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Asymptotic Optimality of the Chi-Square Test with Large Number of Degrees of Freedom within the Class of Symmetric Tests

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Let $x_1, \ldots, x_N$ be independent random variables having normal distributions $\mathfrak{N}(\mu_i, 1)$, $i=1,..., N$. The problem of testing the hypothesis H_0:$\mu=\mathbf{0}$ against the alternative H_1:$\mu\neq\mathbf{0}$, $\mu\in\mathfrak{M}_N=\{\mu:\Sigma\mu_i=0\}$ (where $\underline{\mu}=(\mu_i)_{i=1}^N$) is considered. It is proved that the chi-square test based on the statistic $\Sigma(x_i-\bar{x}_N)^2$, where $\bar{x}_N=\Sigma x_i/N$, is an asymptotically (as $N\to\infty$) most powerful (MP) symmetric (with respect to permutations of $\{x_i\}$) test in the sense that it has asymptotically the same power as the MP symmetric test against a simple alternative $\mu_N\in\mathfrak{M}_N$ for any sequence $\{\mu_N\}$ satisfying some uniform asymptotic negligibility condition on the components of μ_N.

The author's objective is to prove the corresponding property of the chi-square test in the problem of testing equiprobability of outcomes in a multinomial distribution when the number of outcomes, N, grows to infinity together with the number of trials, n. The method of asymptotic analysis of the 'symmetrized' likelihood ratio developed in the present paper, though in a more simple setup, provides us with a basic tool for this problem.

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1. Introduction

Let $x_1, \ldots, x_N$ be independent random variables having normal distributions $\mathfrak{N}(\mu_i, 1)$, $i=1,..., N$, so that the vector $\mathbf{x}=(x_i)_{i=1}^N$ has the normal distribution to be denoted by $P_{N\mu}$ with mean vector $\mu=(\mu_i)_{i=1}^N$ and unit covariance matrix.

Consider the problem of testing the hypothesis H_0:$\mu=\mathbf{0}$ against H_1:$\mu\neq\mathbf{0}$. Having no prior knowledge about the deviations from H_0 it is natural to restrict oneself to the class of symmetric (permutation invariant) tests, i.e. the tests with critical functions, $\phi(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^N$, satisfying

$$\phi(\mathbf{rx}) = \phi(\mathbf{x}), \quad \mathbf{rx} = (x_{r_1}, \ldots, x_{r_N}), \tag{1.1}$$

for any permutation $\mathbf{r}=(r_1, \ldots, r_N)$ of $1,..., N$.

The permutations of coordinates form a subgroup of the group of rotations around the axis with the direction vector $\mathbf{1}=(1, \ldots, 1)$, i.e. the group of orthogonal transformations leaving the vector $\mathbf{1}$ unchanged. Denote by $\mathfrak{M}_{N1}$ and $\mathfrak{M}_{N2}$ the subspaces of $\mathbb{R}^N$ invariant under these rotations, namely, let

$$\mathfrak{M}_{N1} = \{\mathbf{x}:x_1 = \cdots = x_N\}, \ \mathfrak{M}_{N2} = \{\mathbf{x}:\Sigma x_i = 0\}. \tag{1.2}$$

In the problems of testing H_0 against H_{11}:$\mu\in\mathfrak{M}_{N1}$ and H_{12}:$\mu\in\mathfrak{M}_{N2}$ there exist uniformly MP rotation invariant tests based on

$$\bar{x}_N = N^{-1}\,\Sigma\, x_i \text{ and } Z_N^2 = \Sigma\,(x_i - \bar{x}_N)^2 \tag{1.3}$$

respectively. While the former is also a uniformly MP test in the respective problem (it is, in fact, a uniformly MP test in the unrestricted class of tests), the latter fails to be so. Theorem 1.1 to be stated

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below shows, however, that it becomes an asymptotically MP symmetric test for H_0 against H_{12} when $N \to \infty$.

Given α, $0<\alpha<1$, denote by $\phi_N(\mathbf{x})$ the critical function and by $\beta_N(\mu)=E_{N\mu}\phi_N$ the power function of the test which rejects H_0 when Z_N^2 exceeds $\chi^2_{N-1,1-\alpha}$, that is the $(1-\alpha)$-quantile of the χ^2_{N-1}-distribution (chi-square distribution with $N-1$ degrees of freedom);

$$\phi_N(\mathbf{x}) = \mathbf{1}\{Z_N^2 > \chi^2_{N-1,1-\alpha}\},\ \beta_N(\mu) = P_{N\mu}(Z_N^2 > \chi^2_{N-1,1-\alpha}) \tag{1.4}$$

where $\mathbf{1}\{A\}$ denotes the indicator function of the event A. Under $P_{N\mu}$, $\mu \in \mathfrak{M}_{N2}$, Z_N^2 has the $\chi^2_{N-1}(\|\mu\|^2)$ - distribution (noncentral χ^2_{N-1}-distribution with the non-centrality parameter $\|\mu\|^2 = \Sigma\mu_i^2$). This distribution is asymptotically normal as $N \to \infty$ with parameters

$$E_{N\mu}Z_N^2 = N-1+\|\mu\|^2,\ \mathrm{var}_{N\mu}\, Z_N^2 = 2(N-1+2\|\mu\|^2).$$

Therefore

$$\beta_N(\mu) \to \Phi(2^{-1/2}s^2 - u_\alpha) \text{ as } N \to \infty \tag{1.5}$$

with $s = \|\mu\| N^{-1/4}$ where Φ is the normal $\mathfrak{N}(0,1)$ distribution function and $u_\alpha = \Phi^{-1}(1-\alpha)$. The convergence in (1.5) is uniform in $s \geq 0$ because $\beta_N(\mu)$ are monotone increasing and bounded above (by 1) functions of s.

For comparison, the power function of the *MP* test for testing H_0 against a simple alternative $P_{N\mu_1}$, $\mu_1 = (\mu_{1i}^N) \in \mathbb{R}^N$, is $\Phi(\|\mu_1\|^{-1}(\mu_1, \mu) - u_\alpha)$ where $(\mu_1, \mu) = \Sigma \mu_{1i}\mu_i$. Thus for $\mu = t\mu_1$, $t>0$, this test distinguishes alternatives with bounded $\|\mu\|$ (independent of N) whereas it is insensitive to all alternatives $\mu \perp \mu_1$. In contrast to this, Z_N^2 is sensitive to a 'wide range' of alternatives $\mu \in \mathfrak{M}_{2N}$ (dim $\mathfrak{M}_{2N} = N-1$), lying however, sufficiently far from H_0, $\|\mu\| \asymp N^{1/4}$.

$\mathbb{N}$ will denote the set of natural numbers.

THEOREM 1.1. *For any sequence $\mu_N = (\mu_{Ni})_{i=1}^N \in \mathfrak{M}_{N2}$ satisfying the condition*

$$N^{-1}\sum_{i=1}^N |\mu_{Ni}|^r / (N^{-1}\sum_{i=1}^N \mu_{Ni}^2)^{r/2} = 0(1),\ r \in \mathbb{N}, \tag{1.6}$$

and any sequence of size α symmetric tests the inequality

$$\limsup_{N\to\infty} [\beta'_N(\mu_N) - \beta_N(\mu_N)] \leq 0 \tag{1.7}$$

holds where $\beta'_N(\mu)$ denotes the power function of a competing test.

REMARK. The assumption that (1.6) holds for all natural r was adopted in order to facilitate the proof which requires, in fact, only a finite (although rather large) number of relations (1.6). The author conjectures that (1.6) with $r=4$ is sufficient but the proof under this condition should presumably be much more complicated.

The proof of Theorem 1.1 will be given in Section 2. This theorem is an immediate consequence of Theorem 1.2 to be stated below which establishes an asymptotic structure of the 'symmetrized' likelihood ratio (LR).

Namely, to obtain Theorem 1, one has to prove (1.7) with $\beta'_N = \beta_N^*$, the power of the MP symmetric test ϕ_n^*, say, for testing H_0 against the simple alternative $\mu = \mu_N$. This test is given by the Neyman-Pearson lemma applied to the restrictions of P_{N0} and $P_{N\mu_N}$ to the subfield of permutation invariant sets.

Generally, if P_0 and P_1 are two distributions on a sample space $(\mathfrak{X}, \mathfrak{A})$ such that $P_1 \ll P_0$ and $P_0^{\mathfrak{B}}$, $P_1^{\mathfrak{B}}$ their restrictions to a subfield $\mathfrak{B} \subset \mathfrak{A}$, then

$$dP_1^{\mathfrak{B}}/dP_0^{\mathfrak{B}} = E_0[dP_1/dP_0 | \mathfrak{B}]\ P_0 - \text{a.e.}$$

where $E_0[\ldots|\mathfrak{B}]$ is a conditional expectation relative to P_0.

In our case, let $h_N(\mathbf{x})$ be the LR of $P_{N\mu_N}$ and P_{N0}, i.e. the ratio of the corresponding normal densities which is

$$h_N(\mathbf{x}) = \exp[\Sigma \mu_i x_i - \frac{1}{2}\|\mu_N\|^2]. \tag{1.8}$$

Then the LR, $\bar{h}_N(\mathbf{x})$, say, of the restrictions in question is obtained by symmetrization of $h_N(\mathbf{x})$, i.e.

$$\bar{h}_N(\mathbf{x}) = (N!)^{-1} \sum_r h_n(\mathbf{rx}) \tag{1.9}$$

(see LEHMANN (1959), Chapter 2, Section 4). Let $s_N = \|\mu_n\| N^{-1/4}$.

THEOREM 1.2. *Let $\mu_N \in \mathfrak{N}_{N2}$ satisfy (1.6) and $\{s_N\}$ be bounded. Then*

$$\delta_N := E_{N0}|\bar{h}_N(\mathbf{x}) - g_N(\mathbf{x}; s_N)| \quad \to 0 \text{ as } N \to \infty \tag{1.10}$$

where

$$g_N(\mathbf{x}; s) = \exp[W_N(\mathbf{x}; s)] \tag{1.11}$$

with

$$W_N(\mathbf{x}; s) = \frac{s^2(Z_N^2 - N)}{2N^{1/2}} - \frac{s^4}{4}, \tag{1.12}$$

Z_N^2 being defined by (1.3).

The proof will be given in Sections 3 to 15. It is based on an asymptotic analysis of simple linear rank statistics (SLRS), see, e.g., HÁJEK and SIDAK (1967). Namely, let $\mathbb{R}_N = (R_{N1}, \ldots, R_{NN})$ be a random permutation of $(1, \ldots, N)$ independent of $\mathbf{x}$ and taking on any of $N!$ possible values with probability $1/N!$. The corresponding distribution on the set of permutations of $(1, \ldots, N)$ will be denoted by P_{NR} and the joint distribution of $\mathbf{x}$ (under H_0) and $\mathbb{R}_N$, i.e. $P_{N0} \times P_{NR}$, by P_N. The subscript N will mostly be suppressed, so that $P = P_0 \times P_R$. The symbols E_0, E_R and E will denote the expectations relative to P_0, P_R and P respectively.

Let

$$S_N = \sum_{i=1}^{N} x_i m_{NR_i} \tag{1.13}$$

where

$$\mathbf{m}_N = (m_{Ni})_{i=1}^N = \mu_N / \|\mu_N\|. \tag{1.14}$$

Then $\bar{h}_N$ (see (1.8), (1.9)) can be written as

$$\bar{h}_N(\mathbf{x}) = E_R \exp(\|\mu_N\| S_N - \frac{1}{2}\|\mu_N\|^2). \tag{1.15}$$

For a fixed x, S_N is a SLRS with

$$E_R S_N = 0, \ \mathrm{var}_R S_N = Z_N^2/(N-1).$$

It is seen that $\mathrm{var}_R S_N \to 1$ as $N \to \infty$ for the 'most part' of $\mathbf{x}$'s, i.e. on a set of P_0-probability tending to 1. Moreover, it can be shown that for the most part of $\mathbf{x}$'s $S_N \xrightarrow{d} \mathfrak{N}(0,1)$ where d means the convergence in distribution, see HÁJEK and SIDAK (1967), HÁJEK (1961).

The right hand side of (1.15) with S_N replaced by a normal $\mathfrak{N}(0,1)$ random variable equals 1. If the sequence $\{\|\mu_N\|\}$ were bounded, this asymptotic normality could be used to establish a degenerate convergence $\bar{h}_N(\mathbf{x}) \to 1$ thus showing that symmetric tests are asymptotically insensitive to such sequences of alternatives. See CHIBISOV (1961), GVANCELADZE and CHIBISOV (1979) for results obtained by this argument.

In contrast with this, Theorem 2 describes a non-degenerate asymptotic behaviour of the symmetrized LR related to the sequences of alternatives with $\|\mu_N\| \asymp B^{1/4}$ for which symmetric tests attain a non-trivial asymptotic power. Note that in this case the exponent in (1.15) 'blows up' as $N \to \infty$ with variance of order $N^{1/2}$.

The present setup can be viewed as an asymptotic model for the problem of testing the equiprobability of the outcomes in a multinomial distribution when the number of trials tends to infinity. The author's objective is to prove the asymptotic optimality of the chi-square test in this problem within the class of tests symmetric in frequencies when both the number of trials and the number of outcomes tend to infinity. There are results on asymptotic optimality of the chi-square test within certain classes of tests having some specific structure (see, e.g., IVCHENKO and MEDVEDEV (1978)). The class of symmetric tests is, however, the most natural one. (For a related discussion, see CHIBISOV (1991)). Asymptotic analysis of the symmetrized LR developed in the present paper provides us with a principal tool for this problem.

The proof of Theorem 1.2 will be carried out in three stages.

At the first stage (Section 3) the proof of (1.10) by means of a Scheffé type lemma is reduced to the proof of the following assertion: for any $\epsilon>0$

$$P_0\{\bar{h}_N(\mathbf{x};\mu_N) - g_N(\mathbf{x};s_N)(1-\epsilon)\} \to 0 \;\; as \;\; N\to\infty.$$

At the second stage (Section 4) the representation (1.15) is analyzed by introducing several types of conditioning which make S_N a sum of conditionally independent components. (A similar technique was used in CHIBISOV (1991).) This stage results in an inequality of the form

$$\bar{h}_N(\mathbf{x};\mu_N) \geqslant E_R \exp[T_N(\mathbf{x}, \mathbb{R}_N)] \tag{1.17}$$

where T_N is a certain expression depending on conditional moments of the components of S_N (see (4.21)).

At the third stage (Sections 5 to 15) we prove that

$$T_N(\mathbf{x}, \mathbb{R}) - W_N(\mathbf{x}) \xrightarrow{P} 0 \text{ as } N \to \infty. \tag{1.18}$$

We shall show now that the assertions [(1.16) $\Rightarrow$ (1.10)], (1.17) and (1.18) prove Theorem 1.2. Namely, we shall show that (1.17) and (1.18) imply (1.16) and thus Theorem 1.2. Let, for $\epsilon>0$,

$$\pi_N(\mathbf{x}, \epsilon) = P_R\{|T_N(\mathbf{x}, \mathbb{R}) - W_N(\mathbf{x})| > \epsilon\}, \tag{1.19}$$

$$B_N(\epsilon) = \{\mathbf{x} \in \mathbb{R}^N \colon \pi_N(\mathbf{x}, \epsilon) \leqslant \epsilon\}. \tag{1.20}$$

Since

$$P\{|T_N(\mathbf{x}, \mathbb{R}) - W_N(\mathbf{x})| > \epsilon\} = E_0 \pi_N(\mathbf{x}, \epsilon) \geqslant \epsilon P_0(B_N^c(\epsilon))$$

(B^c denoting the complement of B), (1.18) implies that

$$P_0(B_N^c(\epsilon)) \to 0 \quad \text{for any } \epsilon > 0. \tag{1.21}$$

For $\mathbf{x} \in B_N(\epsilon/2)$ we have by (1.1) and definitions (1.11), (1.19), (1.20) (suppressing the arguments $\mathbf{x}$ and $\mathbf{R}$ of T_N and W_N):

$$\bar{h}_N(\mathbf{x}, \mu_N) \geqslant E_R \exp T_N \geqslant E_R[\exp T_N; T_N \geqslant W_N - \frac{\epsilon}{2}]$$

$$\geqslant e^{-\epsilon/2}(1-\frac{\epsilon}{2})\exp W_N \geqslant (1-\epsilon) g_N(x; s_N).$$

Therefore the inequality in (1.16) may only be fulfilled on $B_N^c(\epsilon/2)$, hence (1.16) follows from (1.21).

2. PROOF OF THEOREM 1.1.

By (1.5) for any $\epsilon>0$ one can find $C_\epsilon>0$ and $N_\epsilon \in \mathbb{N}$ such that $\beta_N(\mu) > 1-\epsilon$ whenever $s \geqslant C_\epsilon$ and

$N \geqslant N_\epsilon$. Therefore it is sufficient to prove Theorem 1.1 under an additional assumption that $\{s_N\}$ is bounded so that Theorem 1.2 is applicable.

Let $\phi_N^\star(x)$ be the critical function of the MP symmetric size α test for testing P_{N0} against $P_{N\mu_N}$. Its power under the alternative $P_{N\mu_N}$ is

$$\beta_N^\star = E_0 \phi_N^\star \bar{h}_N. \tag{2.1}$$

Let $\psi_N(\mathbf{x}) = E_0[\phi_N^\star(\mathbf{x}) \mid Z_N^2]$. Then ψ_N is a critical function of some size α test since $E_0\psi_N = E_0\phi_N^\star = \alpha$. Since g_N is a function of Z_N^2 (see (1.11)) one has

$$\tilde{\beta}_N := E_0 \phi_N^\star g_N(\cdot; s_N) = E_0 \psi_N g_N(\cdot; s_N). \tag{2.2}$$

The test ϕ_N (see (1.4)) is the MP one among the tests which are functions of Z_N^2, hence

$$\beta_N := \beta_N(\mu_N) \geqslant E_0 \psi_N \bar{h}_N. \tag{2.3}$$

Using (1.10) we obtain from (2.1)-(2.3)

$$\beta_N^\star \leqslant \tilde{\beta}_N + \delta_N \leqslant E_0\psi_N \bar{h}_N + 2\delta_N \leqslant \beta_N + 2\delta_N$$

which implies the theorem.

3. Proof of Theorem 1.2.

Reduction to (1.16). This part of the proof is based on the following 'one-sided' version of the well-known Scheffé lemma.

Lemma 3.1. *Let U_N and V_N be nonnegative random variables defined for each $N \in \mathbb{N}$ on some probability space $(\mathfrak{X}_N, \mathcal{Q}_N, P_N)$. Assume that the integrals $E_N V_N = \int V_N dP_N$ are uniformly absolutely continuous, i.e.*

$$E_N[V_N; A_N] = \int_{A_N} V_N dP_N \to 0 \textit{ whenever } P_N(A_N) \to 0. \tag{3.1}$$

Moreover, assume that $EU_N \to 1$, $EV_N \to 1$ and

$$P_N\{U_N < V_N - \epsilon\} \to 0 \quad \textit{as } N \to \infty. \tag{3.2}$$

Then

$$E_N |U_N - V_N| \to 0 \quad \textit{as } N \to \infty.$$

Proof. Let $Y_N = U_N - V_N$ and, for an arbitrary $\epsilon > 0$, $B_N = \{Y_N < -\epsilon\}$, $C_N = \{|Y_N| \leqslant \epsilon\}$, $D_N = \{Y_N > \epsilon\}$. Then, by assumptions, $\alpha_N := E_N Y_N \to 0$ and $P_N(B_N) \to 0$ which, by (3.1), implies $\beta_N := E_N[V_N; B_N] \to 0$. Since U_N and V_N are nonnegative, $-V_N \leqslant Y_N \leqslant -\epsilon$ on B_N, hence $E_N[|Y_N|; B_N] \leqslant \beta_N \to 0$. As $U_N \geqslant V_N - \epsilon$ on $C_N \cup D_N = \mathfrak{X}_N \setminus B_N$ one has

$$\liminf E_N[U_N;\ C_N \cup D_N] \geqslant 1 - \epsilon,$$

thus $E_N[U_N; B_N] < 2\epsilon$ for sufficiently larger N. Obviously, $\gamma_N := E_N[|Y_N|;\ C_N] \leqslant \epsilon$. Then for

$$E_N[Y_N; D_N] = E_N Y_N - E_N[Y_N; B_N] - E_N[Y_N; C_N]$$

one has

$$\alpha_N - 2\epsilon - \gamma_N \leqslant E_N[Y_N; D_N] \leqslant \alpha_N + \beta_N + \gamma_N.$$

The relations obtained above imply the lemma.

By means of standard compactness arguments the proof of (1.20) is reduced to the case of convergent sequence $\{s_N\}$, thus we shall assume that $s_N \to s_0 \geqslant 0$. We shall apply Lemma 3.1 with $\bar{h}_N$ and g_N as U_N and V_N respectively. The LR $\bar{h}_N$ satisfies $E_{n0}\bar{h}_N = 1$. The convergence $E_{n0}g_N \to 1$ follows from the explicit expression

$$E_{n0}g_N = (1-\frac{s_N^2}{\nu^{1/2}})^{\nu/2}\exp(-\frac{s_N^2\nu^{1/2}}{2}-\frac{s_N^4}{4}),\ \nu = N-1,$$

which is obtained from the well-known form of the χ_ν^2-distributions. Since under P_{n0}

$$(Z_N^2 - N)(2N)^{-1/2} \xrightarrow{d} \mathfrak{N}(0,1),$$

g_N has the lognormal limit distribution with expectation 1. Therefore g_N are uniformly integrable with respect to P_{n0} and $E_{n0}g_N$ are uniformly absolutely continuous, see Loéve (1969) 9.4.e and 11.4.A. Thus $\bar{h}_N$ and g_N satisfy the conditions of Lemma 3.1 which implies that the relation

$$\forall \epsilon > 0,\ P_0(\bar{h}_N < g_N - \epsilon) \to 0 \ \text{ as } N \to \infty \tag{3.3}$$

ensures (1.10). Since g_N are bounded in P_0-probability, (3.3) follows from (1.16). Thus (1.16) implies (1.10).

4. Conditioning.

Fix some numbers $\kappa, \lambda = 1-\kappa, \kappa'$ and $\lambda' = \kappa - \kappa'$ satisfying the inequalities

$$\frac{2}{3} < \kappa < \frac{3}{4}\left(\frac{1}{4} < \lambda < \frac{1}{3}\right),\ \kappa + \frac{1}{2}\lambda' < 1 \ \text{ and } 0 < \kappa' < \frac{1}{2}. \tag{4.1}$$

(One may take, for example, $\kappa = 2/3 + \alpha$, $\kappa' = 4\alpha$ with an arbitrary $0 < \alpha < 1/12$). Take some sequences of integers $k'_N \asymp N^{\kappa'}$, $l'_N \asymp N^{\lambda'}$, set $k_N = k'_N l'_N$ and define integers l_N and d_N by

$$N = \kappa_N l_N + d_N,\ 0 \leqslant d_N < l_N. \tag{4.2}$$

Then $k_N \asymp N^\kappa$ and $l_N \asymp N^\lambda$. In what follows the subscript N in these notations will be suppressed.

Define the sets of indices

$$I_j = \{(j-1)l + 1, \ldots, jl\},\ j = 1, \ldots, k,\ I_0 = \{kl+1, \ldots, N\}$$

and associate with any permutation $\mathbf{r} = (r_1, \ldots, r_N)$ the (unordered) sets ('blocks')

$$\mathfrak{R}_j = \{r_i, i \in I_j\},\ j = 0, 1, \ldots, k. \tag{4.3}$$

Consider the conditional distributions of $\mathbf{R}$ given the ordered collection of blocks $\mathfrak{R}_0, \mathfrak{R}_1, \ldots, \mathfrak{R}_k$ corresponding to a permutation r. Then conditionally the random permutation $\mathbb{R}$ splits into independent random permutations of elements of r within blocks. This conditioning corresponds to the conditional distribution and expectation, to be denoted by P_{R1} and E_{R1} with respect to the decomposition $\mathfrak{D}_1$ (of the set of permutations) determined by the following equivalence relation: $\mathbf{r}_1 \sim \mathbf{r}_2$ if $\mathfrak{R}_{1j} = \mathfrak{R}_{2j}$, $j = 0, 1, \ldots, k$, i.e. $\mathbf{r}_1$ and $\mathbf{r}_2$ may only differ by the order of their components within blocks. (For the notions of conditional probability and expectation with respect to a decomposition, see, e.g., Shiryayev (1984), Ch. I, § 8.)

Let

$$S_{Nj} = \sum_{i \in I_j} x_i m_{R_i},\ \ j = 0, 1, \ldots, k \tag{4.4}$$

($S_{N0} = 0$ if $d = 0$). Then $S_N = \sum_{j=0}^{k} S_{Nj}$ where $S_{N0}, S_{n1}, \ldots, S_{Nk}$ under P_{R1} are conditionally independent SLRS.

Let, for $j = 0, 1, \ldots, k$ and $r = 2, 3, \ldots$

$$\alpha_j = E_{R1}S_{Nj}.\ \ \beta_{rj} = E_{R1}\tilde{S}_{Nj}^r \tag{4.5}$$

where $\tilde{S}_{Nj} = S_{Nj} - \alpha_j$. Then letting

$$f_{Nj}(\mathbf{x}) = E_{R1}\exp(s_N N^{1/4}\tilde{S}_{Nj}),\ \ j = 0, 1, \ldots, k, \tag{4.6}$$

we have (cf. (1.13), (1.15))

$$\bar{h}_N(\mathbf{x}) = E_R[\exp(s_N N^{1/4} \sum_{j=0}^{k} \alpha_j - \frac{1}{2} s_N^2 N^{1/2}) \prod_{j=0}^{k} f_{Nj}(\mathbf{x}). \tag{4.7}$$

Using the inequality $e^t \geqslant \sum_{r=0}^{m} t^r/r!$, $t \in \mathbb{R}^1$, for an odd m we obtain

$$f_{Nj}(\mathbf{x}) \geqslant 1 + \zeta_{Nj}(\mathbf{x}) \tag{4.8}$$

where (see (4.5), (4.6))

$$\zeta_{Nj}(\mathbf{x}) = \sum_{r=2}^{7} s_N^r N^{r/4} \beta_{rj}/r!. \tag{4.9}$$

Further, using the inequality

$$\ln(1+t) \geqslant t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4, \quad t \geqslant 0,$$

and the fact that $f_{Nj}(\mathbf{x}) \geqslant 1$ by Jensen's inequality, we see from (4.8) that

$$f_{Nj}(\mathbf{x}) \geqslant \exp[\sum_{r=1}^{4} \frac{(-1)^{r-1}}{r} \zeta_{Nj}^r(\mathbf{x})]. \tag{4.10}$$

Therefore (see (4.7))

$$\bar{h}_N(\mathbf{x}) \geqslant E_R \exp[\sum_{j=0}^{k} (s_N N^{1/4} \alpha_j + \sum_{r=1}^{4} \frac{(-1)^{r-1}}{r} \zeta_{Nj}^r(\mathbf{x})) - \tfrac{1}{2} s_N^2 N^{1/2}]. \tag{4.11}$$

We will introduce now the conditional distribution, P_{R2}, of the vector of permutation R with respect to a coarser decomposition $\mathfrak{D}_2$. Split the set of indices $j = 1, \ldots, k$ into k' groups of size l':

$$J_v = \{j : (v-1)l' + 1 \leqslant j \leqslant vl'\}, \quad v = 1, \ldots, k'. \tag{4.12}$$

Let $r_1 \sim r_2$ if for each $v = 1, \ldots, k'$ there exists a permutation $(q_{vj})_{j \in J_v}$ of the set J_v such that $\mathfrak{R}_{10} = \mathfrak{R}_{20}$ and $\mathfrak{R}_{1j} = \mathfrak{R}_{2q_{vj}}$ for $j \in J_v$, $v = 1, \ldots, k'$. In this case the condition on $\mathbb{R}$ specifies blocks $\mathfrak{R}_1, \ldots, \mathfrak{R}_k$, $\mathfrak{R}_0$ and a partition of the set $\{\mathfrak{R}_1, \ldots, \mathfrak{R}_k\}$ into k' groups, so that the blocks (with randomly ordered elements) are arranged in a random order within each group. The order of groups in this case is fixed (unlike $\mathfrak{D}_3$ to be introduced in Section 8).

Let (see (4.5), (4.9) and (4.12))

$$A_v = \sum_{j \in J_v} \alpha_j, \; Z_{tv} = \sum_{j \in J_v} \zeta_j^t, \; t \in \mathbb{N}, \; v = 1, \ldots, k' \tag{4.13}$$

(for brevity, we suppress the dependence on N and x). Note that (A_v, Z_{tv}) for different v are conditionally independent under P_{R2}. Let for $v = 1, \ldots, k'$, $t \in \mathbb{N}$

$$\bar{A}_v = E_{R2} A_v, \; \tilde{A}_v = A_v - \bar{A}_v, \tag{4.14}$$

$$\bar{Z}_{tv} = E_{R2} Z_{tv}, \; \tilde{Z}_{tv} = Z_{tv} - \bar{Z}_{tv}, \tag{4.15}$$

$$Z_v = \sum_{t=1}^{4} \frac{(-1)^{t-1}}{t} z_{tv}, \; \bar{z}_v = E_{R2} Z_v, \; \tilde{Z}_v = Z_v - \bar{Z}_v, \tag{4.16}$$

$$\psi_v = s_N N^{1/4} \tilde{A}_v + \tilde{Z}_v \tag{4.17}$$

Let also $\bar{A}_0 = \alpha_0$, $\bar{Z}_{t0} = \zeta_0^t$. Then the right hand side of (4.11) can be written as

$$E_R\{\exp[\sum_{v=0}^{k'} (s_N N^{1/4} \bar{A}_v + \bar{Z}_v) - \tfrac{1}{2} s_N^2 N^{1/2}] \prod_{v=1}^{k'} E_{R2} e^{\psi_v}\}. \tag{4.18}$$

Note that, by (4.14)-(4.17), $E_{R2}\psi_\nu = 0$. By the arguments similar to those used for deducing (4.10) from (4.6), we get

$$E_{R2}e^{\psi_\nu} \geqslant 1+\omega_\nu \geqslant \exp(\omega_\nu - \tfrac{1}{2}\omega_\nu^2) \tag{4.19}$$

where

$$\omega_\nu = \tfrac{1}{2}E_{R2}\psi_\nu^2 + \tfrac{1}{6}E_{R2}\psi_\nu^3. \tag{4.20}$$

Let also $\omega_0 = 0$. Substituting (4.19) into (4.18) one obtains from (4.11) the inequality (1.17) with

$$T_N(\mathbf{x}, \mathbf{R}_N) = \sum_{\nu=0}^{k'} (s_N N^{1/4}\overline{A}_\nu + \overline{Z}_\nu + \omega_\nu - \tfrac{1}{2}\omega_\nu^2) - \tfrac{1}{2}s_N^2 N^{1/2}. \tag{4.21}$$

5. Proof of (1.18).

In this section (1.18) is reduced to a number of relations that will be proved in Sections 6 to 15.

Let, for $j=1,\ldots,k$ and $r\in\mathbb{N}$

$$X_{rj} = \frac{1}{l}\sum_{i\in I_j} x_i^r,\ \tilde{X}_{rj} = \frac{1}{l}\sum_{i\in I_j}(x_i - X_{1j})^r, \tag{5.1}$$

$$M_{rj} = \frac{1}{l}\sum_{i\in I_j} m_{R_i}^r,\ \tilde{M}_{rj} = \frac{1}{l}\sum_{i\in I_j}(m_{R_i} - M_{1j})^r; \tag{5.2}$$

define $X_{r0}, \tilde{X}_{r0}, M_{r0}, \tilde{M}_{r0}$ in a similar way with l replaced by d.

Recall that, for a SLRS $S_n = \sum_1^n c_i d_{R_i}$

$$ES_n = n\bar{c}\bar{d},\ \ var\,S_n = \frac{n^2}{n-1}(\overline{c^2} - \bar{c}^2)(\overline{d^2} - \bar{d}^2) \tag{5.3}$$

where $\bar{c} = \Sigma c_i/n$, $\overline{c^2} = \Sigma c_i^2/n$, etc. (see Hájek and Sidak (1967)).

Applying (5.3) to S_{Nj} (cf. (4.4), (4.5)) we obtain for $j=1,\ldots,k$

$$\alpha_j = lX_{1j}M_{1j},\ \beta_{2j} = \frac{l^2}{l-1}\tilde{X}_{2j}\tilde{M}_{2j}. \tag{5.4}$$

We need also β_{rj}, $3\leqslant r\leqslant 7$. The following formula is obtained from the one in Hájek and Sidak (1967), Chapter 2, Problems and Complements, no. 25:

$$\beta_{4j} = \frac{l^2}{(l-2)(l-3)}\Big[\frac{l(l+1)}{l-1}\tilde{X}_{4j}\tilde{M}_{4j} - 3l(\tilde{X}_{2j}^2\tilde{M}_{4j} + \tilde{X}_{4j}\tilde{M}_{2j}^2) + \frac{3(l^2-3l+3)l}{l-1}\tilde{X}_{2j}^2\tilde{M}_{2j}^2\Big]. \tag{5.5}$$

Note the formula to be used in the sequel which follows from (5.5), (5.4) and the inequalities $\tilde{X}_{2j}^2 \leqslant \tilde{X}_{4j}$, $\tilde{M}_{2j}^2 \leqslant \tilde{M}_{4j}$:

$$\beta_{4j} - 3\beta_{2j}^2 = l[(\tilde{X}_{4j} - 3\tilde{X}_{2j}^2)(\tilde{M}_{4j} - 3\tilde{M}_{2j}^2) - 6\tilde{X}_{2j}^2\tilde{M}_{2j}^2] + 0(1)\tilde{X}_{4j}\tilde{M}_{4j}. \tag{5.6}$$

In all other cases we shall use the following lemma.

Lemma 5.1. *For $r=2,3,\ldots$ and $j=1,\ldots,k$*

$$\beta_{rj} = \sum C_l(\{p_i\},\{q_i\})\prod_{i=1}^{a}\tilde{X}_{p_i,j}\prod_{i=1}^{b}\tilde{M}_{q_i,j} \tag{5.7}$$

where C_l's depend only on the indicated arguments, $C_l = 0(l^a)$, the summation extends over some set of groups of indices $\{p_i\}_{i=1}^a$, $\{q_i\}_{i=1}^b$ such that $\Sigma p_i = \Sigma q_i = r$, $p_i, q_i \geqslant 2$, all p_i, q_i being even if r is even. In the case r is even the coefficient $C_l(\{2,\ldots,2\},\{2,\ldots,2\})$ of $\tilde{X}_{2j}^{r/2}\tilde{M}_{2j}^{r/2}$ is

$$\frac{r!}{(r/2)!2^{r/2}}\, l^{r/2}(1+0(1/l)). \tag{5.8}$$

The proof will be omitted. The arguments are similar to those used in WALD and WOLFOWITZ (1944).

For any set of (not necessarily different) natural numbers $\{r_i\}_{i=1}^{t}$, $t \in \mathbb{N}$, $r_i \geqslant 2$, let

$$B_v(r_1, \ldots, r_t) = \sum_{j \in J_v} \beta_{r_1 j} \cdots \beta_{r_t j},\ v = 1,\ldots, k'. \tag{5.9}$$

Let, suppressing the arguments $r_1, \ldots, r_t$,

$$\overline{B}_v = E_{R2} B_v,\ \tilde{B}_v = B_v - \overline{B}_v. \tag{5.10}$$

Then by (4.9) and (4.13)

$$Z_{tv} = \sum_{2 \leqslant r_i, \ldots, r_t \leqslant 7} C(r_1, \ldots, r_t) s_N^{\Sigma r_i} N^{\Sigma r_i/4} B_v(r_1, \ldots, r_t) \tag{5.11}$$

where

$$C(r_1, \ldots, r_t) = t!/\prod_{\rho=2}^{7} [\tau_\rho!(\rho!)^{\tau_\rho}]$$

with $\tau_\rho = \#\{r_i = \rho,\ i = 1,\ldots, t\}, 0! = 1$. In particular, $C(r) = 1/r!$. We shall also need

$$C(6) = \frac{1}{6!},\ C(2,2) = \frac{1}{4},\ C(2,4) = \frac{1}{4!},\ C(2,2,2) = \frac{1}{8}. \tag{5.13}$$

By the definitions (4.15) and (5.10), $\overline{Z}_{1v}$ is given by (5.11) with B_v replaced by $\overline{B}_v$.

Note that any product $\beta_{r_1} \cdots \beta_{r_t}$, $r_t \geqslant 2$, has again the form of the right hand side of (5.7) with $\Sigma p_i = \Sigma q_i = \Sigma r_i$. Hence, due to (5.9) and (5.11), for $t \in \mathbb{N}$

$$Z_{tv} = \sum s_N^n N^{n/4} 0(l^{a \wedge b}) \sum_{j \in J_v} (\prod_{i=1}^{a} \tilde{X}_{p_i j} \prod_{i=1}^{b} \tilde{M}_{q_i j}) \tag{5.14}$$

where the outer summation extends over some finite set depending on t of groups of indices $\{p_i\}$, $\{q_i\}$, $p_i, q_i \geqslant 2$, $2t \leqslant n = \Sigma p_i = \Sigma q_i$. Obviously, Z_v (see (4.16)) is again a sum of the form (5.14).

The expressions (5.14) for different t may contain terms with the same $\{p_i\}$, $\{q_i\}$ which only differ by the factor depending on l. Consider, in particular, the sum $\sum_{j \in J_v} \tilde{X}_{2j}^3 \tilde{M}_{2j}^3$. It enters into $B_v(6)$, $B_v(2,4)$ and $B_v(2,2,2)$, the respective coefficients of $l^3(1+0(1/l))$ being 15, 3 and 1 (see (5.7)-(5.9)). By (5.11), (5.13) Z_{1v}, Z_{2v} and Z_{3v} contain this sum with a factor $s_N^6 N^{3/2} l^3(1+0(1/l))$ and coefficients $15/6! = 1/48$, $3/4! = 1/8$ and $1/8$ respectively. Hence this sum enters into $Z_v = \Sigma(-1)^{t-1} Z_{tv}/t$ with a factor $s_N^6 N^{3/2} 0(l^2)$ since the coefficients of l^3 cancel. (Note that a similar cancellation occurred to $\tilde{X}_{2j}^2 \tilde{M}_{2j}^2$ in (5.6)). The factor $\tilde{M}_{2j}^3$ combines with $\tilde{X}_{2j} \tilde{X}_{4j}$ and $\tilde{X}_{6j}$ where $a = 2$ and 1 (see (5.14)), thus the order of l in the corresponding terms is also $0(l^2)$.

We shall use the following notation for averaging over $j \in J_v$ (see (4.12)): for any $a_1, \ldots, a_k \in \mathbb{R}^1$

$$\langle a_j \rangle_v = \frac{1}{l'} \sum_{j \in J_v} a_j,\ v = 1,\ldots, k'. \tag{5.15}$$

Recall that under P_{R2} the blocks in each group J_v are arranged in a random order, hence by (5.3)

$$\overline{Z}_{tv} = \sum s_N^n N^{n/4} 0(l^{a \wedge b}) l' \langle \prod_{i=1}^{a} \tilde{X}_{p_i j} \rangle_v \langle \prod_{i=1}^{b} \tilde{M}_{q_i j} \rangle_v \tag{5.16}$$

with the same summation as in (5.14). By the above argument, the terms containing $\langle \tilde{M}_{2j}^3 \rangle_v$ enter into $\overline{Z}_v$ with a factor of order $0(l^2)$.

Taking into account (4.21), the analogue of (5.11) for $\overline{Z}_{tv}$ and the above analysis of $\overline{Z}_v$, it is seen that (1.18) is implied by the following relations (5.17)-(5.23) (where the convergence takes place in probability $P = P_0 \times P_R$, in summations v runs over $1,\ldots,k'$):

$$N^{1/4} \sum_v \bar{A}_v \xrightarrow{P} 0; \tag{5.17}$$

$$N^{1/2} \sum_v \bar{B}_v(2) - N^{-1/2} Z_N^2 + lN^{-1/2} Z_N^2 + lN^{-1/2} \sum_{j=1}^{k} X_{1j}^2 \xrightarrow{P} 0; \tag{5.18}$$

$$\frac{N}{24} \sum_v [\bar{B}_v(4) - 3\bar{B}_v(2,2)] + \frac{1}{4} \xrightarrow{P} 0; \tag{5.19}$$

$$N^{3/2} l^2 l' \sum_v \langle \prod_i \tilde{X}_{p_i,j} \rangle_v \langle \tilde{M}_{2j}^3 \rangle_v \xrightarrow{P} 0 \tag{5.20}$$

where $\prod_i \tilde{X}_{p_i,j}$ is either $\tilde{X}_{6,j}$ or $\tilde{X}_{2j}\tilde{X}_{4j}$ or $\tilde{X}_{2j}^3$;

$$N^{n/4} l^b l' \sum_v \langle \prod_{i=1}^{a} \tilde{X}_{p_i,j} \rangle_v \langle \prod_{i=1}^{b} \tilde{M}_{q_i,j} \rangle_v \xrightarrow{P} 0 \tag{5.21}$$

for any $\{p_i\}, \{q_i\}$ with $p_i, q_i \geqslant 2$, $\Sigma p_i = \Sigma q_i = n \geqslant 3$ except $n = 4$ and the terms with $(q_1, q_2, q_3) = (2,2,2)$:

$$\sum_v \omega_v - \frac{1}{2} s_N^2 N^{-1/2} l \sum_{j=1}^{k} X_{1j}^2 \xrightarrow{P} 0 \tag{5.22}$$

$$\sum_v \omega_v^2 \xrightarrow{P} 0. \tag{5.23}$$

In (5.17)-(5.23) we neglect the terms of (4.21) with $v = 0$. Following the same methods the reader can check that their main contribution is $N^{-1/2} \sum_{I_0} x_i^2 \xrightarrow{P} 0$.

Next we shall reduce (5.22) and (5.23) to some relations of more specific form. First, we introduce the following notation.

For random variables S_N and a sequence $\Delta_N > 0$ we shall write: $S_N = 0_P(\Delta_N)$ if S_N/Δ_N is bounded in probability, i.e.

$$\forall \epsilon > 0,\ \exists C(\epsilon),\ P\{|S_N| > C(\epsilon)\Delta_N\} < \epsilon\};$$

$S_N = o_P(\Delta_N)$ if $S_N/\Delta_N \xrightarrow{P} 0$;

$S_N = o_P^{\epsilon}(\Delta_N)$ if $S_N = o_p(\Delta_N N^{\epsilon})$ for any $\epsilon > 0$;

$S_N = \Omega(\Delta_N)$ if, for any $\epsilon > 0$ and $n > 0$,

$$P\{|S_N| > \Delta_N N^{\epsilon}\} = o(N^{-n}). \tag{5.26}$$

Obviously, $[S_N = \Omega(\Delta_N)] \Rightarrow [S_N = o_p^{\epsilon}(\Delta_N)]$. Moreover, if $S_{n1}, \ldots, S_{Nk}$ are identically distributed, $S_{N1} = \Omega(\Delta_N)$ and $k = 0(N^p)$ for some $p > 0$, then

$$\max_{1 \leqslant j \leqslant k} |S_{Nj}| = \Omega(\Delta_N) \text{ and } \sum_{j=1}^{k} S_{Nj} = \Omega(k\Delta_N).$$

We shall show now that the following relations (5.27)-(5.30) imply (5.22) and (5.23) ($v = 1,\ldots,k'$):

$$N^{1/2} \sum_v E_{R2} \tilde{A}_v^2 - N^{-1/2} l \sum_{j=1}^{k} X_{1j}^2 \xrightarrow{P} 0; \tag{5.27}$$

$$N^{3/4} \sum_v E_{R2} \tilde{A}_v^3 \xrightarrow{P} 0,\quad N^{3/2} \sum_v (E_{R2} \tilde{A}_v^3)^2 \xrightarrow{P} 0; \tag{5.28}$$

$$E_R \tilde{A}_1^r = \Omega((l')^{r/2}/N^{r/2}),\ \ r = 2,4; \tag{5.29}$$

$$E_R|\tilde{Z}_v^r| = \Omega(l'/N),\ \ r = 2,3,4. \tag{5.30}$$

By (4.17) and (4.20) there exists a constant $C>0$ such that

$$|\omega_v - E_{R2}(\tfrac{1}{2}s_N^2 N^{1/2}\tilde{A}_v^2 + \tfrac{1}{6}s_N^3 N^{3/4}\tilde{A}_v^3)| \leqslant$$

$$\leqslant C\, E_{R2}(N^{1/4}|\tilde{A}_v\tilde{Z}_v| + \tilde{Z}_v^2 + N^{1/2}|\tilde{A}_v^2\tilde{Z}_v| + N^{1/4}|\tilde{A}_v\tilde{Z}_v^2| + |\tilde{Z}_v^3|). \tag{5.31}$$

Since $E_R E_{R2}(...) = E_R(...)$, making use of the Cauchy-Schwarz inequality one obtains from (5.29) and (5.30) that

$$E_R \sum_v E_{R2}|\tilde{Z}_v^r| = o_p^\epsilon(k/N) \xrightarrow{P} 0,\ r = 2,3, \tag{5.32}$$

$$E_R \sum_v E_{R2}(N^{1/4}|\tilde{A}_v\tilde{Z}_v^r|) = o_p^\epsilon(k/N^{3/4}) \xrightarrow{P} 0,\ r = 1,2,... \tag{5.33}$$

and

$$E_R \sum_v E_{R2}(N^{1/2}|\tilde{A}_v^2\tilde{Z}_v|) = o_p^\epsilon(k(l')^{1/2}/N) \xrightarrow{P} 0 \tag{5.34}$$

(recall that $k \asymp N^\kappa$, $l' \asymp N^{\lambda'}$ with $\kappa + (\lambda'/2) < 1$, see (4.1)). Therefore, in view of (5.31), (5.27)-(5.30) imply (5.22).

Due to the inequality $(\sum_1^n a_i)^2 \leqslant n \sum_1^n a_i^2$, (5.23) follows from the relations:

$$N^{r/2}\sum_v (E_{R2}\tilde{A}_v^r)^2 \xrightarrow{P} 0,\ \ r = 2,3, \tag{5.35}$$

$$\sum_v (E_{R2}\tilde{Z}_v^r)^2 \xrightarrow{P} 0,\ \ r = 2,3, \tag{5.36}$$

$$N^{1/2}\sum_v (E_{R2}\tilde{A}_v\tilde{Z}_v^r)^2 \xrightarrow{P} 0,\ \ r = 1,2, \tag{5.37}$$

and

$$N\sum_v (E_{R2}\tilde{A}_v^2\tilde{Z}_v^2)^2 \xrightarrow{P} 0. \tag{5.38}$$

The relation (5.35) with $r=2$ follows from (5.29) with $r=4$ according to the inequality $(E_{R2}\tilde{Z}_v^2)^2 \leqslant E_{R2}\tilde{Z}_v^4$ and (5.35) with $r=3$ is a part of (5.28). Finally, (5.32)-(5.34) imply (5.36)-(5.38) by means of the inequality $\Sigma a_i^2 \leqslant (\Sigma|a_i|)^2$.

Thus the proof of (1.18) reduces to that of (5.17)-(5.21), (5.27)-(5.30).

6. Auxiliary results.

Set for $r \in \mathbb{N}$

$$K_r = N^{r/2} E_R m_{R_1}^r = N^{\frac{r}{2}-1}\sum_{i=1}^N m_i^r. \tag{6.1}$$

By the assumption $\mu_N \in \mathfrak{M}_{N2}$ (see (1.2)), the definition (1.1), and the condition (1.6),

$$K_1 = 0,\ K_2 = 1,\ K_r = 0(1)\ \ as\ N \to \infty. \tag{6.2}$$

Applying to $M_{rj}, j = 1,..., k, r \in \mathbb{N}$, (see (5.2)) the formulae (5.3) with $n = N$, $c_i = 1(0)$ for $i \in I_j\ (i \notin I_j)$

and $d_i = m_i$ one obtains

$$E_R M_{rj} = \frac{K_r}{N^{r/2}}, \ \mathrm{var}_R\, M_{rj} = \frac{N-l}{N-1}\, \frac{K_{2r} - K_r^2}{N^r l}. \tag{6.3}$$

Moreover (see, e.g., HÁJEK and SIDAK (1967), Ch. 2, § 3.1, formula (25)), for $j_1 \neq j_2$, $r \in \mathbb{N}$

$$\mathrm{cov}(M_{rj_1}, M_{rj_2}) = -(K_{2r} - K_r^2)/N^r(N-1). \tag{6.4}$$

The asymptotic behaviour of higher order (product) moments of M_{rj} is given by the following lemma. It is stated for some 'initial' set of indices j ($j = 1,\ldots, a+b$) but it holds for any set of j's because of symmetric dependence of $\{M_{rj}\}$ on j.

LEMMA 6.1. *Under the condition (1.6) for any integers $a, b \geqslant 0, c_j \geqslant 0, p_j \geqslant 0$ such that $p_j \geqslant 2$ if $c_j = 0, j = 1,\ldots, b$, and $q_{ij} \geqslant 1, r_{ij} \geqslant 2, i = 1,\ldots, c_j, j = 1,\ldots, b$, one has*

$$E_R[\prod_{j=1}^{a} M_{1j} \prod_{j=1}^{b} (M_{1,a+j}^{p_j} \prod_{i=1}^{c_j} M_{r_{ij},a+j}^{q_{ij}}) =$$

$$= \prod_{j=1}^{b} (C(p_j) \prod_{i=1}^{c_j} C(q_{ij}))(1 + 0(l^{-1}))l^{-\Sigma(p_j - [p_j/2])} N^{-(3a/2) + [a/2] - (\Sigma p_j + \Sigma\Sigma r_{ij} q_{ij})/2} \tag{6.5}$$

where

$$\begin{aligned} C(p) &= 1 && \text{for } p = 1, \\ &= \frac{(2n)!}{n!2^n} && \text{for } p = 2n,\ n \in \mathbb{N}, \\ &= \frac{(2n+1)!}{(n-1)!2^n \cdot 3} && \text{for } p = 2n+1,\ n \in \mathbb{N}. \end{aligned} \tag{6.6}$$

The proof will be omitted. It consists in finding the main term in the expression for the expectation in question and is based on the results and methods of WALD and WOLFOWITZ (1944).

In the following special case, the formula with a better order of the remainder is obtained by a direct computation:

$$E_R\, M_{11}^2\, M_{12}^2 = (Nl)^{-2}(1 + 0(1/k)). \tag{6.7}$$

Set

$$X'_{rj} = X_{rj} - Ex_1^r,\ r \in \mathbb{N},\ j = 1,\ldots, k. \tag{6.8}$$

In the following lemma the notation introduced in Section 5 (see (5.26)) is used.

LEMMA 6.2. *For any $r \in \mathbb{N}$, $X'_{rj} = \Omega(l^{-1/2})$.*

PROOF. This follows from the fact that the normal distribution has finite moments of any order. By using, e.g., the formula (6.6) of PETROV (1987), Chapter 3, one obtains from $E_0 x_1^{rp} < \infty$

$$P_0\{|X'_{rj}| > N^{\epsilon} l^{-1/2}\} = O(N^{-\epsilon p} l^{1-(p/2)})$$

which implies the lemma since p can be chosen arbitrarily large.

Using the expressions of $\tilde{X}_{rj}$ (see (5.1)) through X_{pj}, $1 \leqslant p \leqslant r$, one obtains

COROLLARY 6.1. *For any $r \in \mathbb{N}$, $\tilde{X}_{rj} - E\kappa_1^r = \Omega(l^{-1/2})$.*

COROLLARY 6.2. *For an even $r \in \mathbb{N}$ one has $P_0\{X_{rj} > L\} = O(N^{-n})$ for any $L > E\kappa_1^r$ and $n \in \mathbb{N}$, whence $\max_{1 \leqslant j \leqslant k} X_{rj} = O_p(1)$ and $\sum_{j=1}^{k} X_{rj} = O_p(k)$. The same assertions hold for $\tilde{X}_{rj}$.*

Set for $r\in\mathbb{N}, j=1,\ldots,k$

$$X^{*}_{rj}=\frac{1}{l}\sum_{I_j}|x_i|^r,\quad M^{*}_{rj}=\frac{1}{l}\sum_{I_j}|m_{R_i}|^r. \tag{6.9}$$

By well-known inequalities for averages

$$(X^{*}_{pj})^{r/p}\leq X^{*}_{rj},\quad (M^{*}_{pj})^{r/p}\leq M^{*}_{rj},\quad 1\leq p\leq r. \tag{6.10}$$

Using the expressions for $\tilde{X}_{rj}$ trough X_{pj}, $1\leq p\leq r$, and the inequalities $|X_{pj}|\leq X^{*}_{pj}$ and (6.10) one obtains (as well as for $\tilde{M}_{rj}$) that there exist constants $C_r>0$ depending only on r such that

$$|\tilde{X}_{rj}|\leq C_r X^{*}_{rj},\quad |\tilde{M}_{rj}|\leq C_r M^{*}_{rj}. \tag{6.11}$$

The relations (5.17)-(5.21), (5.27)-(5.30) are of the form

$$S_N\xrightarrow{P}0 \tag{6.12}$$

with some sequences of random variables S_N. They will be proved by proving either that

$$ES_N^2\rightarrow 0 \tag{6.13}$$

($ES_N\rightarrow 0$ if $S_N\geq 0$) or that

$$E_R S_N^2\xrightarrow{P_0}0 \tag{6.14}$$

($E_R S_N\rightarrow 0$ if $S_N\geq 0$).

In the former case (6.12) follows from the Chebyshev (Markov) inequality. In the latter case one can find $\alpha_N\downarrow 0$ such that $P_0(B_N^c)\rightarrow 0$ for $B_N=\{x:E_R S_N^2\leq\alpha_N\}$. Then, for any $\epsilon>0$,

$$\sup_{x\in B_N}P_R\{|S_N|>\epsilon\}\leq\alpha_N/\epsilon^2\rightarrow 0$$

which implies that

$$P\{|S_N|>\epsilon\}=E_0 P_R\{|S_N|>\epsilon\}\rightarrow 0$$

so that (6.12) holds.

7. Proof of (5.17).

By (4.13) and (5.4),

$$A_\nu=l\sum_{J_\nu}X_{1j}\,M_{1j}. \tag{7.1}$$

Under P_{R2} the statistics A_ν, $\nu=1,\ldots,k'$, are independent SLRS and by (5.3)

$$\bar{A}_\nu=ll'\langle X_{1j}\rangle_\nu\langle M_{1j}\rangle_\nu \tag{7.2}$$

(see (5.15)). To prove (5.17) we shall show that

$$N^{1/2}E(\sum_\nu\bar{A}_\nu)^2\rightarrow 0. \tag{7.3}$$

Using (7.2), mutual independence of $\langle X_{1j}\rangle_\nu$, $\nu=1,\ldots,k'$, and the fact that $E_0\langle X_{1j}\rangle_\nu=0$, $\nu=1,\ldots,k'$, we obtain

$$E(\sum_\nu\bar{A}_\nu)^2=(ll')^2k'E_R\langle M_{1j}\rangle_1^2E_0\langle X_{1j}\rangle_1^2. \tag{7.4}$$

By (5.2) and (5.16), $\langle M_{1j}\rangle_1=(ll')^{-1}\sum_{i=1}^{ll'}m_{R_i}$ has the same distributional properties as M_{11} with $l:=ll'$. In particular, by (6.2), (6.3)

$$E_R\langle M_{1j}\rangle_1^2 = (N-ll')/(N-1)Nll'. \tag{7.5}$$

Moreover

$$E_0\langle X_{1j}\rangle_1^2 = (1/l')var_0 X_{1j} = 1/ll'.$$

Hence $E(\sum_v \bar{A}_v)^2 = O(k'/N)$ and the left hand side of (7.3) is $O(k'N^{-1/2}) \to 0$ due to the choice of $k' \asymp N^{\kappa'}$, $0<\kappa'<1/2$, see (4.1).

8. Proof of (5.18).

Let for brevity

$$U_N = \sum_v \bar{B}_v(2). \tag{8.1}$$

We shall prove the convergence (6.14) for the left hand side of (5.18) which amounts to

$$N^{1/2}E_R U_N - N^{-1/2}Z_N^2 + lN^{-1/2}\sum_{j=1}^k X_{1j}^2 \xrightarrow{P_0} 0 \tag{8.2}$$

and

$$N \operatorname{var}_R U_N \xrightarrow{P_0} 0. \tag{8.3}$$

By (5.4) and (5.10)

$$B_v(2) = \frac{l^2}{l-1}\sum_{J_v} \tilde{X}_{2j}\tilde{M}_{2j}. \tag{8.4}$$

Under P_{R2} it is SLRS and by (5.11) and (5.3) (see notation (5.16))

$$\bar{B}_v(2) = \frac{l^2 l'}{l-1}\langle \tilde{X}_{2j}\rangle_v \langle \tilde{M}_{2j}\rangle_v. \tag{8.5}$$

Taking into account that $\tilde{X}_{2j} = X_{2j} - X_{1j}^2$, $\tilde{M}_{2j} = M_{2j} - M_{1j}^2$ and using (6.2), (6.3) we obtain (8.1).

For the proof of (8.3) we shall introduce one more conditioning P_{R3}, E_{R3}, with respect to decomposition $\mathfrak{D}_3$ which specifies block $\mathfrak{R}_1, \ldots, \mathfrak{R}_k, \mathfrak{R}_0$ and a partition of the set of blocks $\{\mathfrak{R}_1, \ldots, \mathfrak{R}_k\}$ into groups of size l', so that a random permutation consists of consecutive groups arranged in a random order, the blocks of randomly ordered elements being randomly ordered within each group. Then putting (8.5) into (8.1) we obtain a sum of products where $\langle \tilde{M}_{2j}\rangle_v$ are randomly ordered with respect to $\langle X_{2j}\rangle_v$, $v=1,\ldots,k'$. Thus U_N is a SLRS under P_{R3}. According to the relation between conditional and unconditional variance

$$\operatorname{var}_R U_N = \operatorname{var}_R E_{R3} U_N + E_R \operatorname{var}_{R3} U_N. \tag{8.6}$$

Hence in order to prove (8.3) we have to show that

$$N \operatorname{var}_R E_{R3} U_N \xrightarrow{P_0} 0 \text{ and } NE_R \operatorname{var}_{R3} U_N \xrightarrow{P_0} 0. \tag{8.7}$$

By (5.3) and (8.5),

$$E_{R3} U_N = \frac{l^2 l'}{(l-1)k'}\sum_{v=1}^{k'}\langle \tilde{X}_{2j}\rangle_v \sum_{v=1}^{k'}\langle \tilde{M}_{2j}\rangle_v = O(l/k)\sum_{j=1}^k \tilde{X}_{2j}\sum_{j=1}^k \tilde{M}_{2j}. \tag{8.8}$$

Since $\tilde{M}_{2j} = M_{2j} - M_{1j}^2$ and $\sum_{j=0}^k M_{2j} = A_2/l = 1/l$ (see (6.1), (6.2)), one has

$$\operatorname{var}_R \sum_{j=1}^k \tilde{M}_{2j} = \operatorname{var}_R\left(M_{20} + \sum_{j=1}^k M_{1j}^2\right) \leq 2\left(\operatorname{var}_R M_{20} + \operatorname{var}_R \sum_{j=1}^k M_{1j}^2\right). \tag{8.9}$$

To estimate the last term in (8.9), we will first estimate

$$E_R(\sum_1^k M_{1j}^2)^2 = k\, E_R\, M_{11}^4 \;+ k(k-1)E_R\, M_{11}^2\, M_{12}^2.$$

By Lemma 6.1 and (6.7),

$$E_R\, M_{11}^4 = O((Nl)^{-2}), E_R\, M_{11}^2\, M_{12}^2 = (Nl)^{-2}(1+O(k^{-1})), \tag{8.10}$$

hence

$$E_R(\sum_1^k M_{1j}^2)^2 = k^2(Nl)^{-2}(1+O(k^{-1})). \tag{8.11}$$

By (6.3)

$$E_R \sum_1^k M_{1j}^2 = k(Nl)^{-1}(1+O(k^{-1})). \tag{8.12}$$

Now (8.11) and (8.12) imply that $\mathrm{var}_R \sum_1^k M_{1j}^2 = O(k/l^2N^2)$. By (6.3) with $l := d$, $\mathrm{var}_R\, M_{20} = O(N^{-2}) = O(k/l^2N^2)$ (see (4.2)). Therefore one has from (8.9)

$$\mathrm{var}_R \sum_1^k \tilde{M}_{2j} = O(k/l^2N^2). \tag{8.13}$$

By Corollary 6.2, $\sum_{j=1}^k \tilde{X}_{2j} = O_{p_0}(k)$. Hence (8.8) and (8.13) imply

$$N\, \mathrm{var}_R\, E_{R3}\, U_N = O_p(1/l) \tag{8.14}$$

which proves the first relation in (8.7).

Now we turn to the second relation in (8.7). By (5.3) (see (8.1), (8.5)) one has

$$\mathrm{var}_{R3}\, U_N = O(l^2(l')^2/k')\Sigma_1\Sigma_2 \tag{8.15}$$

where

$$\Sigma_1 = \sum_{v=1}^{k'} (\langle\tilde{X}_{2j}\rangle_v - \frac{1}{k'}\sum_{u=1}^{k'} \langle\tilde{X}_{2j}\rangle_u)^2, \tag{8.16}$$

$$\Sigma_2 = \sum_{v=1}^{k'} (\langle\tilde{M}_{2j}\rangle_v - \frac{1}{k'}\sum_{u=1}^{k'} \langle\tilde{M}_{2j}\rangle_u)^2. \tag{8.17}$$

Let $\xi_N = \max_{1\leqslant j\leqslant k} (|X_{2j}'| \vee |X_{1j}|)$ (see (6.8)). By Lemma 6.2, $\xi_N = \Omega(l^{-1/2})$. By a direct computation one obtains

$$\Sigma_1 \leqslant k'(2\xi_N - \xi_N^2)^2 = \Omega(k'/l). \tag{8.18}$$

Applying the formula $\Sigma_1^n(a_i-\bar{a})^2 = \Sigma a_i^2 - n\bar{a}^2$ to Σ_2 one has

$$E_R\, \Sigma_2 = (k'-1)[\, E_R\langle\tilde{M}_{2j}\rangle_1^2 - E_R\, \langle\tilde{M}_{2j}\rangle_1\, \langle\tilde{M}_{2j}\rangle_2].$$

Using the relation $\tilde{M}_{2j} = M_{2j} - M_{1j}^2$, after some algebra one arrives at

$$E_R\, \Sigma_2 = [(k'-1)/l'](V_1 - 2V_2 + V_3) \tag{8.19}$$

where

$$V_1 = E_R M_{21}^2 - E_R M_{21} M_{22},\;\; V_2 = E_R M_{21} M_{11}^2 - E_R M_{21} M_{12}^2,\;\; V_3 = E_R M_{11}^4 - E_R M_{11}^2 M_{12}^2.$$

By (6.3), (6.4), $V_1 = \mathrm{var}_R M_{21} - \mathrm{cov}_R(M_{21}, M_{22}) = O(1/lN^2)$.

Lemma 6.1 applied to each term of V_2 and V_3 implies $V_2=O(1/lN^2)$, $V_3=O(1/l^2N^2)$. Therefore

$$E_R\Sigma_2 = O(k'/l'lN^2). \tag{8.20}$$

It follows now from (8.15), (8.18) and (8.20) that

$$NE_R\, var_{R3}\, U_N = \Omega(1/l) \to 0$$

which proves the second relation of (8.7) and thus completes the proof of (5.18).

9. PROOF OF (5.19).

According to (5.9), (5.10), we have to prove that

$$NE_{R2} \sum_{j=1}^{k} (\beta_{4j} - 3\beta_{2j}^2) + 6 \xrightarrow{P} 0. \tag{9.1}$$

Using (5.6) we see that (9.1) will follow from the relations:

$$Nl\, E_{R2}\, \Sigma\, \tilde{X}_{2j}^2\, \tilde{M}_{2j}^2 - 1 \xrightarrow{P_0} 0, \tag{9.2}$$

$$NlE_{R2}\, \Sigma\, (\tilde{X}_{4j} - 3\tilde{X}_{2j}^2)(\tilde{M}_{4j} - 3\tilde{M}_{2j}^2) \xrightarrow{P_0} 0, \tag{9.3}$$

$$N\, E_{R2}\, \Sigma\, \tilde{X}_{4j}\, \tilde{M}_{4j} \xrightarrow{P_0} 0. \tag{9.4}$$

To prove (9.2) we will show that

$$Nl\, E_R\, V_N - 1 \xrightarrow{P_0} 0 \tag{9.5}$$

and

$$N^2l^2 var_R\, E_{R2}\, V_N \xrightarrow{P_0} 0 \text{ with } V_N = \Sigma\, \tilde{X}_{2j}^2\, \tilde{M}_{2j}^2. \tag{9.6}$$

One has $E_R\, V_N = E_R \tilde{M}_{21}^2\, \Sigma \tilde{X}_{2j}^2$. Using the equality $\tilde{M}_{21} = M_{21} - M_{11}^2$ and (6.3), (6.5) we obtain

$$E_R\, \tilde{M}_{21}^2 = N^{-2}(1 + 0(1/l)). \tag{9.7}$$

It follows from Corollary 6.1 that $\Sigma\, \tilde{X}_{2j}^2 = k(1 + o_{P_0}(1))$. These relations imply (9.5).

Each sum $\sum_{j\in J_v} \tilde{X}_{2j}^2 \tilde{M}_{2j}^2$, $v = 1,\ldots, k'$, is a SLRS under P_{R2}, hence by (5.3)

$$E_{R2}\, V_N = l' \sum_{v=1}^{k'} \langle \tilde{M}_{2j}^2 \rangle_v \langle \tilde{X}_{2j}^2 \rangle_v.$$

Let

$$W_N = l' \sum_{v=1}^{k'} \langle \tilde{M}_{2j}^2 \rangle_v = \sum_{j=1}^{k} \tilde{M}_{2j}^2.$$

It follows from Corollary 6.1 that $\max_v |\langle \tilde{X}_{2j}^2 \rangle_v - 1| = o_{P_0}(1)$, consequently $E_{R2} V_N = W_N(1 + o_{P_0}(1))$ and (9.6) is reduced to

$$N^2 l^2\, var_R\, W_N^2 \to 0. \tag{9.8}$$

Applying lemma 6.1 to the sum of squares of $\tilde{M}_{2j} = M_{2j} - M_{1j}^2$ we get

$$E_R\, W_N^2 = k^2 E_R M_{21}^2 M_{22}^2 (1 + o(1)) = k^2 N^{-4} (1 + o(1))$$

which implies that $N^2 l^2 E_R W_N^2 \to 1$. Together with $NlE_R\, W_N \to 1$ which is a consequence of (9.7) this proves (9.8) and thus (9.2).

To prove (9.3) we shall show that

$$N^2 l^2 E_R (E_{R2} U_N)^2 \xrightarrow{P_0} 0$$

where U_N denotes the sum in (9.3). By means of relations $(E_{R2}U_N)^2 \leqslant E_{R2}U_N^2$ and $E_R E_{R2} U_N^2 = E_R U_N^2$ this will follow from

$$N^2 l^2 E_R U_N^2 \xrightarrow{P_0} 0. \tag{9.9}$$

Let $\xi_j = \tilde{X}_{4j} - 3\tilde{X}_{2j}^2$. By Corollary 6.1, $\max_{1 \leqslant j \leqslant k} |\xi_j| \xrightarrow{P_0} 0$. Applying lemma 6.1 to the expressions $\tilde{M}_{41} = M_{41} - 4M_{31}M_{11} + 6M_{21}M_{11}^2 - 3M_{11}^4$ and $\tilde{M}_{21} = M_{21} - M_{11}^2$ we get

$$E_R \tilde{M}_{41} \tilde{M}_{4\tau},\ E_R \tilde{M}_{21}^2 \tilde{M}_{2\tau}^2,\ E_R \tilde{M}_{41} \tilde{M}_{2\tau}^2 = O(N^{-4}),\ \tau = 1, 2.$$

Therefore

$$E_R U_N^2 = E_R (\tilde{M}_{41} - 3\tilde{M}_{21}^2)^2 \sum_{j=1}^{k} \xi_j^2 + E_R (\tilde{M}_{41} - 3\tilde{M}_{21}^2)(\tilde{M}_{42} - 3\tilde{M}_{22}^2) \sum_{j_1 \neq j_2} \xi_{j_1} \xi_{j_2} = O(k^2/N^4) o_{P_0}(1)$$

which immediately implies (9.9) and thus (9.3).

To prove (9.4) we use the relations $E_0 \tilde{X}_{4j} = O(1)$ and $E_R \tilde{M}_{4j} = O(N^{-2})$. Since $\tilde{X}_{4j} \geqslant 0, \tilde{M}_{4j} \geqslant 0$, (9.4) follows from

$$NE \Sigma \tilde{X}_{4j} \tilde{M}_{4j} = NkO(N^{-2}) = O(1/l) \to 0.$$

10. Proof of (5.20).

Since the left-hand side of (5.20) is non-negative, we shall estimate

$$E \sum_{v} \langle \Pi \tilde{X}_{p_i,j} \rangle_v \langle \tilde{M}_{2j}^3 \rangle_v = k' E_R \langle \tilde{M}_{2j}^3 \rangle_1 E_0 \langle \Pi \tilde{X}_{p_i,j} \rangle_1.$$

Applying the inequality $\tilde{M}_{2j} \leqslant M_{2j}$ and Lemma 6.1 we obtain $E_R \langle \tilde{M}_{2j}^3 \rangle = E_R \tilde{M}_{21}^3 = O(N^{-3})$. Using the estimate $\tilde{X}_{rj} \leqslant C_r X_{rj}$ for an even r (see (6.11)) we find that $E_0 \langle \Pi \tilde{X}_{p_i,j} \rangle_1 = E_0 \Pi \tilde{X}_{p_i,j} = O(1)$. Therefore the expectation of the left-hand side of (5.20) is $O(lN^{-1/2}) \to 0$ which proves (5.20).

11. Proof of (5.21).

We shall prove the relation (6.13) for the left-hand side of (5.21). (In case where all p_i, q_i are even one can proceed as in Section 10.) To that end we shall show that

$$N^{n/2} l^{2b} (l')^2 k' E_R \langle \prod_{i=1}^{b} \tilde{M}_{q_i,j} \rangle_1^2 E_0 \langle \prod_{i=1}^{a} \tilde{X}_{p_i,j} \rangle_1^2 \to 0 \tag{11.1}$$

and

$$N^{n/2} l^{2b} (l')^2 (k')^2 E_R \langle \Pi \tilde{M}_{q_i,j} \rangle_1 \langle \Pi \tilde{M}_{q_i,j} \rangle_2 E_0 \langle \Pi \tilde{X}_{p_i,j} \rangle_1 \langle \Pi \tilde{X}_{p_i,j} \rangle_2. \tag{11.2}$$

Using (6.11) and Lemma 6.1 we obtain

$$E_R \langle \Pi \tilde{M}_{q_i,j} \rangle_1 \langle \Pi \tilde{M}_{q_i,j} \rangle_\tau = O(N^{-n}),\ \tau = 1,2. \tag{11.3}$$

Moreover, we have

$$E_0 \langle \Pi \tilde{X}_{p_i,j} \rangle_1 \langle \Pi \tilde{X}_{p_i,j} \rangle_\tau = O(1),\ \tau = 1,2. \tag{11.4}$$

Since $n = \Sigma_{i=1}^{b} q_i$ and $q_i \geqslant 2$ one has $2b \leqslant n$ for an even n and $2b \leqslant n-1$ for an odd n. Hence (11.3) and (11.4) imply that the left hand sides of (11.2) and (a fortiori) (11.3) are $O(l^{n-2} N^{2-(n/2)})$ for an even n and $O(l^{n-3} N^{2-(n/2)})$ for an odd n. Hence taking into account that $l \asymp N^\lambda, \lambda < 1/3$ (see (4.1), (4.2)) one obtains (11.1), (11.2) for $n \geqslant 7$. In the case of $n=6$ one has $b \leqslant 2$ because the combination $(q_1, q_2, q_3) = (2,2,2)$ is excluded, and by the same argument one obtains $O(l^2/N) \to 0$ for the left-hand sides of (11.1) and (11.2). Thus it remains to consider the cases of $n=3$ and 5.

When $n=3$ one has $p_1=q_1=3(a=b=1)$ and when $n=5$ the combinations $p_1=5$ and $(p_1,p_2)=(2,3)$, $q_1=5$ and $(q_1,q_2)=(2,3)$ are possible. In all these cases the left hand side of (11.2) is 0.

One has

$$E\langle \Pi \tilde{X}_{p_i,j}\rangle_1^2 = (l')^{-1} E\,\Pi X_{p_i,1}^2 = O(1/ll').$$

The last estimate follows from the fact that $l^{2a}\prod_1^a X_{p_i,1}^2 = \prod_1^a(\sum_{\nu\in I_1} x_\nu^{p_i})^2$ is a sum of l^{2a} products containing at least $l(l-1)...(l-2a+1)$ summands with zero expectations (all indices n are different). Since $2b\leqslant n-1$ this relation together with (11.3) implies that the left-hand side of (11.1) is $O(l^{n-3}N^{1-(n/2)})\to 0$ for $n=3$ and 5.

12. Proof of (5.27).

We shall prove two relations which imply (5.27):

$$N^{1/2}E_R\,U_N - N^{-1/2}\,l\sum_{j=1}^{k} X_{1j}^2 \xrightarrow{P_0} 0 \tag{12.1}$$

and

$$N\,var_R\,U_N \xrightarrow{P_0} 0 \text{ with } U_N = \sum_{\nu=1}^{k'} E_{R2}\tilde{A}_\nu^2. \tag{12.2}$$

For any numbers $a_1,\ldots,a_k$ let

$$\hat{a}_j = a_j - \langle a_j\rangle_\nu \text{ as } j\in J_\nu,\ \nu=1,\ldots,k'. \tag{12.3}$$

Then (see (4.14), (7.1), (7.2))

$$\tilde{A}_\nu = l\sum_{J_\nu} \hat{X}_{1j}\hat{M}_{1j}. \tag{12.4}$$

In a way similar to (7.2) we obtain by (5.3)

$$E_{R2}\tilde{A}_\nu^2 = var_{R2}A_\nu = \frac{l^2(l')^2}{l'-1}\langle \hat{X}_{1j}^2\rangle_\nu\langle \hat{M}_{1j}^2\rangle_\nu. \tag{12.5}$$

Hence (see (12.2))

$$E_R\,U_N = \frac{l^2(l')^2}{l'-1}E_R\langle \hat{M}_{1j}^2\rangle_1\sum_{\nu=1}^{k'}\langle \hat{X}_{1j}^2\rangle_\nu. \tag{12.6}$$

By (6.3), $E_R\langle M_{1j}^2\rangle_1 = E_R M_{1j}^2 = (N-l)/(N-1)Nl$. Together with (7.5) this gives

$$E_R\langle \hat{M}_{1j}^2\rangle_1 = E_R(\langle M_{1j}^2\rangle_1 - \langle M_{1j}\rangle_1^2) = (l'-1)/(N-1)ll'. \tag{12.7}$$

Taking into account that $\Sigma_\nu\langle X_{1j}^2\rangle_\nu = (l')^{-1}\sum_{j=1}^{k} X_{1j}^2$ we obtain

$$N^{1/2}E_R\,U_N = N^{-1/2}l\sum_1^k X_{1j}^2 + O(N^{-3/2})l\sum_1^k X_{1j}^2 - O(N^{-1/2})ll'\sum_\nu\langle X_{1j}\rangle_\nu^2.$$

Since $\sum_1^k X_{1j}^2 \overset{d}{=} \chi_k^2/l$ and $\sum_\nu\langle X_{1j}\rangle_\nu^2 \overset{d}{=} \chi_{k'}^2/ll'$, the two last terms are distributed as $O(N^{-3/2})\chi_k^2 \xrightarrow{P} 0$ and $O(N^{-1/2})\chi_{k'}^2 \xrightarrow{P} 0$ which proves (12.1).

Turn to the proof of (12.2). In view of (12.5) we have

$$var_R\, U_N = l^4(l')^2(1 + O(1/l'))(T_1 + T_2) \tag{12.8}$$

where

$$T_1 = var_R\, \langle \hat{M}_{1j}^2\rangle_1 \sum_{v=1}^{k'} \langle \hat{X}_{1j}^2\rangle_v^2, \tag{12.9}$$

$$T_2 = \mathrm{cov}_R(\langle \hat{M}_{1j}^2\rangle_1, \langle \hat{M}_{1j}^2\rangle_2) \sum_{u\neq v} \langle \hat{X}_{1j}^2\rangle_u \langle \hat{X}_{1j}^2\rangle_v. \tag{12.10}$$

To study $var_R\, \langle \hat{M}_{1j}^2\rangle_1$ consider first

$$E_R\, \langle \hat{M}_{1j}^2\rangle_1^2 = E_R(\langle M_{1j}^2\rangle_1 - \langle M_{1j}\rangle_1^2)^2 = E_R(\langle M_{1j}^2\rangle_1^2 - 2\langle M_{1j}^2\rangle \langle M_{1j}\rangle_1^2 + \langle M_{1j}\rangle_1^4). \tag{12.11}$$

Applying Lemma 6.1 we have

$$E_R\, \langle M_{1j}^2\rangle_1^2 = (1/l')E_R M_{11}^4 + (1+O(1/l'))E_R M_{11}^2 M_{12}^2 = (Nl)^{-2}(1 + O(1/l')). \tag{12.12}$$

In a similar way we obtain that all other terms in (12.11) are of order $O(1/N^2l^2l')$. Subtracting the square of (12.7) we find that

$$\mathrm{var}_R\, \langle \hat{M}_{1j}^2\rangle_1 = O(1/N^2l^2l'). \tag{12.13}$$

Taking into account that $0\leqslant \langle \hat{X}_{1j}^2\rangle_v \leqslant \langle X_{1j}^2\rangle_v \overset{d}{=} \chi_{l'}^2/ll'$ we obtain that the sum in (12.9) is $O_{p_0}(k'/l^2)$. Therefore the contribution of the term with T_1 into $N\, var_R\, U_N$ (see (12.2), (12.8)) is $O_{p_0(1/l)}\to 0$.

In a similar way, using Lemma 6.1 and (6.7), we obtain

$$E_R\, \langle \hat{M}_{1j}^2\rangle_1 \langle \hat{M}_{1j}^2\rangle_2 = E_R(\langle M_{1j}^2\rangle_1 - \langle M_{1j}\rangle_1^2)(\langle M_{1j}^2\rangle_2 - \langle M_{1j}\rangle_2^2)$$
$$= \frac{1}{N^2l^2} - \frac{2}{N^2l^2l'} + O(\frac{1}{N^2l^2}(\frac{1}{k} + \frac{1}{(l')^2})).$$

Subtracting the square of (12.7) we get

$$\mathrm{cov}_R(\langle \hat{M}_{1j}^2\rangle_1, \langle \hat{M}_{1j}^2\rangle_2) = O(\frac{1}{N^2l^2}(\frac{1}{k} + \frac{1}{(l')^2}))$$

In the same way as above, the sum in (12.10) is $O_{p_0}((k')^2/l^2)$. Therefore the contribution of the term with T_2 into $N\, var_R\, U_N$ is $O_{p_0}((k+(k')^2)/N\to 0$ which together with the estimate obtained above implies (12.2).

13. Proof in (5.28).
We shall show that

$$N^{3/2}E\sum_v (E_{R2}\tilde{A}_v^3)^2 \to 0. \tag{13.1}$$

This directly implies the second relation in (5.28); it is seen from (13.2) below that $E(E_{R2}\tilde{A}_u^3)(E_{R2}\tilde{A}_v^3)=O$ for $u\neq v$, hence (13.1) implies

$$N^{3/2}E(\sum_v E_{R2}\tilde{A}_v^3)^2 \to 0$$

whence the first relation of (5.28) follows.

By the formula (14) of Hájek and Sidak (1967), Chapter 2, Problems and Complements, applied to $\tilde{A}_v$ as in (12.4)

$$E_{R2}\tilde{A}_v^3 = l^3O(l')\langle \hat{X}_{1j}^3\rangle_v \langle \hat{M}_{1j}^3\rangle_v. \tag{13.2}$$

Hence (13.1) is equivalent to

$$N^{3/2}l^6(l')^2k'E_R\, \langle \hat{M}_{1j}^3\rangle_1^2 E_0 \langle \hat{X}_{1j}^3\rangle_1^2 \to 0. \tag{13.3}$$

We shall show first that

$$E_R \hat{M}_{11}^6 = O(1/l^3N^3),\ E_R\hat{M}_{11}^3\ \hat{M}_{12}^3 = O(1/l^3N^3). \tag{13.4}$$

Applying Lemma 6.1 to the binomial expansion of $\hat{M}_{11}^6 = (M_{11} - \langle M_{1j}\rangle_1)^6$ we obtain $E_R M_{11}^6 = O(1/l^3N^3)$ and $E_R M_{11}^{6-r} \langle M_{1j}\rangle_1^r = O(1/l^3N^3), r=1,\ldots, 6$, in the latter case because it is an average of $(l')^r$ terms having this order. Hence the first relation in (13.4) holds. The second one is proved in a similar way (the order in it is, in fact, $O(1/l^4N^3)$)

It follows from (13.4) that

$$E_R \langle \hat{M}_{1j}^3\rangle_1^2 = O(1/l^3N^3) \tag{13.5}$$

because it is an average of $(l')^2$ terms as in (13.4).

By a direct analysis of arising terms one finds that

$$E_0 \hat{X}_{11}^6 = O(1/l^3),\ E_0 \hat{X}_{11}^3\ \hat{X}_{12}^3 = O(1/l'l^3)$$

whence

$$E_0 \langle \hat{X}_{1j}^3\rangle_1^2 = O(1/l'l^3). \tag{13.6}$$

Using (13.5) and (13.6) one obtains that the left-hand side of (13.3) is $O(k/N^{3/2}) \to 0$ which proves (13.1).

14. Proof of (5.29)

Since $E_R \tilde{A}_1^r = E_R(E_{R2}\tilde{A}_1^r)$, in case $r=2$ we use (12.5), (12.7) and the fact that $\langle \tilde{X}_{1j}^2\rangle_1 \leqslant \langle X_{1j}^2\rangle_1 = \sum_{j=1}^{l'} X_{1j}^2/l'$ where $X_{1j}l^{1/2} \overset{d}{=} \mathfrak{N}(0,1)$ whence by Corollary 6.2

$$\langle \hat{X}_{1j}^2\rangle_1 = \Omega(1/l). \tag{14.1}$$

These relations imply (5.29) for $r=2$.

Further, in a way similar to (5.5) one has

$$\begin{aligned} E_{R2}\tilde{A}_1^4 &\leqslant l^4[O(l')\langle \hat{X}_{1j}^4\rangle_1 \langle \hat{M}_{1j}^4\rangle_1 + O((l')^2 \langle \hat{X}_{1j}^2\rangle_1 \langle \hat{M}_{1j}^2\rangle_1^2\rangle \\ &\leqslant l^4 O((l')^2) \langle \hat{X}_{1j}^4\rangle_1 \langle \hat{M}_{1j}^4\rangle_1. \end{aligned} \tag{14.2}$$

In the same way as (6.11), one obtains $\langle \hat{X}_{1j}^4\rangle_1 \leqslant C_4 \langle X_{1j}^4\rangle_1$ $\langle \hat{M}_{1j}^4\rangle_1 \leqslant C_4\langle M_{1j}^4\rangle_1$. By Lemma 6.1 $E_R\langle M_{1j}^4\rangle_1 = E_R M_{11}^4 = O(1/l^2N^2)$. Like (14.1), $\langle X_{1j}^4\rangle_1 = \Omega(1/l^2)$. These relations applied to (14.2) imply (5.29) for $r=4$.

15. Proof of (5.30).

By (5.14), (5.16) and the definitions (4.15), (4.16), $\tilde{Z}_1$ is a finite sum of terms of the form

$$s_N^n N^{n/4} O(l^{a\wedge b}) \sum_{j\in J_1} (X)_j(M)_j \tag{15.1}$$

where

$$(X)_j = \prod_{i=1}^{a} \tilde{X}_{p_i j} - \langle \prod_{i=1}^{a} \tilde{X}_{p_i j}\rangle_1, \tag{15.2}$$

$$(M)_j = \prod_{i=1}^{b} \tilde{M}_{q_i j} - \langle \prod_{i=1}^{b} \tilde{M}_{q_i j}\rangle_1 \tag{15.3}$$

and $n = \Sigma p_i = \Sigma q_i$, $p_i, q_i \geqslant 2$. Therefore, to prove (5.30) it is sufficient to prove that corresponding relations hold for each term of (15.1), i.e. to show that for any $2 \leqslant r \leqslant 4$ and $\{p_i\}, \{q_i\}$ in (15.2), (15.3),

$$N^{rn/4} l^{r(a\wedge b)} E_R (\sum_{j\in J_1} (X)_j(M)_j)^r = \Omega(l'/N). \tag{15.4}$$

Consider first the case of $r=2$. By (5.3)

$$E_{R2}(\sum_{j\in J_1}(X)_j(M)_j)^2 = O(l')\langle (X)_j^2\rangle_1 \langle (M)_j^2\rangle_1 \tag{15.5}$$

where according to (15.2), (15.3)

$$\langle (X)_j^2\rangle_1 = \langle \prod_{i=1}^{a} \tilde{X}_{p_i j}^2\rangle_1 - \langle \prod_{i=1}^{a} \tilde{X}_{p_i j}\rangle_1^2, \tag{15.6}$$

$$\langle (M)_j^2\rangle_1 = \langle \prod_{i=1}^{b} \tilde{M}_{q_i j}^2\rangle_1 - \langle \prod_{i=1}^{n} \tilde{M}_{q_i j}\rangle_1^2. \tag{15.7}$$

In case $b=1, q_1=2$ one obtains after some calculations (cf. (8.19) and the subsequent estimates):

$$E_R\langle (M)_j^2\rangle_1 = (1-(1/l'))(V_1-2V_2+V_3) = O(1/l^2N^2). \tag{15.8}$$

In other cases one has by Lemma 6.1

$$E_R\langle (M)_j^2\rangle_1 \leqslant E_R\langle \prod_1^b \tilde{M}_{q_i j}^2\rangle = E_R\prod_1^b \tilde{M}_{q_i j}^2 = O(N^{-n}). \tag{15.9}$$

By Corollary 6.1, $\xi_j := \tilde{X}_{2j}-1=\Omega(l^{-1/2})$. In case $a=2, p=2$ one has for $j\in J_1$ (see (15.2))

$$(X)_j = \tilde{X}_{2j} - \langle \tilde{X}_{2j}\rangle_1 = \xi_j - \langle \xi_j\rangle_1 = \Omega(l^{-1/2}). \tag{15.10}$$

In case $a=2, p_1=p_2=2$ one has for $j\in J_1$

$$(X)_j = \tilde{X}_{2j}^2 - \langle \tilde{X}_{2j}^2\rangle_1 = 2(\xi_j - \langle \xi_j\rangle_1) + \xi_j^2 - \langle \xi_j^2\rangle_1 = \Omega(l^{-1/2}), \tag{15.11}$$

Hence in both these cases

$$\langle (X)_j^2\rangle = \Omega(1/l). \tag{15.12}$$

In case $a=1, p_1=3$ one has by Corollary 6.1 $\tilde{X}_{3j}=\Omega(l^{-1/2})$ and

$$\langle (X)_j^2\rangle_1 \leqslant \langle \tilde{X}_{3j}\rangle_1^2 = \Omega(1/l). \tag{15.13}$$

In other cases the estimate $\tilde{X}_{p_i j}=\Omega(1)$ is used which implies

$$\langle (X)_j^r\rangle_1 = \Omega(1),\ r\in\mathbb{N}. \tag{15.14}$$

Now (15.4) with $r=2$ immediately follows from (15.5)-(15.9), (15.12), (15.13): consider all possible combinations of $\{p_i\}$, $\{q_i\}$ for $n=2,3$ and 4, and use (15.9) and (15.13) for $n\geqslant 5$ taking into account that $2b\leqslant n$ or $n-1$ for an even or odd n respectively.

For $r-3,4$ one has (cf. (13.2), (14.2)):

$$E_{R2}(\sum_{J_1}(X)_j(M_j))^3 = O(l')\langle (X)_j^3\rangle_1 \langle (M)_j^3\rangle_1, \tag{15.15}$$

$$E_{R2}(\sum_{J_1}(X)_j(M_j))^4 \leqslant O(l')\langle (X)_j^4\rangle_1 \langle (M)_j^4\rangle_1 + O((l')^2)\langle (X)_j^2\rangle_1^2 \langle (M)_j^2\rangle_1^2. \tag{15.16}$$

In a way similar to (15.9) one can show that

$$E_R\langle (M)_j^r\rangle = O(N^{-rn/2})$$

Using (15.10) in case $a=1, p_1=1$ and (15.14) otherwise one obtains (15.4) from (15.15) and (15.16).

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