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A.J. Cabo, P. Groeneboom

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Limit Theorems for Functionals of Convex Hulls

A.J. Cabo
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

P. Groeneboom
Delft University of Technology
Faculty of Technical Mathematics and Informatics
Mekelweg 4, 2600 GA Delft, The Netherlands

Abstract. In [4], a central limit theorem for the number of vertices of the convex hull of a uniform sample from the interior of a convex polygon is derived. This is done by approximating the process of vertices of the convex hull by the process of extreme points of a Poisson point process and by considering the latter process of extreme points as a Markov process (for a particular parametrization). We show that this method can also be applied to derive limit theorems for the boundary length and for the area of the convex hull. This extends results of Rényi and Sulanke (1963) and Buchta (1984), and shows that the boundary length and the area have a strikingly different probabilistic behavior.

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1. Introduction.

In 1963, Rényi and Sulanke derived asymptotic expressions for the expected boundary length and the expected area of the convex hull of a uniform sample from the unit square (see [5]). They stated in their introduction that the computations for more general convex polygons are rather complicated ("ziemlich unübersichtliche Rechnungen"). Moreover they noted that the expected boundary length and the expected area of the convex hull behave surprisingly differently, at least in a first analysis ("Hier ergiebt sich die auf ersten Blick überraschende Tatsache, daß sich Flächeninhalt und Umfang asymptotisch verschieden verhalten").

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A paper by Buchta (see [1]), shows that indeed the computations of the first moment measures become quite complicated for polygons more general than the unit square, and proceeding in this way to the computation of higher moments seems an extraordinarily hard task. We show that the computation of asymptotic expressions for the first moments becomes rather easy, if one looks at the process *locally* instead of *globally*, using an approach which is most conveniently summarized by saying that "everything happens in the corners".

Moreover, we will derive the actual limiting behavior (after rescaling), and show that the area of the region between the convex hull of the sample and the boundary of the convex polygon satisfies a central limit theorem, in contrast to the boundary length of the convex hull. In fact we will show that the dominating asymptotic behavior of the boundary length of the convex hull depends on a number of edges that remains bounded, as the sample size tends to infinity, whereas the dominating asymptotic behavior of the area of the region between the convex hull of the sample and the boundary of the convex polygon will involve a number of edges tending to infinity, as the sample size tends to infinity. In this sense the asymptotic behavior of the boundary length is even *more* local than the behavior of the area.

Another expression of the phenomenon just mentioned, is that there is a natural stationary process, describing the limiting behavior of the area of the region between the convex hull of the sample and the boundary of the convex polygon, whereas no such stationarity holds for the limiting behavior of the boundary length. However, we expect that this striking difference in behavior would disappear when samples from convex figures with a smooth boundary are considered. This would also provide an explanation for the observations in [5] on the differences in this respect between samples from the unit square and samples from convex figures with a smooth boundary.

The present paper is structured in the following way. Since we will relate the behavior of the functionals of the finite sample process to the behavior of corresponding functionals of a limiting Poisson point process, we first study the functionals of the limiting process. This is done in section 2. Here we already see the difference in behavior of the boundary length and the area: for the "area functional" of the convex hull of the Poisson point process a central limit theorem is obtained in contrast to the "length functional".

In section 3 we relate the results for the Poisson process to the finite sample behavior. Computations of the relevant second moments are given in the appendix.

Since the computations, involving *second* moments, are considerably more complicated than for first moments, we treat (for reasons of space) the case of uniform samples from convex polygons separately. Moreover, this case has some pecularities which are not present in say, the case of samples from convex figures with smooth boundaries or samples from absolutely continuous distributions with infinite support, such as a two-dimensional normal distribution.

2. Functionals of the convex hull of a Poisson point process.

We shall study functionals of the Poisson point process $\mathcal{P}$ on $\mathbb{R}^2_+$, with intensity Lebesgue measure. The functionals will depend on $\mathcal{P}$ via another process, which is defined by Definition 2.2 in [4]. For convenience, this definition is repeated below.

Definition 2.1. For each $a > 0$, $W(a)$ is the point of a realization of the Poisson point process $\mathcal{P}$ on $\mathbb{R}_+^2$, such that all points of the realization lie to the right of the line $x+ay = c$, which passes through $W(a)$. If there are several of these points, we take the point with the smallest y-coordinate.

We first consider the functional, corresponding to "area". To this end, we introduce the following process, describing a "growing area", as a function of the parameter a in Definition 2.1.

Definition 2.2. Let $0 < a < b < \infty$. Then $A(a, b)$ is the area of the region, bounded on the right and left by vertical lines through the x-coordinates of the points $W(a)$ and $W(b)$, and bounded from below and above by the line $y = 0$ and the (left lower) boundary of the convex hull of $\mathcal{P}$, respectively.

For each $a_0 > 0$, we introduce the increasing filtration $\{\mathcal{F}_{[a_0,a]} : a \geq a_0\}$ of σ-algebras

$$\mathcal{F}_{[a_0,a]} = \sigma\{W(c) : c \in [a_0, a]\}. \tag{2.1}$$

Then the process $\{(W(a), A(a_0, a)) : a \geq a_0\}$ is a Markov process with respect to this filtration. This process has the following martingale characterization.

THEOREM 2.1. *Let C_0 be the set of continuous functions $f : \mathbb{R}_+^3 \to \mathbb{R}$, with compact support contained in $(0,\infty)^2 \times [0,\infty)$, and let, for each $a > 0$, the linear operator $L_a : C_0 \to C_0$ be defined by*

$$[L_a f](x,y,z) = \int_0^y u\left\{f\left(x + au, y - u, z + \tfrac{1}{2}au^2 + au(y-u)\right) - f(x,y,z)\right\} du, \tag{2.2}$$

for $(x, y, z) \in (0,\infty)^2 \times [0,\infty)$. Then, for each $f \in C_0$ and each $a_0 > 0$, the process

$$X_f(a) = f(W(a), A(a_0, a)) - \int_{a_0}^a [L_c f](W(c), A(a_0, c))\, dc,\ a \geq a_0,$$

is a martingale with respect to the filtration $\{\mathcal{F}_{[a_0,a]} : a \geq a_0\}$.

PROOF: We have to show that, for $a > 0$:

$$\lim_{h\downarrow 0} h^{-1}E\left\{f(W(a+h), A(a_0, a+h)) - f(W(a), A(a_0,a)) \mid (W(a), A(a_0,a)) = (x,y,z)\right\}$$
$$= [L_a f](x,y,z).$$

But, referring to Figure 2.1, it is easily seen that

$$A(a_0, a+h) - A(a_0, a) = h\left\{\tfrac{1}{2}au^2 + au(y-u)\right\} + o(h),\ h \downarrow 0,$$

if $W(a) = (x,y)$ and a jump from (x,y) to $(x,y) + (au, -u) + (\epsilon_1(h), \epsilon_2(h))$ occurs in the "time interval" $[a, a+h]$, where $\|(\epsilon_1(h), \epsilon_2(h))\| = \mathcal{O}(h)$.

The remaining part of the proof is the same as the proof of Theorem 2.1 in [4]. ∎

It is convenient to write (2.2) in the following form:

$$[L_a f](x,y,z) = \int_{\mathbb{R}^3} \{f(w+z) - f(w)\} M(a,w;dz), \tag{2.3}$$

where, for $a > 0$, the jump measure $M(a, w; \cdot)$ is defined by

$$M(a,w;B) = \int_0^y u 1_B\left(au, -u, \tfrac{1}{2}au^2 + au(y-u)\right) du, \tag{2.4}$$

where $w = (x,y,z)$ and $B \subset \mathbb{R}^3$ is a Borel set.

We now transform the process $\{(W(a), A(a_0,a)) : a \geq a_0\}$ into a Markov process with stationary transition probabilities. First, as in [4], we introduce the process $\{Z(a) : a \in \mathbb{R}\}$, by defining

$$\begin{aligned} X(a) &= \left(U(e^a) + e^a V(e^a)\right)/\exp\left\{\tfrac{1}{2}a\right\}, \\ Y(a) &= e^{\frac{1}{2}a} V(e^a), \\ Z(a) &= (X(a), Y(a)). \end{aligned}$$

The process $\{Z(a) : a \in \mathbb{R}\}$ has the one-dimensional marginal distributions

$$P\{X(a) \in dx,\, Y(a) \in dy\} = \exp\left\{-\tfrac{1}{2}x^2\right\} dx dy, \tag{2.5}$$

see (2.26) and (2.27) in [4]. Next, defining $\bar{A}(a_0,a)$ by

$$\bar{A}(a_0,a) = A(e^{a_0}, e^a),\ a \geq a_0$$

and using the 1-1 correspondence between $Z(a)$ and $W(e^a)$, we obtain

$$\begin{aligned} &\lim_{h \downarrow 0} h^{-1} E\left\{f(Z(a+h), \bar{A}(a_0,a+h)) - f(x,y,z) \mid (Z(a), \bar{A}(a_0,a)) = (x,y,z)\right\} \\ &= \int_0^y u \left\{f\left(x, y-u, z + \tfrac{1}{2}u^2 + uy\right) - f(x,y,z)\right\} du \\ &\quad + \left(y - \tfrac{1}{2}x\right)\frac{\partial}{\partial x} f(x,y,z) + \tfrac{1}{2}y \frac{\partial}{\partial y} f(x,y,z), \qquad a \geq a_0. \end{aligned} \tag{2.6}$$

As a corollary to Theorem 2.3 in [4], we get the following result.

THEOREM 2.2. *Let, for each $a_0 \in \mathbb{R}$, the process $\{(X(a), Y(a), \bar{A}(a_0, a)) : a \geq a_0\}$ be defined by*

$$X(a) = (U(e^a) + e^a V(e^a)) / \exp\{\tfrac{1}{2}a\},$$
$$Y(a) = e^{\frac{1}{2}a} V(e^a),$$
$$\bar{A}(a_0, a) = A(e^{a_0}, e^a),$$

where $a \geq a_0$. Then, for each $a_0 \in \mathbb{R}$, this process is a Markov process with stationary transition probabilities, and with an infinitesimal generator given by (2.6). Moreover, the process is strongly mixing in the following sense. Defining the σ-algebra

$$\bar{\mathcal{F}}_{[a,\infty)} = \sigma\{(X(c), Y(c)) : c \in [a, \infty)\},$$

we have

$$|P(A \cap B) - P(A)P(B)| \leq c \cdot e^{-\frac{1}{2}b}, \tag{2.7}$$

if $A \in \bar{\mathcal{F}}_{[a_0,a]}$, $B \in \bar{\mathcal{F}}_{[a+b,\infty)}$, where $c > 0$ is a fixed constant.

PROOF: The statement about the (stationary) Markov structure immediately follows from (2.6), and (2.7) follows from (2.30) in [4]. ∎

Theorem 2.2 implies that the sequence $X_1, X_2, \ldots$, defined by

$$X_i = \bar{A}(0, i) - \bar{A}(0, i-1),\ i = 1, 2, \ldots, \tag{2.8}$$

is a stationary sequence of random variables, satisfying the mixing condition

$$|P(A \cap B) - P(A)P(B)| \leq c \cdot e^{-\frac{1}{2}n},$$

if $A \in \sigma\{X_1, \ldots, X_k\}$ and $B \in \sigma\{X_{k+m} : m > n\}$, where $k, n \geq 1$ and c is as in (2.7). If we can show that $EX_1^k < \infty$, for all $k \geq 1$, we would get

$$\{\bar{A}(0, n) - nEX_1\} / \sqrt{\mathrm{Var}(A(0, n))} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \tag{2.9}$$

i.e., $\bar{A}(0, n)$ would converge in distribution, after standardization, to a standard normal distribution.

The finiteness of all moments EX_1^k follows from the following lemma.

LEMMA 2.1. *For each $a \geq 1$ the moment generating function*

$$\lambda \to E\exp\{\lambda A(1, a)\}$$

is finite for λ in a neighborhood of the origin.

PROOF: Conditionally on $(U(1), V(1)) = (x,y)$, the area $A(1,a)$ is bounded above by $\frac{1}{2}(a-1)y^2$ (the area of the triangle with vertices (x,y), $(x+y,0)$ and $(x+ay,0)$). In Lemma 2.4 ([4]), the marginal distribution of the Markov process $\{W(a) : a > 0\}$, was computed. By part (i) of this Lemma, we have

$$P\{W(a) \in (dx, dy)\} = \exp\left\{-\frac{(x+ay)^2}{2a}\right\} dxdy, \quad 0 < x, y < \infty.$$

¿From this, we obtain

$$E\exp\{\lambda A(1,a)\} \le \int_{\mathbb{R}_+^2} \exp\left\{-\tfrac{1}{2}(x+y)^2 + \tfrac{1}{2}\lambda(a-1)y^2\right\} dx\,dy, \tag{2.10}$$

and the right side of (2.10) is clearly finite for λ in a neighborhood of zero (depending on a). ∎

Since $X_1 = A(1,e)$, it follows that the moment generating function of X_1 exists in a neighborhood of the origin. So it is clear that all conditions for a central limit theorem of the form (2.9) are fulfilled, and all that is left to do is to compute first and second moments. We will compute these moments for the process $\{A(a_0, a) : a \ge a_0\}$ in its original parametrization, since the computations are somewhat simpler in appearance, and can be transferred immediately to the process in its stationary form (and hence to properties of the sequence $X_1, X_2, \ldots$).

We have the following result, which is quite similar to Theorem 2.4 in [4].

THEOREM 2.3. *Let $a > 1$ and $\alpha = a - 1$. Then:*

(i) $$EA(1,a) = \tfrac{1}{3}\log a,$$

(ii) $$\mathrm{Var}(A(1,a)) = \frac{50}{189}\log a + \frac{1}{9}\left(\tan^{-1}\sqrt{\alpha}\right)^2 + \frac{2}{9}\left\{3\alpha^{-2} + 7\alpha^{-1} + 4\right\}\frac{\tan^{-1}(\sqrt{\alpha})}{\sqrt{\alpha}} - \frac{2}{9}\left\{3\alpha^{-2} + 6\alpha^{-1}\right\}.$$

We note in passing that part (ii) of Theorem 2.4 in [4] contains a typing error: $\frac{4}{9}\left(\tan^{-1}(b/a)\right)^2$ should be $\frac{4}{9}\left(\tan^{-1}\left(\sqrt{(b/a-1)}\right)\right)^2$ (the correct formula is in fact given on p. 365 of [4]).

The proof of Theorem 2.3 will be given in the appendix. As a corollary we obtain the following central limit theorem.

COROLLARY 2.1. *For any sequence (a_n) such that $a_n \geq 1$, for each n, and $\lim_{n\to\infty} a_n = \infty$, we have*

$$\left\{A(1,a_n) - \tfrac{1}{3}\log a_n\right\} / \sqrt{\tfrac{50}{189}\log a_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \quad \text{as } n \to \infty, \tag{2.11}$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution.

PROOF: We note that, by Theorem 2.3,

$$\lim_{n\to\infty} n^{-1}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{50}{189},$$

where X_i is defined by (2.8). It follows that we can write $\bar{A}(0,n)$ as a series that has a finite second moment and whose terms are α-mixing and stationary. Using a generalized form of the classical Central Limit Theorem, we may now complete the proof, in exactly the same way, as was done in the proof of Corollary 2.3 in [4] (with $N(1,a_n)$ everywhere replaced by $A(1,a_n)$). ∎

We briefly comment on the more general situation of a Poisson process $\mathcal{P}_\lambda$ with intensity Lebesgue measure in the region R_λ, defined by

$$R_\lambda = \{(x,y) : y \geq 0, x - \lambda y \geq 0\},\ \lambda \in \mathbb{R}.$$

Taking $\lambda = 0$ we get the case we have considered so far. Here we only consider the case $\lambda > 0$, since the case $\lambda < 0$ is quite similar. Defining $W(a)$ as before (see Definition 2.1), but with the parameter a varying over the bigger interval $(-\lambda, \infty)$, we get the following marginal density for $W(a)$:

$$P\{U(a) \in dx,\ V(a) \in dy\} = \exp\left\{-\frac{(x+ay)^2}{2(a+\lambda)}\right\} dxdy,\ (x,y) \in R_\lambda. \tag{2.12}$$

This follows from the fact that the probability that a realization of the Poisson process has no point in the triangle with vertices $(\lambda(x+ay)/(a+\lambda), (x+ay)/(a+\lambda))$, $(0,0)$, $(x+ay,0)$ is given by $\exp\left\{-\frac{1}{2}(x+ay)^2/(a+\lambda)\right\}$.

We next redefine $A(a,b)$ in the following way.

Definition 2.3. Let $-\lambda < a < b < \infty$. Then $A'(a,b)$ is the area of the region, bounded on the right and left by lines parallel to the line $x = \lambda y$ through the points $W(a)$ and $W(b)$, and bounded from below and above by the line $y = 0$ and the boundary of the convex hull of $\mathcal{P}_\lambda$, respectively.

Instead of Theorem 2.1 we now get:

THEOREM 2.4. *Let C_0 be the set of continuous functions $f: \mathbb{R}_+^3 \to \mathbb{R}$, with compact support contained in $(0,\infty)^2 \times [0,\infty)$, and let, for each $a > -\lambda$, the linear operator $L_a : C_0 \to C_0$ be defined by*

$$\begin{aligned}&[L_a f](x,y,z)\\ &= \int_0^y u\left\{f\left(x+au, y-u, z+\tfrac{1}{2}(a+\lambda)u^2+(a+\lambda)u(y-u)\right)-f(x,y,z)\right\}du,\end{aligned}$$

for $(x,y,z) \in (0,\infty)^2 \times [0,\infty)$. Then, for each $f \in C_0$ and each $a_0 > \lambda$, the process

$$X_f(a) = f\left(W(a), A'(a_0,a)\right) - \int_{a_0}^{a} [L_c f]\left(W(c), A'(a_0,c)\right) dc,\ a \geq a_0,$$

is a martingale with respect to the filtration $\{\mathcal{F}_{[a_0,a]} : a \geq a_0\}$.

The process $\{(U(a),V(a),A'(a_0,a)) : a \geq -\lambda\}$ can now be transformed into a Markov process with stationary transition probabilities, just as before. First, we introduce the process $\{Z(a) : a \in \mathbb{R}\}$, by defining

$$\begin{aligned}X(a) &= \{U(e^a-\lambda)-\lambda V(e^a-\lambda)+e^a V(e^a-\lambda)\}/\exp\{\tfrac{1}{2}a\},\\ Y(a) &= e^{\frac{1}{2}a}V(e^a-\lambda),\\ Z(a) &= (X(a),Y(a)).\end{aligned}$$

This process has the same structure as before, in particular (2.5) and (2.6) are satisfied. It follows that $A'(a,b)$ has the same distribution as $A(a+\lambda, b+\lambda)$, and we obtain the same central limit theorem as before (with the parameters a and b shifted to $a+\lambda$ and $b+\lambda$).

In section 3 we shall derive from Corollary 2.1 a central limit theorem for the area of the convex hull of a uniform sample from a convex polygon. We now first turn to the other functional: the boundary length. In analogy with the area, we introduce a process, describing the "growing boundary length".

Definition 2.4. Let $1 \leq a < b < \infty$. Then $L(a,b)$ is the boundary length of the convex hull of $\mathcal{P}$ between the points $W(a)$ and $W(b)$ minus $U(b)-U(a)$.
For $0 < a < b < 1$, we define $L(a,b)$ to be the boundary length of the convex hull of $\mathcal{P}$ between the points $W(a)$ and $W(b)$ minus $V(a)-V(b)$.

So, for $1 \leq a < b < \infty$, $L(a,b)$ is the boundary length of the convex hull of $\mathcal{P}$ between the points $W(a)$ and $W(b)$ minus the projection of this part of the boundary on the x-axis, whereas for $0 < a < b < 1$, we compare the boundary length of the convex hull of $\mathcal{P}$ with its projection on the y-axis. As before, we get that the process $\{(W(a), L(a_0,a)) : a \geq a_0\}$ is a Markov process with respect to the filtration $\{\mathcal{F}_{[a_0,a]} : a \geq a_0\}$. Moreover, we get the following martingale characterization.

THEOREM 2.5. *Let C_0 be the set of continuous functions $f : \mathbb{R}_+^3 \to \mathbb{R}$, with compact support contained in $(0,\infty)^2 \times [0,\infty)$, and let, for each $a \geq 1$, the linear operator $L_a : C_0 \to C_0$ be defined by*

$$(2.13) \quad [L_a f](x,y,z) = \int_0^y u\left\{f\left(x+au, y-u, z+u\sqrt{1+a^2}-au\right) - f(x,y,z)\right\} du,$$

for $(x,y,z) \in (0,\infty)^2 \times [0,\infty)$. Then, for each $f \in C_0$ and each $a_0 \geq 1$, the process

$$Y_f(a) = f\big(W(a), L(a_0,a)\big) - \int_{a_0}^a [L_c f]\big(W(c), L(a_0,c)\big)\, dc,\ a \geq a_0,$$

is a martingale with respect to the filtration $\left\{\mathcal{F}_{[a_0,a]} : a \geq a_0\right\}$.

In this case it does not seem possible to transform the process to a Markov process with stationary transition probabilities. Nonetheless Theorem 2.4 is helpful in computing the first and second moment measure of the process.

THEOREM 2.6. *For the boundary length $L = L(0,\infty)$, of the convex hull of the Poisson process $\mathcal{P}$, we have*

(i) $$EL(0,\infty) \approx 1.13283$$

(ii) $$VarL(0,\infty) \approx 1.88969$$

The proof follows from numerical integration of the expressions for the moments of $L(\frac{1}{t},t)$, $t > 0$, as t tends to infinity. It is given in Appendix B.

3. Functionals of the convex hull of a uniform sample.

The results of the previous section and the strong approximation result, Lemma 2.2 in [4], yield limit theorems for the functionals of the finite sample.

First we need to recall the definition of the process, running through the vertices of the left-lower boundary of the convex hull of a uniform sample from the interior of the unit square. (Definition 2.1 in [4])

Definition 3.1 For each $a > 0$, $W_n(a)$ is the point of the sample, such that all points of the sample lie to the right of the line $x + ay = c$, which passes through $W_n(a)$. If there are several of such points, we take the one with the smallest y-coordinate.

In [4], it was shown that the point process $\{\sqrt{n}W_n(a) : a > 0\}$ converges in distribution to the point process $\{W(a) : a > 0\}$.

Let A_n denote the difference between the area of the unit square and the area of the convex hull of n uniform points in this square. Analogously, L_n denotes the difference between the boundary length of the unit square and the boundary length of the convex hull of the uniform sample. It is clear from Theorem 2.3, that A_n is of order $\frac{\log n}{n}$, as $n \to \infty$. However, it follows from Theorem 2.5, that the expectation of L_n tends to zero at a rate of $\frac{1}{\sqrt{n}}$, whereas the variance decreases as $\frac{1}{n}$, for $n \to \infty$.

The rate for the area A_n is not surprising, since, as was shown in [4], EN_n is of order $\log n$, for $n \to \infty$. Moreover, by a well-known relation (see e.g. [3]) between the expected area and the expected number N_n of vertices of the convex hull, we have

$$EA_n = \frac{1}{n} EN_{n-1}.$$

To give an idea of the proof of our main theorem, we have to recall some of the results in [4], on which the theorem heavily depends.

Consider the region R_n of the unit square, that lies to the left and below the curve $C(\frac{3\log n}{n})$, where $C(\alpha)$ is defined by

$$C(\alpha) = \bigcup_{i=1}^{3} C_i(\alpha),$$

with

$$C_1(\alpha) = \left\{(x,y) : \frac{1}{2} < y \le 1, x = \alpha\right\}$$

$$C_2(\alpha) = \left\{(x,y) : xy = \frac{1}{2}\alpha, \alpha \le x \le \frac{1}{2}\right\}$$

$$C_3(\alpha) = \left\{(x,y) : \frac{1}{2} < x \le 1, y = \alpha\right\}$$

It was shown in [4], that the vertices $\{W_n(a) : a > 0\}$ of the convex hull of the uniform sample, belong to the region R_n with a probability tending to 1, as $n \to \infty$. (Corollary 2.1 in [4])

Now consider a Poisson point process ξ_n on $\mathbb{R}_+^2$ with intensity $n\times$ Lebesgue measure and let η_n be the sample point process corresponding to the sample of size n from the unit square. Then, it was also shown in [4], that there exists a probability space such that the probability that the realizations from $\eta|_{R_n}$ (η_n restricted to the region R_n) and from $\xi|_{R_n}$ differ, tends to 1, as $n \to \infty$. (Lemma 2.2 in [4])

Let us introduce the following notation:

$\phi(\zeta_n)$ is a functional of the convex hull of the process ζ_n, where in our case, ϕ is one of the following

N_n	the number of vertices
A_n	the remaining area
L_n	the remaining length,

where A_n and L_n are defined more precisely at the beginning of this section.

A suffix β_n, as in $\phi_{\beta_n}(\zeta_n)$, is used to express the fact that we consider the functional ϕ only for values of the timeparameter in the interval $[\beta_n, 1/\beta_n]$ and $\beta_n = \frac{\log n}{n}$.

The crucial argument is the following. Summarizing the results from [4], we get, speaking very loosely,

$$\phi_{\beta_n}(\eta_n) = \phi_{\beta_n}(\eta_n|_{R_n}) = \alpha(n)^{-1}\phi_{\beta_n}(\xi_n|_{R_n}) = \alpha(n)^{-1}\phi_{\beta_n}(\xi_n), \tag{3.1}$$

for some function $\alpha(n)$ of n, and with the equalities only holding on a set having a probability mass, tending to 1 and for n tending to infinity. In words: If we want to study a functional of the convex hull of the sample point process, we might just as well study the same functional of the convex hull a Poisson point process, as long as we restrict our attention to those lines generating the convex hulls with slopes in the interval $[\beta_n, 1/\beta_n]$ and only for n large enough.

The function $\alpha(n)$ is as follows:

$$\alpha(n) = \begin{cases} 1 & \text{if } \phi = N_n \\ n & \text{if } \phi = A_n \\ \sqrt{n} & \text{if } \phi = L_n. \end{cases}$$

First, we show that it *is* sufficient to consider the part ϕ_{β_n} of ϕ. By symmetry, it is enough to prove the following lemma.

LEMMA 3.1. *With $\beta_n = \frac{\log n}{n}$, we have*

$$EA_n(0,\beta_n) \sim c_1 \frac{\log\log n}{n}, \quad \text{as } n \to \infty \tag{i}$$

$$EL_n(0,\beta_n) \sim c_2 \frac{\log n}{n}, \quad \text{as } n \to \infty, \tag{ii}$$

for some constants c_1, c_2.

The proof is given in Appendix C.

We may now state the main result of this section.

THEOREM 3.1. *We have, as $n \to \infty$,*

$$\{A_n - 4b_n\}/2c_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where $b_n = \frac{2}{3}\frac{\log n}{n}$ and $c_n = \sqrt{\frac{100}{189}\frac{\log n}{n}}$.
$\mathcal{N}(0,1)$ denotes the standard normal distribution.

PROOF: From Theorem 2.4 and the 'equalities' (3.1), we may deduce that

$$\{A_n(\beta_n, 1/\beta_n) - b_n\}/2c_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \text{as} \quad n \to \infty. \tag{3.2}$$

Furthermore, by Markov's inequality and Lemma 3.1,

$$P\left\{\frac{A_n(0,\beta_n)}{c_n} \geq \epsilon\right\} \leq \frac{1}{\epsilon}\frac{EA_n(0,\beta_n)}{c_n} \to 0, \quad n \to \infty.$$

Hence

$$\begin{aligned}\frac{A_n - b_n}{c_n} &= \frac{A_n(\beta_n, 1/\beta_n) - b_n}{c_n} + o_P(1) \\ &\xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \quad \text{by (3.2)}.\end{aligned}$$

Of course we may proceed in the same way for the other corners. The only thing left to show is that the random variables $A_n(\beta_n, 1/\beta_n)$, for the different corners, are asymptotically independent. Since this involves exactly the same argument as the one used in the corresponding Corollary 2.4 in [4], we refer to the last part of the proof of that Corollary. This proves the Theorem. ∎

Next, consider a uniform sample from a convex plane polygon with $k(\geq 3)$ vertices. In the same way that led us to Theorem 3.1, we may derive from Theorem 2.1 and the remarks made thereafter, the following more general result.

THEOREM 3.2. *Let $A(C_n)$ denote the area of the convex hull of a uniform sample of size n from the interior of a convex polygon C with $k(\geq 3)$ vertices and area $A(C)$. Then, for the remaining area*

$$A_n := A(C) - A(C_n),$$

we have, as $n \to \infty$,

$$\frac{\left\{A_n - \frac{2}{3}k\frac{\log n}{n}\right\}}{\sqrt{\frac{100}{189}k\frac{\log n}{n}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution.

The behavior of L_n is rather different. Essential in Theorem 3.1 is the asymptotic negligibility (see [3]) of the individual components of A_n. However, as is seen from Theorem 2.5, the dominating asymptotic behavior of L_n depends on a bounded number of parts. Another way of expressing this, is saying that the pieces of nA_n constitute a series that diverges as $n \to \infty$, whereas the pieces of $\sqrt{n}L_n$ constitute a series that converges in distribution to a (random) summable infinite series. Although the latter infinite series can be considered as a functional of the limiting Poisson point process, this characterization does not seem to lead to a simple limit theorem for L_n, like Theorem 3.2 for A_n. The only information we have so far about the limiting distribution is given in Theorem 2.5, where the first two moments are specified.

Appendix A. (Proof of Theorem 2.3).

ad (i). As shown in the proof of Theorem 2.1,

$$
\begin{aligned}
&\lim_{h\downarrow 0}\frac{1}{h}E\{A(a,a+h)|W(a)=(x,y)\}\\
(5.1)\qquad &=\int_0^y (auy-\frac{1}{2}au^2)du\\
&=\frac{5}{24}ay^4.
\end{aligned}
$$

It follows that, for $a \geq 1$,

$$EA(1,a)=\frac{5}{24}\int_1^a sEV(s)^4ds=\frac{1}{3}\log a$$

since $EV(s)^4=\frac{8}{5s^2}$, by Lemma 2.4 in [4].

ad (ii). We consider the second moment of $A(1,a)$:

$$
\begin{aligned}
&\lim_{h\downarrow 0}\frac{1}{h}E\left\{A(1,a+h)^2-A(1,a)|\mathcal{F}_{[1,a]}\right\}\\
&=\lim_{h\downarrow 0}\frac{1}{h}E\left\{A(a,a+h)^2+2A(1,a)A(a,a+h)|\mathcal{F}_{[1,a]}\right\}\\
&=\int_0^{V(a)} u\left\{yau-\frac{1}{2}au^2\right\}^2du+\frac{5}{12}aA(1,a)V(a)^4.
\end{aligned}
$$

Hence

$$(5.2)\qquad EA(1,a)^2=\frac{11}{120}\int_1^a s^2EV(s)^6ds+\frac{5}{12}\int_1^a sE(1,s)V(s)^4ds.$$

The first integral of (5.2) is easily computed to be $\frac{22}{35}\log a$.

For the second integral, we use a time reversal argument similar to the one given on page 343 in [4].

It follows that

$$(5.3)\qquad E\{A(1,s)|W(s)\}=\frac{5}{24}\int_1^s\frac{1}{r^3}E\left\{U(r)^4|W(s)]\right\}dr,$$

where we use (5.1).

So we have to compute $EU(r)^4V(s)^4$. We first take $r=1$. From Lemma 2.4 in [4], it follows that

$$
\begin{aligned}
EU(1)^4V(s)^4 &= \frac{4}{5}EV(1)^4U(1)^4\exp\left\{-\frac{1}{2}\sigma V(1)^2\right\}\\
(5.4)\qquad &-\frac{4}{5\sigma}EV(1)^2U(1)^4\exp\left\{-\frac{1}{2}\sigma V(1)^2\right\}\\
&-\frac{4}{5\sigma^2}EU(1)^4\left\{\exp\left\{-\frac{1}{2}\sigma V(1)^2\right\}-1\right\},
\end{aligned}
$$

where $\sigma := s - 1$.
We introduce the following notation:

$$I_k := \begin{cases} \int_0^\infty y^k \, e^{-\frac{1}{2}\sigma y^2} \left(\int_y^\infty e^{\frac{1}{2}t^2} dt\right) dy & , k \text{ even} \\ \int_0^\infty y^k \, e^{-\frac{1}{2}sy^2} dy & , k \text{ odd} . \end{cases}$$

While performing the computation of the expectations in (5.4), one is led to the use of the integrals I_k, that are easy to evaluate. In fact, we can write (5.4) as

$$\begin{aligned} &\frac{5\sigma^2}{4}\{\sigma^2(I_8 - I_7) + (6\sigma^2 - \sigma)I_6 + (\sigma - 5\sigma^2)I_5 \\ &\quad + (3\sigma^2 - 6\sigma - 2)I_4 + (5\sigma + 2)I_3 + (-3\sigma - 12)I_2 \\ &\quad 10I_1 - 6I_0 + 2EU(1)^4\}. \end{aligned}$$

Elementary computations show that for $k = 1, 2, \ldots$

$$\begin{aligned} I_0 &= \frac{\tan^{-1}\sqrt{\sigma}}{\sqrt{\sigma}} \\ I_1 &= \frac{1}{\sigma} \\ I_{2k} &= \frac{2k-1}{\sigma} I_{2k-2} - \frac{1}{\sigma} I_{2k-1} \end{aligned}$$

Using these recurrent relations, we may now conclude that

$$EV(s)^4U(1)^4 = \frac{16}{5}\left(\frac{21}{\sigma^4} + \frac{15}{\sigma^3}\right)\frac{\tan^{-1}\sqrt{\sigma}}{\sqrt{\sigma}} - \frac{16}{5}\left(\frac{21}{\sigma^4} + \frac{8}{\sigma^3} - \frac{4}{5\sigma^2}\right). \tag{5.5}$$

To simplify the notation slightly, we define the function $\varphi(\sigma)$ to be equal to the right-hand side of (5.5). Then, by the definition and the stationarity of the process $\{Z(a) : a \in \mathbb{R}\}$, we get

$$EV(s)^4U(r)^4 = \varphi(\frac{s}{r} - 1) \quad , s > r > 0. \tag{5.6}$$

Remark: Using Lemma 2.4 in [4], one may compute that

$$EU(1)^4V(1)^4 = \frac{64}{105}.$$

From (5.5), it follows that

$$EU(1)^4V(1)^4 = \lim_{\sigma\downarrow 0}\varphi(\sigma)$$

and this limit may indeed be shown to be equal to $\frac{64}{105}$.

Now, the second term in (5.2) equals

$$
\begin{aligned}
&\frac{25}{288}\int_1^a s\,ds\int_1^s \frac{1}{r^3}\varphi(\frac{s}{r}-1)\,dr \\
(5.7)\qquad &=\frac{25}{288}\int_1^a \frac{1}{r}\,dr\int_1^{a/r} u\varphi(u-1)\,du,
\end{aligned}
$$

by Fubini's theorem and a change of variables.
To find a primitive of $u\varphi(u-1)$, we write it as $(u-1)\varphi(u-1)+\varphi(u-1)$.

We shall use the indefinite integrals given by (4.2), (4.3) and (4.4) in [4], as well as the following:

$$
\begin{aligned}
\int \tan^{-1}\sqrt{u-1}(u-1)^{-9/2} &= -\frac{2}{7}\tan^{-1}\sqrt{u-1}(u-1)^{-7/2} \\
&\quad -\frac{1}{7}\left(\frac{1}{3(u-1)^3}-\frac{1}{2(u-1)^2}+\frac{1}{u-1}+\log\frac{u-1}{u}\right) \\
\int \tan^{-1}\sqrt{u-1}(u-1)^{-7/2} &= -\frac{2}{5}\tan^{-1}\sqrt{u-1}(u-1)^{-5/2} \\
&\quad -\frac{1}{5}\left(\frac{1}{2(u-1)^2}-\frac{1}{(u-1)}-\log\frac{u-1}{u}\right).
\end{aligned}
$$

Then we have

$$
(5.8)\qquad \int_1^{a/r} u\varphi(u-1)\,du = \Phi(\frac{a}{r}-1)-\Phi(0),
$$

$$
\begin{aligned}
\text{where } \Phi(x) :=& -\frac{16}{5}\left\{\frac{6}{x^3}+\frac{72}{5x^2}+\frac{10}{x}\right\}\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} \\
&+\frac{16}{5}\left\{\frac{6}{x^3}+\frac{62}{5x^2}+\frac{32}{5x}+\frac{4}{5}\log x+1\right\}, \quad x>0 \\
\text{and } \Phi(0) :=& \lim_{x\downarrow 0}\Phi(x)=\frac{4^4.43}{5^3.21}.
\end{aligned}
$$

For the double integral in (5.7), this yields

$$
\begin{aligned}
&\int_1^a \frac{\Phi(r-1)}{r}dr-\Phi(0)\log a \\
=&\frac{64}{25}\left(\frac{3}{(a-1)^2}+\frac{7}{a-1}+4\right)\frac{\tan^{-1}\sqrt{a-1}}{\sqrt{a-1}}-\frac{64}{25}\left(\frac{3}{(a-1)^2}+\frac{6}{a-1}\right) \\
&+\frac{32}{25}\left(\tan^{-1}\sqrt{a-1}\right)^2+\frac{32}{25}(\log a)^2-\frac{4^4.43}{5^3.21}\log a,
\end{aligned}
$$

whence the theorem follows.

Appendix B. (Proof of Theorem 2.5).

To be able to do the computations, observe that, by Definition 2.4

$$L(\frac{1}{a},a) = L(\frac{1}{a},1) + L(1,a) + U(1) + V(a), \quad a > 1. \tag{5.9}$$

ad (i). From Theorem 2.5 we get

$$\begin{aligned}
&\lim_{h\downarrow 0} \frac{1}{h} E\{L(a,a+h)|W(a)=(x,y)\} \\
&= \int_0^y u^2 \left\{\sqrt{1+a^2}-a\right\} du \\
&= \frac{1}{3} y^3 \left(\sqrt{1+a^2}-a\right).
\end{aligned}$$

Hence, for $a \geq 1$, we have

$$\begin{aligned}
EL(1,a) &= \frac{1}{3}\int_1^a \left(\sqrt{1+s^2}-s\right) EV(s)^3 dx \\
&= \sqrt{2\pi}\int_1^a \frac{\sqrt{1+s^2}-s}{8s^{3/2}} dx,
\end{aligned} \tag{5.10}$$

since $EV(s)^3 = \dfrac{3\sqrt{2\pi}}{8s^{3/2}}$.

Moreover,

$$EU(1) = \int_0^\infty \int_0^\infty y \exp\left\{-\frac{1}{2}(x+y)^2\right\} dxdy = \frac{1}{4}\sqrt{2\pi}. \tag{5.11}$$

Since $EL(\frac{1}{a},1) = EL(1,a)$, we get, combining (5.10) and (5.11),

$$\begin{aligned}
EL(0,\infty) &= 2\left\{\lim_{a\to\infty} E\left(L(1,a)+U(1)\right)\right\} \\
&= \frac{1}{2}\sqrt{2\pi}\left\{1 - \int_1^\infty \frac{\sqrt{1+s^2}-1}{2s^{3/2}} ds\right\}
\end{aligned}$$

ad (ii). Using (5.9), observe that, by symmetry and the fact that the processes $\{W(a) = (U(a)) : a > 0\}$ and $\{(V(\frac{1}{a}), U(\frac{1}{a})) : a > 0\}$ are identically distributed,

$$\begin{aligned}
\text{Var } L(\frac{1}{a},a) &= 2EL(\frac{1}{a},1)^2 + 4EL(1,a)\{U(1)+V(1)\} \\
&\quad + 2EL(\frac{1}{a},1)L(1,a) + E\{U(1)+V(1)\}^2 \\
&\quad - \left\{EL(\frac{1}{a},a)\right\}^2.
\end{aligned} \tag{5.12}$$

First, we have

$$E\{U(1)+V(1)\}^2 = \int_0^\infty \int_0^\infty (x+y)^2 e^{-\frac{1}{2}(x+y)^2}\,dx\,dy = 2.$$

Next, to compute $EL(1,a)\{U(1)+V(1)\}$, remark that

$$E\{L(1,a)|W(1)\} = \frac{1}{3}\int_1^a \left(\sqrt{1+s^2}-s\right) E\{V(s)^3|W(1)\}\,ds.$$

Hence

$$EL(1,t)\{U(1)+V(1)\} = \frac{1}{3}\int_1^t \left(\sqrt{1+s^2}-s\right) EV(s)^3\{U(1)+V(1)\}\,ds.$$

¿From Lemma 2.4 in [4], we get

$$\begin{aligned} E\{V(s)^3|W(1)=(x,y)\} &= \frac{3}{4}y_1^2 e^{-\frac{1}{2}\sigma y_1^2} \\ &\quad - \frac{3}{4\sigma}y_1 e^{-\frac{1}{2}\sigma y_1^2} + \frac{3}{4\sigma}\int_0^{y_1} e^{-\frac{1}{2}\sigma u^2}\,du, \end{aligned} \tag{5.13}$$

where, again, $\sigma := s-1$.

Consequently,

$$\begin{aligned} EV(s)^3\{U(1)+V(1)\} &= \frac{3}{4}EV(1)^2\{U(1)+V(1)\}e^{-\frac{1}{2}\sigma V(1)^2} \\ &\quad - \frac{3}{4\sigma}EV(1)\{U(1)+V(1)\}e^{-\frac{1}{2}\sigma V(1)^2} \\ &\quad + \frac{3}{4\sigma}E\{U(1)+V(1)\}\int_0^{V(1)} e^{-\frac{1}{2}\sigma u^2}\,du. \end{aligned}$$

Similar computations as in Appendix A, yield

$$EV(s)^3\{U(1)+V(1)\} = \chi(s-1),$$

where

$$\chi(x) := \frac{1}{2(x+1)^2} - \frac{1}{4x(x+1)} + \frac{1}{4x}\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}.$$

Hence

$$EL(1,a)\{U(1)+V(1)\} = \int_1^a \chi(s-1)\left\{\sqrt{1+s^2}-s\right\}ds. \tag{5.14}$$

Next, we need the following analogue of (5.2):

$$EL(1,a)^2 = \frac{2}{5}\int_1^a \left\{\frac{\sqrt{1+s^2}-s}{s}\right\}^2 ds + \frac{2}{3}\int_1^a \left\{\sqrt{1+s^2}-s\right\} EV(s)^3 L(1,s)ds \tag{5.15}$$

The second integral can be written as

$$\frac{2}{9}\int_1^a \left\{\sqrt{1+s^2}-s\right\}\int_1^s \frac{\sqrt{1+r^2}-r}{r^3} EV(s)^3 U(r)^3 dr, \tag{5.16}$$

using, again, a time reversal argument.
To compute the expectation, first take $r = 1$. From Lemma 2.4 in [4] and (5.13), we may conclude

$$EU(1)^3 V(s)^3 = \psi(s-1),$$

where

$$\psi(x) := \frac{9}{16}\left(\frac{15}{x^3}+\frac{1}{x^2}\right) - \frac{9}{16}\left(\frac{15}{x^3}+\frac{6}{x^2}-\frac{1}{x}\right)\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}, \quad x > 0.$$

As is easily seen, we now also have

$$EU(r)^3 V(s)^3 = \psi\left(\frac{s}{r}-1\right) \quad , \quad s > r > 0.$$

Hence, (5.16) equals

$$\frac{2}{9}\int_1^a \left(\sqrt{1+s^2}-s\right)\int_1^s \frac{\sqrt{1+r^2}-r}{r^3}\psi\left(\frac{s}{r}-1\right) drds. \tag{5.17}$$

Finally, proceeding along the same lines, we obtain

$$EL\left(\frac{1}{a},1\right)L(1,a) = \frac{1}{9}\int_1^a \left\{\sqrt{1+s^2}-s\right\}\int_{\frac{1}{a}}^1 \frac{\sqrt{1+r^2}-1}{r^3}\psi\left(\frac{s}{r}-1\right) drds. \tag{5.18}$$

Although the integration may be pushed a bit further, we have to evaluate the integrals (5.14), (5.15) and (5.17), and (5.18) numerically as $t \to \infty$. Together with (5.12), this yields the theorem.

Appendix C. (Proof of Lemma 3.1).

Let ϕ stand for either A_n or L_n. Then

$$E\{\phi(0,a+h)-\phi(0,a)\mid W_n(a)\}=(n-1)hf_a(y)+O(h^2),$$

for $0\le a\le 1$, $0<x,y<1$, $x+ay<1$ and for some function f_a, depending on a. Hence

$$E\phi(0,\beta_n)\sim\binom{n}{2}\int_0^{\beta_n}da\iint_{0<x+ay<1}f_a(y)(1-A_a(x,y))^{n-1}\,dxdy,$$

where $A_a(x,y)$ denotes the area of the region cut off from the unit square to the left of the line $l_a: x'+ay'=c$ through (x,y).

We now have

$$\int_0^{\beta_n}da\iint_{0<x+ay<1}f_a(y)(1-A_a(x,y))^{n-1}\,dxdy$$

(5.19)
$$=\int_0^{\beta_n}da\iint_{0<x+ay<a}f_a(y)\left\{1-\frac{1}{2}\frac{(x+ay)^2}{a}\right\}^{n-1}dxdy$$
$$+\int_0^{\beta_n}da\iint_{a\le x+ay<1}f_a(y)\left\{1-x-ay+\frac{1}{2}a\right\}^{n-1}dxdy$$

It can be shown that we only need to consider the first integral (see Appendix A3 in ([4]).

(i) Take $\phi=A_n$.

Then, it follows from the proof of Theorem 2.3, that

$$f_a(y)=\frac{5}{24}ay^4.$$

Hence, the first integral in (5.19) equals

(5.20)
$$\frac{5}{24}\int_0^{\beta_n}a\,da\int_{\substack{0<u<a\\0<y<u/a}}y^4\left(1-\frac{u^2}{2a}\right)^{n-1}dudy$$
$$=\frac{1}{24}\int_0^{\beta_n}\frac{1}{a}da\int_0^{\frac{1}{2}a}v^2(1-v)^{n-1}\,dv$$

But

$$\int_0^{\frac{1}{2}a}v^2(1-v)^{n-1}\,dv=-\frac{a^2}{4n}\left(1-\frac{1}{2}a\right)^n-\frac{a}{n(n+1)}\left(1-\frac{1}{2}a\right)^{n+1}$$
$$+\frac{2}{n(n+1)(n+2)}\left\{1-\left(1-\frac{1}{2}a\right)^{n+2}\right\}$$

Hence (5.20) is asymptotically equivalent to

$$-\frac{1}{96}\int_0^{\beta_n} ae^{-\frac{1}{2}na}\,da - \frac{1}{24n(n-1)}\int_0^{\beta_n} e^{-\frac{1}{2}na}\,da$$
$$+\frac{1}{12n(n+1)(n+2)}\int_0^{\beta_n}\frac{1}{a}\left\{1-\exp\{-\frac{1}{2}na\}\right\}da$$
$$\sim c\frac{\log\log n}{n^3},\quad \text{as } n\to\infty.$$

This yields part (i) of the Lemma, since we now have

$$EA_n(0,\beta_n)\sim c\binom{n}{2}\frac{\log\log n}{n^3}\sim c_1\frac{\log\log n}{n},\quad \text{as } n\to\infty$$

(ii) If $\phi = L_n$, then, for $a<1$,

$$f_a(y)=\frac{1}{3}\frac{\sqrt{1+a^2}-1}{a^3}y^3.$$

Hence

$$EL_n(0,\beta_n)=\binom{n}{2}\int_0^{\beta_n}\frac{\sqrt{1+a^2}-1}{a^3}\,da\int_{0<x+ay<1}\frac{1}{3}y^3(1-A_a(x,y))^{n-1}\,dxdy$$

Again, it is enough to consider the following integral:

$$\int_0^{\beta_n}\frac{\sqrt{1+a^2}-1}{a^3}\,da\int_{\substack{0<u<a\\ o<y<u/a}} y^3(1-\frac{u^2}{2a})^{n-1}\,dudy$$
$$=\frac{1}{\sqrt{2}}\int_0^{\beta_n}\frac{\sqrt{1+a^2}-1}{a^{9/2}}\,da\int_0^{\frac{1}{2}a} v^{3/2}(1-v)^{n-1}\,dv$$
$$\le\frac{1}{\sqrt{2}}\int_0^{\beta_n}\frac{\sqrt{1+a^2}-1}{a^{9/2}}\,da\int_0^{\frac{1}{2}a} v^{3/2}\,dv,\quad \text{since } 1-v\le 1,$$
$$=c_0\int_0^{\beta_n}\frac{\sqrt{1+a^2}-1}{a^2}\,da$$
$$\sim c_2\frac{\log n}{n},\quad n\to\infty.$$

This completes the proof of the Lemma.

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