quad \mathrm{E}\{\xi_j^2\} < \infty, \quad j = 0,1,2,$

iv. $-\infty < \frac{D_b}{D_i} < 0;$

if i and ii do apply but iii or iv do not then the $\mathbf{z}_n$*-process is null recurrent (if. i. of ii. do not apply see theorem 6.1 of [1]).*

COROLLARY 2.1. *If the* $\mathbf{z}_n$*-process is positive recurrent then the first moments of the hitting point are given by*

$$\mathrm{E}\{\mathbf{h}_1\} = \frac{1}{2(1-\mu_1)}[\mathrm{E}\{\mathbf{u}^2\} - (M_B - 1)\alpha_1], \tag{2.13}$$

$$\mathrm{E}\{\mathbf{h}_2\} = \frac{1}{2(1-\nu_2)}[\mathrm{E}\{\mathbf{v}^2\} - (M_B - 1)\alpha_2],$$

and

$$M_B = 1 - \frac{D_b}{D_i}. \tag{2.14}$$

Proof of theorem 2.1. We prove this theorem directly, i.e. without using theorem 6.2 of [1]. Suppose the $\mathbf{z}_n$-process is positive recurrent, so that the hitting point process $\mathbf{k}_m$, $m=1,2,...$, is also positive recurrent, i.e. (2.9), (2.12)i and ii hold, further we have

$$\infty > 1 + \frac{1}{-\alpha_{12}} \mathrm{E}\{(\mathbf{h}_1 + \xi_1 - 1)\eta_1 | \mathbf{h}_1 > 0,\ \mathbf{h}_2 = 0\} > 1, \tag{2.15}$$

since the middle term in (2.15) represents the first moment of the first entrance time into B when starting at a point $\mathbf{h}_1 \in B_{10}$, here theorem 2.3 of [1] has been used. Since (2.9) holds and $1 > K_1 > 0$ it follows from (2.15) and theorem 6.1 of [1] that $\mathrm{E}\{\mathbf{h}_1\} < \infty$, and similarly, $\mathrm{E}\{\mathbf{h}_2\} < \infty$, and so $\mathrm{E}\{\mathbf{h}_1\}$, $\mathrm{E}\{\mathbf{h}_2\}$ is a finite, unique solution of (2.7), i.e. cf. (2.8), $D_i \neq 0$. Hence $\mathrm{E}\{\mathbf{h}_1\}$ and $\mathrm{E}\{\mathbf{h}_2\}$ are given by the righthand sides of (2.8) and so we obtain from (2.10) and (2.11),

$$M_B = 1 - \frac{\Delta}{-\alpha_{12}} = 1 - \frac{D_b}{D_i}. \tag{2.16}$$

Because the definition of M_B implies that $\infty > M_B > 1$, if the $\mathbf{z}_n$-process is stationary, it is seen that (2.12)iii and iv hold.

Next suppose that the conditions (2.12) hold. It then follows from (2.12)i and theorem 6.1 of [1] that the $\mathbf{k}_m$-process is positive recurrent and so (2.9) holds and the equations (2.7) have a finite, unique solution because of (2.12)iii and iv. It follows that for this solution (2.15) holds and that $\infty > \Delta > 0$ and so, cf. (2.10), $\infty > M_B > 1$,Hence the $\mathbf{z}_n$-process is positive recurrent. □

Proof of corollary 2.1. The relations (2.13) and (2.14) follow immediate from the first part of the preceding proof. □

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