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An M/G/1 Queue with Dependence between Interarrival and Service Times

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This paper analyses a variant of the $M/G/1$ queue in which the service times of arriving customers depend on the length of the interval between their arrival and the previous arrival. Such a dependence structure arises, e.g., when individual customers arrive according to a Poisson process, while customer collectors are sent out according to a Poisson process to collect the customers and to bring them to the service facility. In this case collected numbers of customers, and hence total collected service requests, are positively correlated with the corresponding collect intervals. Viewing a batch of collected customers as one (super)customer gives rise to an $M/G/1$ queue with a positive correlation between service times of such customers and their interarrival times. Both for individual customers and for batches of collected customers we derive the transforms of the sojourn time, waiting time and queue length distributions. We also compare our results with those for the ordinary $M/G/1$ queue without dependence.

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1 Introduction

Consider the following situation. Customers arrive at pick-up points according to independent Poisson processes. At these pick-up points they wait for a bus to bring them to a single-server service facility (e.g., the check-in counter of a hotel). Busses with unlimited customer capacity move according to a fixed route along the pick-up points, with fixed speed, collecting all waiting customers that they encounter and finally delivering all collected customers at the service facility. The intervals between the starts of successive bus tours are exponentially distributed. Because of the fixed tour length, the arrival process of busses at the service facility is a Poisson process. Viewing the batch of customers brought to the service facility by a bus as one supercustomer, the service facility very closely resembles an ordinary $M/G/1$ queue; the only difference is that *the service time of a supercustomer depends on the previous interarrival time.* Indeed, if two consecutive busses arrive at a relatively long (short) interval, then the second bus is likely to pick up relatively many (few) customers: the

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interarrival time and the size of the picked-up batch are positively correlated, and hence so are the interarrival time and the supercustomer service time.

In the present paper we analyse the $M/G/1$ queue with the above-sketched structure of the correlation between interarrival and service times. Our motivation for this analysis is twofold. Firstly, in the vast literature on single server queues it is almost exclusively assumed that there is no dependence between arrival intervals, between service times and between interarrival and service times, although dependencies between these quantities occur in a very natural way. The main reason for ignoring these dependencies seems the mathematical complexity that they almost invariably give rise to. However, the present model allows a very detailed analysis of most performance measures of interest, thus giving valuable insight into the effect that dependencies between interarrival and service times may have on those performance measures. Our second motive for the analysis is that the correlation structure under consideration seems quite natural in many situations in which customers are collectively brought to a central service facility. Examples are mail pick-up and the pick-up of customers at airport terminals. In computer-communications, one might think of the collection of packets in a 'train' in a Local Area Network with interconnected rings, to be delivered at a bridge queue. Furthermore, modern reservation protocols for the use of transmission slots in high-speed Local and Metropolitan Area Networks also may give rise to customer collection. In particular, in a recent performance evaluation by Boxma, Levy and Yechiali [1990] of the Cyclic-Reservation Multiple-Access (CRMA) protocol a quite similar customer collection procedure occurs. In Boxma, Levy and Yechiali [1990] it is assumed that the collection of customers takes place at fixed intervals, thus giving rise to a $D/G/1$ queue - in which obviously the interarrival and service times are *not* dependent. A more detailed model of the CRMA protocol, taking its backpressure mechanism into account (Nassehi [1989]) would lead to a model that is similar to ours, but in which the arrival process is closer to a deterministic process than to a Poisson process. It would be worthwhile to combine the known $D/G/1$ results and the results from the present paper, to study the performance of CRMA with backpressure. Performance measures like the sojourn time distributions might be approximated by a weighted sum of these distributions for the present $M/G/1$ case and the $D/G/1$ case with weight factors the squared coefficient of variation of the interarrival times and one minus this coefficient.

In the remainder of this section we present a model description, a brief survey of related literature, and an overview of the paper.

Model description

Individual customers require service from a service facility with a single server. Their service times are independent, identically distributed stochastic variables with distribution $B(\cdot)$, with mean β, second moment $\beta^{(2)}$ and Laplace-Stieltjes Transform (LST) $\beta(\cdot)$; $B(0+) = 0$. These individual customers arrive at a pick-up point according to a Poisson process with rate λ. They are collected by a collector and delivered in batches at the service facility at times $\mathbf{t}_1, \mathbf{t}_2, \dots$. The collect intervals $\boldsymbol{\sigma}_i = \mathbf{t}_i - \mathbf{t}_{i-1}, i = 1, 2, \dots$, with $\mathbf{t}_0 = 0$, are independent, negative exponentially distributed stochastic variables with mean $1/\gamma$. A delivered batch of customers can be viewed as one supercustomer. Batches (supercustomers) apparently arrive at the service facility according to a Poisson process with rate γ. In the sequel the travel time of the collector from the pick-up point to the service facility is assumed to be zero. Thus the

arrival of the batch customer at the service facility coincides with the arrival of the collector at the pick-up point. A non-zero travel time can easily be implemented into our model, by just adding it to the waiting times and sojourn times of individual customers.
Define the total offered traffic load as $\rho := \lambda\beta$. In a more general framework Loynes [1962] showed that $\rho < 1$ is a necessary and sufficient condition for the waiting times to have a proper limiting distribution. Without proof we claim that in our model the same holds for the other quantities under consideration: sojourn times, queue lengths and busy periods.
After this global model description we consider batch sizes, service times of batch customers and the correlation structure of the model in some more detail. The number of individual customers, $\mathbf{K}_i$, constituting the i-th batch customer generated in a collect interval $\boldsymbol{\sigma}_i$ of length u has conditional distribution:

$$Pr\{\mathbf{K}_i = n \,|\, \boldsymbol{\sigma}_i = u\} = e^{-\lambda u}\frac{(\lambda u)^n}{n!}, \qquad u > 0, n = 0, 1, \ldots, \tag{1.1}$$

so

$$\begin{aligned} Pr\{\mathbf{K}_i = n\} &= \int_{u=0}^{\infty} \gamma e^{-\gamma u} Pr\{\mathbf{K}_i = n \,|\, \boldsymbol{\sigma}_i = u\} du \\ &= \frac{\gamma}{\gamma+\lambda}\left(\frac{\lambda}{\gamma+\lambda}\right)^n, \qquad n = 0, 1, \ldots . \end{aligned} \tag{1.2}$$

The service time $\boldsymbol{\tau}_i$ of the i-th batch customer generated in a collect interval $\boldsymbol{\sigma}_i$ of length u has conditional distribution

$$\begin{aligned} Pr\{\boldsymbol{\tau}_i < t \,|\, \boldsymbol{\sigma}_i = u\} &= \sum_{n=0}^{\infty} Pr\{\mathbf{K}_i = n \,|\, \boldsymbol{\sigma}_i = u\} B^{n*}(t) \\ &= \sum_{n=0}^{\infty} e^{-\lambda u}\frac{(\lambda u)^n}{n!} B^{n*}(t), \qquad t \geq 0, u > 0. \end{aligned} \tag{1.3}$$

Hence

$$E(e^{-\omega \boldsymbol{\tau}_i} \,|\, \boldsymbol{\sigma}_i = u) = e^{-\lambda(1-\beta(\omega))u}, \qquad Re\,\omega \geq 0, u > 0. \tag{1.4}$$

It follows that

$$Pr\{\boldsymbol{\tau}_i < t\} = \sum_{n=0}^{\infty} \frac{\gamma}{\gamma+\lambda}\left(\frac{\lambda}{\gamma+\lambda}\right)^n B^{n*}(t), \qquad t \geq 0, \tag{1.5}$$

$$E(e^{-\omega \boldsymbol{\tau}_i}) = \frac{\gamma}{\gamma + \lambda(1-\beta(\omega))}, \qquad Re\,\omega \geq 0. \tag{1.6}$$

The service times $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \ldots$ of batch customers are independent, identically distributed stochastic variables. It should be noted that a batch may be empty, and hence a super-customer may have zero service time:

$$Pr\{\mathbf{K}_i = 0 \,|\, \boldsymbol{\sigma}_i = u\} = Pr\{\boldsymbol{\tau}_i = 0 \,|\, \boldsymbol{\sigma}_i = u\} = e^{-\lambda u}, \qquad u > 0, \tag{1.7}$$

$$Pr\{\mathbf{K}_i = 0\} = Pr\{\boldsymbol{\tau}_i = 0\} = \frac{\gamma}{\gamma+\lambda}. \tag{1.8}$$

From (1.6) it follows that

$$E\boldsymbol{\tau}_i = \frac{\lambda\beta}{\gamma}, \qquad E\boldsymbol{\tau}_i^2 = \frac{\lambda\beta^{(2)}}{\gamma} + 2(\frac{\lambda\beta}{\gamma})^2. \tag{1.9}$$

The bivariate LST of $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ follows from (1.3):

$$E(e^{-\zeta\boldsymbol{\sigma}_i-\omega\boldsymbol{\tau}_i}) = \frac{\gamma}{\gamma+\zeta+\lambda(1-\beta(\omega))}, \qquad Re\,\zeta \geq 0, Re\,\omega \geq 0, \tag{1.10}$$

yielding

$$Cov(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) = \frac{\lambda\beta}{\gamma^2}, \tag{1.11}$$

$$correl(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) = [1 + \frac{\gamma\beta^{(2)}}{\lambda\beta^2}]^{-1/2} \geq 0. \tag{1.12}$$

Note that $correl(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) \to 1$ for $\gamma \downarrow 0$ and also for $\lambda \to \infty$, whereas $correl(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) \to 0$ for $\gamma \to \infty$ and for $\lambda \downarrow 0$. For $\gamma \to \infty$ the queue approaches an ordinary $M/G/1$ queue, as individual customers are collected instantaneously.

Remark 1.1 Instead of assuming that there is a single Poisson arrival stream of individual customers with service time distribution $B(\cdot)$, we could also have allowed several independent Poisson arrival streams at various pick-up points, with different service time distributions. For the analysis of the $M/G/1$ queue with batch customers, one should then first aggregate the various independent Poisson streams into one Poisson stream, with service time distribution of an arbitrary individual customer a weighted sum of the service time distributions of the various streams. A distinction between the various arrival streams may be useful in certain applications, where e.g. waiting times of individual customers must be determined taking into account the location of the pick-up point and the travel time of the collector to the service facility. This refinement can easily be implemented into our model, without seriously complicating the analysis.

□

Related literature

Very few studies have appeared that analyse a queueing system with correlation between the interarrival and service times. We are not aware of any studies that explicitly consider a correlation between the service time of a customer and the *subsequent* interarrival time, but the waiting time for such a correlated queue can be analysed by studying the recurrence relation for the waiting time,

$$\mathbf{W}_{i+1} = \max(0, \mathbf{W}_i + \boldsymbol{\tau}_i - \boldsymbol{\sigma}_{i+1}),$$

in an ordinary $GI/GI/1$ queue with similarly distributed $\boldsymbol{\tau}_i - \boldsymbol{\sigma}_{i+1}$ and with $\mathbf{W}_i$ and $\boldsymbol{\tau}_i - \boldsymbol{\sigma}_{i+1}$ independent.

Like the present paper, a number of papers has been devoted to the single server queue with correlation between the service time of a customer and the *preceding* interarrival time. Conolly [1968] and Conolly and Hadidi [1969] consider an $M/M/1$ queue in which the service time and the preceding interarrival time are linearly related. Conolly and Choo [1979] study an $M/M/1$ queue in which $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ have a bivariate exponential distribution with density

$$g(s,t) = \lambda\mu(1-r)e^{-\lambda s-\mu t}I_0[2\{\lambda\mu rst\}^{1/2}], \tag{1.13}$$

where $I_0[z]$ is a zero order modified Bessel function of the first kind, and where $r \in [0,1)$ is the correlation between $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$. For $r = 0$ the queue reduces to an ordinary $M/M/1$ queue. The marginal distributions of $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ are negative exponential. Conolly and Choo analyse the waiting time distribution for this correlated $M/M/1$ queue, showing that its density can be expanded in a series of partial fraction terms. Their numerical calculations reveal that the positive correlation leads to a considerable reduction in mean waiting time. For the same model, (i) Hadidi [1981] shows that the waiting times are hyperexponentially distributed; (ii) Hadidi [1985] examines the sensitivity of the waiting time distribution to the value of the correlation coefficient; (iii) Langaris [1987] studies the busy period distribution. However, the starting point of the latter study seems to be wrong: it is assumed that a customer who starts a general busy period has an ordinary service time, whereas in this correlated queue a customer who starts a busy period is likely to have a relatively long interarrival time and hence a relatively long service time. Jacobs [1980] obtains heavy traffic results for the waiting time in queues with sequences of ARMA correlated negative exponentially distributed interarrival and service times.

Organization of the paper

In section 2 (3) we derive the LST of the sojourn (waiting) time distributions of batch customers as well as of individual customers. The mean waiting time of batch customers, $E\mathbf{W}$, is compared with the mean waiting times in (i) the $M/G/1$ queue without collection, in which individual customers do not wait to be picked up but immediately join one common queue for service ($E\mathbf{W}_{M/G/1}$), and (ii) the $M/G/1$ queue of supercustomers in the case of *independence* of interarrival and service times ($E\mathbf{W}_I$). It is proven that $E\mathbf{W}_I \geq E\mathbf{W} \geq E\mathbf{W}_{M/G/1}$.

In section 4 we obtain the joint distribution of the numbers of those individual customers waiting to be collected and those already collected but waiting for service, immediately after the departure of an individual customer. The queue length distribution of batch customers is also derived. Section 5 is devoted to a discussion concerning the busy period distribution in the $M/G/1$ queue with dependence. The mean busy period is easily obtained; the busy period *distribution* gives rise to mathematical difficulties which are briefly discussed but not solved. Section 6 contains some numerical results, exposing the influence of the dependence on waiting time means and busy period variance. Conclusions and suggestions for further research are presented in section 7.

2 The sojourn time

Denote by $\mathbf{R}_i$ the sojourn time of the i-th batch customer, $i = 1, 2, \ldots$, i.e. the time from the arrival of the batch customer until the completion of the service of the last individual customer belonging to the batch customer. It is easily seen that

$$\mathbf{R}_i = \max(0, \mathbf{R}_{i-1} - \boldsymbol{\sigma}_i) + \boldsymbol{\tau}_i, \qquad i = 2, 3, \ldots . \tag{2.1}$$

We now derive the LST of the sojourn time distribution. Starting from the recurrence relation (2.1),

$$\begin{aligned}
Pr\{\mathbf{R}_i < t\} &= Pr\{\mathbf{R}_{i-1} < \boldsymbol{\sigma}_i, \boldsymbol{\tau}_i < t\} + \\
&\quad Pr\{\mathbf{R}_{i-1} \geq \boldsymbol{\sigma}_i, \mathbf{R}_{i-1} - \boldsymbol{\sigma}_i + \boldsymbol{\tau}_i < t\} \\
&= \int_{u=0}^{\infty} \gamma e^{-\gamma u} Pr\{\mathbf{R}_{i-1} < u\} Pr\{\boldsymbol{\tau}_i < t \mid \boldsymbol{\sigma}_i = u\} du + \qquad (2.2) \\
&\quad \int_{u=0}^{\infty} \gamma e^{-\gamma u} \int_{x=0}^{t} Pr\{u \leq \mathbf{R}_{i-1} < u + t - x\} d_x Pr\{\boldsymbol{\tau}_i < x \mid \boldsymbol{\sigma}_i = u\} du.
\end{aligned}$$

Let $r_i(\omega) := E(e^{-\omega \mathbf{R}_i})$ for $Re\,\omega \geq 0$, $i = 1, 2, \ldots$. Using (1.3) and (1.4),

$$\begin{aligned}
r_i(\omega) &= \int_{u=0}^{\infty} \gamma e^{-\gamma u} e^{-\lambda(1-\beta(\omega))u} Pr\{\mathbf{R}_{i-1} < u\} du + \qquad (2.3) \\
&\quad \int_{u=0}^{\infty} \gamma e^{-\gamma u} \sum_{n=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^n}{n!} \int_{t=0}^{\infty} e^{-\omega t} d_t \int_{x=0}^{t} Pr\{u \leq \mathbf{R}_{i-1} < u + t - x\} dB^{n*}(x) du.
\end{aligned}$$

Consider both terms in the right-hand side. Partial integration shows that the first term equals

$$\frac{\gamma}{\gamma + \lambda(1 - \beta(\omega))} r_{i-1}(\gamma + \lambda(1 - \beta(\omega))).$$

The second term equals

$$\begin{aligned}
&\int_{u=0}^{\infty} \gamma e^{-\gamma u} \sum_{n=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^n}{n!} \int_{t=0}^{\infty} e^{-\omega t} \int_{x=0}^{t} dB^{n*}(x) d_t Pr\{\mathbf{R}_{i-1} < u + t - x\} du \\
&= \int_{y=0}^{\infty} e^{-\omega y} \int_{u=0}^{y} \gamma e^{-(\gamma - \omega)u} \sum_{n=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^n}{n!} \int_{x=0}^{\infty} e^{-\omega x} dB^{n*}(x) du\, d_y Pr\{\mathbf{R}_{i-1} < y\} \\
&= \int_{y=0}^{\infty} e^{-\omega y} \int_{u=0}^{y} \gamma e^{-(\gamma - \omega + \lambda(1-\beta(\omega)))u} du\, d_y Pr\{\mathbf{R}_{i-1} < y\} \\
&= \frac{\gamma}{\gamma - \omega + \lambda(1 - \beta(\omega))} [r_{i-1}(\omega) - r_{i-1}(\gamma + \lambda(1 - \beta(\omega)))].
\end{aligned}$$

Here the first equality follows from the substitution $y = u + t - x$ and the last equality follows from a straightforward evaluation of successively the inner and outer integral. Adding both terms gives

$$\begin{aligned}
r_i(\omega) &= \frac{\gamma}{\gamma + \lambda(1 - \beta(\omega))} r_{i-1}(\gamma + \lambda(1 - \beta(\omega))) + \qquad (2.4) \\
&\quad \frac{\gamma}{\gamma - \omega + \lambda(1 - \beta(\omega))} [r_{i-1}(\omega) - r_{i-1}(\gamma + \lambda(1 - \beta(\omega)))].
\end{aligned}$$

As observed in the Introduction, for $\rho < 1$ the sojourn times $\mathbf{R}_i$ have a proper limiting distribution for $i \to \infty$. Denote by $\mathbf{R}$ a stochastic variable with this distribution.

Let $r(\omega) := E(e^{-\omega \mathbf{R}})$ for $Re\,\omega \geq 0$. From (2.4) we obtain, as an important intermediate result of our analysis:

$$r(\omega) = \frac{\gamma\omega}{\omega - \lambda(1-\beta(\omega))} \frac{r(\gamma + \lambda(1-\beta(\omega)))}{\gamma + \lambda(1-\beta(\omega))}, \qquad Re\,\omega \geq 0. \tag{2.5}$$

We shall solve this functional equation for $r(\cdot)$ below.

Remark 2.1 Letting $\omega \to \infty$ in (2.5) we obtain

$$Pr\{\mathbf{R} = 0\} = \frac{\gamma}{\gamma + \lambda} r(\gamma + \lambda). \tag{2.6}$$

This formula may also be obtained directly from (2.1) and (1.8):

$$\begin{aligned} Pr\{\mathbf{R}_i = 0\} &= \int_{u=0}^{\infty} \gamma e^{-\gamma u} Pr\{\boldsymbol{\tau}_i = 0, \mathbf{R}_{i-1} < \boldsymbol{\sigma}_i \mid \boldsymbol{\sigma}_i = u\} du \\ &= \int_{u=0}^{\infty} \gamma e^{-\gamma u} e^{-\lambda u} Pr\{\mathbf{R}_{i-1} < u\} du \\ &= \frac{\gamma}{\gamma + \lambda} r_{i-1}(\gamma + \lambda). \end{aligned}$$

□

Remark 2.2 Letting $\omega \downarrow 0$ in (2.5) we obtain

$$r(\gamma) = 1 - \lambda\beta. \tag{2.7}$$

Again, this formula may also be obtained directly:

$$\begin{aligned} r_{i-1}(\gamma) &= \int_{t=0}^{\infty} e^{-\gamma t} dPr\{\mathbf{R}_{i-1} < t\} \\ &= Pr\{\mathbf{R}_{i-1} < \boldsymbol{\sigma}_i\}. \end{aligned}$$

The latter term equals the probability that an arriving batch customer sees the server idle, while obviously the probability that the server is idle equals $1 - \lambda\beta$. Because of the PASTA property both probabilities are equal.

□

We now solve the functional equation for $r(\cdot)$. Define

$$f(\omega) := \frac{\gamma\omega}{\omega - \lambda(1-\beta(\omega))}, \qquad Re\,\omega \geq 0, \tag{2.8}$$

$$g(\omega) := \gamma + \lambda(1-\beta(\omega)), \qquad Re\,\omega \geq 0. \tag{2.9}$$

Then we can write

$$r(\omega) = \frac{f(\omega)}{g(\omega)} r(g(\omega)). \tag{2.10}$$

Let

$$\begin{aligned} g^{(0)}(\omega) &:= \omega, & Re\,\omega \geq 0, \\ g^{(h)}(\omega) &:= g(g^{(h-1)}(\omega)), & Re\,\omega \geq 0,\ h = 1, 2, \dots . \end{aligned}$$

Iterating (2.10) we find

$$r(\omega) = r(g^{(M+1)}(\omega)) \prod_{h=0}^{M} \frac{f(g^{(h)}(\omega))}{g^{(h+1)}(\omega)}, \qquad Re\,\omega \geq 0, \tag{2.11}$$

for any non-negative integer M.

Lemma 2.1

(i). The equation $\omega = g(\omega)$, $Re\,\omega \geq 0$ has a unique solution ω^*. ω^* is real.

(ii). $\lim_{M \to \infty} g^{(M)}(\omega) = \omega^*$ for all ω with $Re\,\omega \geq 0$.

(iii). $\prod_{h=0}^{\infty} \frac{f(g^{(h)}(\omega))}{g^{(h+1)}(\omega)}$ converges for all ω with $Re\,\omega \geq 0$.

Proof

See appendix A.

□

Lemma 2.1 implies that

$$r(\omega) = r(\omega^*) \prod_{h=0}^{\infty} \frac{f(g^{(h)}(\omega))}{g^{(h+1)}(\omega)}, \qquad Re\,\omega \geq 0. \tag{2.12}$$

Putting $\omega = 0$ in (2.12) we find

$$r(\omega^*) = 1 / \prod_{h=0}^{\infty} \frac{f(g^{(h)}(0))}{g^{(h+1)}(0)}, \tag{2.13}$$

which leads to our main result:

Theorem 2.1

The LST of the sojourn time distribution of a batch customer is

$$r(\omega) = \prod_{h=0}^{\infty} \frac{f(g^{(h)}(\omega))}{g^{(h+1)}(\omega)} / \frac{f(g^{(h)}(0))}{g^{(h+1)}(0)}, \qquad Re\,\omega \geq 0, \tag{2.14}$$

with $f(\cdot)$ and $g(\cdot)$ given by (2.8) and (2.9).

The LST in (2.14) could be numerically inverted using a procedure outlined in Abate and Whitt [1991]. Differentiating (2.14),

$$\mathbf{ER} = \sum_{h=0}^{\infty} \frac{g^{(h)'}(0)[f(g^{(h)}(0))g'(g^{(h)}(0)) - f'(g^{(h)}(0))g(g^{(h)}(0))]}{f(g^{(h)}(0))g(g^{(h)}(0))}, \tag{2.15}$$

which shall be used for numerical calculations.

We now clarify the meaning of the terms composing (2.5), thus relating $\mathbf{R}$ to the waiting time in an ordinary $M/G/1$ queue. Denote by $\mathbf{W}_{M/G/1}$ a stochastic variable with distribution the stationary distribution of the waiting time in an ordinary $M/G/1$ queue with arrival rate λ and service time distribution function $B(\cdot)$. In the sequel we refer to this queue as the corresponding $M/G/1$ queue without collection. From Cohen [1982] p. 255, we have:

$$E(e^{-\omega \mathbf{W}_{M/G/1}}) = \frac{(1-\lambda\beta)\omega}{\omega - \lambda(1-\beta(\omega))}, \qquad Re\,\omega \geq 0. \tag{2.16}$$

Denote by $\mathbf{H}$ the sojourn time of a batch customer leaving no batch customers behind. Such a sojourn time has distribution function $H(\cdot)$ with

$$dH(t) = \frac{e^{-\gamma t}dR(t)}{\int_{u=0}^{\infty} e^{-\gamma u}dR(u)}, \qquad t > 0,$$

and, cf. (2.6)

$$H(0+) = \frac{\gamma}{\gamma+\lambda}\frac{r(\gamma+\lambda)}{r(\gamma)}.$$

Denote by $\mathbf{U}$ the amount of work arriving during such a sojourn time.

$$E(e^{-\omega \mathbf{U}}) = \frac{\int_{t=0}^{\infty} e^{-\lambda(1-\beta(\omega))t}e^{-\gamma t}dR(t)}{\int_{u=0}^{\infty} e^{-\gamma u}dR(u)} = \frac{r(\gamma+\lambda(1-\beta(\omega)))}{r(\gamma)}, \qquad Re\,\omega \geq 0. \tag{2.17}$$

So we can write, using (2.7)

$$r(\omega) = E(e^{-\omega \mathbf{W}_{M/G/1}})E(e^{-\omega \boldsymbol{\tau}})E(e^{-\omega \mathbf{U}}), \qquad Re\,\omega \geq 0. \tag{2.18}$$

From (2.18), using (2.7) and (2.17),

$$\begin{aligned} \mathbf{ER} &= E\mathbf{W}_{M/G/1} + E\boldsymbol{\tau} + E\mathbf{U} \\ &= E\mathbf{W}_{M/G/1} + E\boldsymbol{\tau} + \frac{\lambda\beta}{1-\lambda\beta}\int_{t=0}^{\infty} te^{-\gamma t}dR(t), \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} Var(\mathbf{R}) &= Var(\mathbf{W}_{M/G/1}) + Var(\boldsymbol{\tau}) + \frac{(\lambda\beta)^2}{1-\lambda\beta}\int_{t=0}^{\infty} t^2e^{-\gamma t}dR(t) + \\ &\quad \frac{\lambda\beta^{(2)}}{1-\lambda\beta}\int_{t=0}^{\infty} te^{-\gamma t}dR(t) - (\frac{\lambda\beta}{1-\lambda\beta}\int_{t=0}^{\infty} te^{-\gamma t}dR(t))^2. \end{aligned} \tag{2.20}$$

Remark 2.3 Using (2.8) and (2.9) we find that the factor for $h = 0$ of the infinite product (2.14) equals $E(e^{-\omega \mathbf{W}_{M/G/1}})E(e^{-\omega \boldsymbol{\tau}})$. So the remainder of the infinite product equals $\frac{r(\gamma+\lambda(1-\beta(\omega)))}{r(\gamma)}$. Similarly we find that the term for $h = 0$ of the infinite sum (2.15) equals $E\mathbf{W}_{M/G/1} + E\boldsymbol{\tau}$. So, using (2.7), the remainder of the infinite sum equals $\frac{\lambda\beta}{1-\lambda\beta}\int\limits_{t=0}^{\infty} te^{-\gamma t}dR(t)$.

□

To provide additional insight, we now give a more intuitive derivation of (2.18). The sojourn time of a batch customer consists of two phases, viz.:
(i). its waiting time, i.e. the time needed to do the work associated with the individual customers present at the server upon the batch customer's arrival;
(ii). its service time, i.e. the time needed to do the work associated with the individual customers present at the bus stop upon the batch customer's arrival. (Remember that the arrival of the batch customer at the server coincides with the arrival of the collector at the bus stop.)
So the sojourn time of a batch customer equals the amount of work associated with the individual customers present upon its arrival (at the bus stop as well at the server). Denote by $\mathbf{V}$ the steady state amount of work associated with the individual customers (at the bus stop as well as at the server). Because of the PASTA property ($\stackrel{d}{=}$ denoting equality in distribution)

$$\mathbf{R} \stackrel{d}{=} \mathbf{V}. \tag{2.21}$$

Denote by $\mathbf{V}_{M/G/1}$ a stochastic variable with distribution the stationary distribution of the amount of work in the corresponding $M/G/1$ queue without collection. Denote by $\mathbf{Y}$ a stochastic variable, independent of $\mathbf{V}_{M/G/1}$, with distribution the stationary distribution of the amount of work associated with the individual customers at an arbitrary epoch in a non-serving interval, i.e. the amount of work associated with the individual customers present at the bus stop when the server is idle. Now the following work decomposition property holds, cf. Boxma [1989]:

$$\mathbf{V} \stackrel{d}{=} \mathbf{V}_{M/G/1} + \mathbf{Y}. \tag{2.22}$$

Because of the PASTA property

$$\mathbf{V}_{M/G/1} \stackrel{d}{=} \mathbf{W}_{M/G/1}. \tag{2.23}$$

The amount of work associated with individual customers at an arbitrary epoch in a non-serving interval, $\mathbf{Y}$, consists of two components, viz.:
(i). the amount of work associated with individual customers that have arrived during the sojourn time of the last batch customer (possibly empty), $\mathbf{Y}^{(i)}$. This sojourn time has distribution function $H(\cdot)$. So

$$\mathbf{Y}^{(i)} \stackrel{d}{=} \mathbf{U}. \tag{2.24}$$

(ii). the amount of work associated with individual customers that have arrived during the past non-serving period since the departure of the last batch customer (possibly empty), $\mathbf{Y}^{(ii)}$. This past non-serving period is negative exponentially distributed with parameter γ, since the non-serving period is a (residual) collect interval. So

$$\mathbf{Y}^{(ii)} \stackrel{d}{=} \boldsymbol{\tau}. \tag{2.25}$$

Moreover, $\mathbf{Y}^{(i)}$ and $\mathbf{Y}^{(ii)}$ are independent, since the individual customers arrive according to a Poisson process.
Combining (2.21) - (2.25) yields (2.18).

We now study the sojourn time of a non-empty batch customer. Denote by $\mathbf{K}$ the number of individual customers constituting a batch customer.

$$E(e^{-\omega\mathbf{R}}\,|\,\mathbf{K}>0) = \frac{r(\omega) - E(e^{-\omega\mathbf{R}} I_{\{\mathbf{K}=0\}})}{Pr\{\mathbf{K}>0\}}. \tag{2.26}$$

Similarly to (2.4),

$$E(e^{-\omega\mathbf{R}} I_{\{\mathbf{K}=0\}}) = \frac{\gamma(\gamma+\lambda)r(\omega) - \gamma\omega r(\gamma+\lambda)}{(\gamma+\lambda)(\gamma+\lambda-\omega)}. \tag{2.27}$$

So

$$E(e^{-\omega\mathbf{R}}\,|\,\mathbf{K}>0) = \frac{(\gamma+\lambda)(\lambda-\omega)r(\omega) + \gamma\omega r(\gamma+\lambda)}{\lambda(\gamma+\lambda-\omega)}. \tag{2.28}$$

From (2.28) we obtain

$$E(\mathbf{R}\,|\,\mathbf{K}>0) = E\mathbf{R} + \frac{\gamma}{\lambda(\gamma+\lambda)}(1 - r(\gamma+\lambda)). \tag{2.29}$$

Remember that a non-empty batch customer is likely to have a relatively long service time, but also a relatively short waiting time. From (2.29) we see that the on average longer service time outweighs the on average shorter waiting time.

We finally study the sojourn time of an *individual* customer, $\tilde{\mathbf{R}}$. Let $\tilde{r}(\omega) := E(e^{-\omega\tilde{\mathbf{R}}})$ for $Re\,\omega \geq 0$. First we study the number of individual customers, $\tilde{\mathbf{N}}$. We shall find $\tilde{r}(\omega)$ from $\tilde{r}(\lambda(1-z)) = E(z^{\tilde{\mathbf{N}}}), |\, z \,| \leq 1$. Denote by $\mathbf{N}_{M/G/1}$ a stochastic variable with distribution the stationary distribution of the number of customers in the corresponding $M/G/1$ queue without collection. From Cohen [1982] p. 247, we have:

$$E(z^{\mathbf{N}_{M/G/1}}) = \frac{(1-\lambda\beta)(1-z)\beta(\lambda(1-z))}{\beta(\lambda(1-z)) - z}, \qquad |\, z \,| \leq 1. \tag{2.30}$$

Denote by $\mathbf{X}^{(i)}$ the number of individual customers present at an arbitrary epoch in a non-serving interval that have arrived during the sojourn time of the last batch customer (possibly empty). This sojourn time has distribution function $H(\cdot)$. So

$$E(z^{\mathbf{X}^{(i)}}) = \frac{r(\gamma+\lambda(1-z))}{r(\gamma)}, \qquad |\, z \,| \leq 1. \tag{2.31}$$

Denote by $\mathbf{X}^{(ii)}$ the number of individual customers present at an arbitrary epoch in a non-serving interval that have arrived during the past non-serving period since the departure of the last batch customer (possibly empty). This past non-serving period is negative exponentially distributed with parameter γ, since the non-serving period is a (residual) collect interval. So

$$E(z^{\mathbf{X}^{(ii)}}) = \frac{\gamma}{\gamma + \lambda(1-z)}, \qquad |\, z \,| \leq 1. \tag{2.32}$$

Observe that $\mathbf{X}^{(i)}$ and $\mathbf{X}^{(ii)}$ are independent, since the individual customers arrive according to a Poisson process and the non-serving period is a (residual) collect interval, not depending on the sojourn time of the last batch customer. Now the following queue length decomposition holds, cf. Fuhrmann & Cooper [1985]:

$$E(z^{\tilde{\mathbf{N}}}) = E(z^{\mathbf{N}_{M/G/1}})E(z^{\mathbf{X}^{(i)}})E(z^{\mathbf{X}^{(ii)}}) \qquad |\, z \,| \leq 1. \tag{2.33}$$

Denote by $\mathbf{R}_{M/G/1}$ a stochastic variable with distribution the stationary distribution of the sojourn time in the corresponding $M/G/1$ queue without collection. Substituting $\omega = \lambda(1-z)$ in (2.33) leads to:

Theorem 2.2
The LST of the sojourn time distribution of an individual customer is

$$\begin{aligned}
\tilde{r}(\omega) &= E(e^{-\omega \mathbf{R}_{M/G/1}})E(e^{-\omega \boldsymbol{\sigma}})E(e^{-\omega \mathbf{H}}) \\
&= E(e^{-\omega \mathbf{W}_{M/G/1}})\beta(\omega)\frac{\gamma}{\gamma+\omega}\frac{r(\gamma+\omega)}{r(\gamma)}, \qquad Re\, \omega \geq 0,
\end{aligned} \tag{2.34}$$

with $r(\cdot)$ given by Theorem 2.1.

From (2.34), using (2.7),

$$\begin{aligned}
E\tilde{\mathbf{R}} &= E\mathbf{W}_{M/G/1} + \beta + E\boldsymbol{\sigma} + E\mathbf{H} \\
&= E\mathbf{W}_{M/G/1} + \beta + \frac{1}{\gamma} + \frac{1}{1-\lambda\beta}\int_{t=0}^{\infty} te^{-\gamma t}dR(t),
\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}
Var(\tilde{\mathbf{R}}) &= Var(\mathbf{W}_{M/G/1}) + \beta^{(2)} - \beta^2 + \frac{1}{\gamma^2} + \\
&\quad \frac{1}{1-\lambda\beta}\int_{t=0}^{\infty} t^2 e^{-\gamma t}dR(t) - (\frac{1}{1-\lambda\beta}\int_{t=0}^{\infty} te^{-\gamma t}dR(t))^2.
\end{aligned} \tag{2.36}$$

3 The waiting time

Denote by $\mathbf{W}_i$ the waiting time of the i-th batch customer, $i = 1, 2, \ldots$, i.e. the time from arrival of the batch customer until the start of the service of the first individual customer belonging to the batch customer. It is easily seen that

$$\mathbf{W}_i = \max(0, \mathbf{R}_{i-1} - \boldsymbol{\sigma}_i), \qquad i = 2, 3, \ldots . \tag{3.1}$$

We now derive the LST of the waiting time distribution. Starting from the recurrence relation (3.1),

$$\begin{aligned} Pr\{\mathbf{W}_i < t\} &= Pr\{\mathbf{R}_{i-1} - \boldsymbol{\sigma}_i < t\} \\ &= \int\limits_{u=0}^{\infty} \gamma e^{-\gamma u} Pr\{\mathbf{R}_{i-1} < t + u\} du, \qquad t > 0, \end{aligned} \tag{3.2}$$

and

$$Pr\{\mathbf{W}_i = 0\} = \int\limits_{u=0}^{\infty} \gamma e^{-\gamma u} Pr\{\mathbf{R}_{i-1} < u\} du = r_{i-1}(\gamma). \tag{3.3}$$

Let $w_i(\omega) := E(e^{-\omega \mathbf{W}_i})$ for $Re\,\omega \geq 0$, $i = 1, 2, \ldots$.

$$\begin{aligned} w_i(\omega) &= E(e^{-\omega \mathbf{W}_i} I_{\{\mathbf{W}_i = 0\}}) + E(e^{-\omega \mathbf{W}_i} I_{\{\mathbf{W}_i > 0\}}) \\ &= Pr\{\mathbf{W}_i = 0\} + \int\limits_{t=0}^{\infty} e^{-\omega t} d_t [\int\limits_{u=0}^{\infty} \gamma e^{-\gamma u} Pr\{\mathbf{R}_{i-1} < t + u\} du] \\ &= r_{i-1}(\gamma) + \int\limits_{y=0}^{\infty} e^{-\omega y} \int\limits_{u=0}^{y} \gamma e^{-(\gamma - \omega) u} du \, d_y Pr\{\mathbf{R}_{i-1} < y\} \\ &= \frac{\gamma r_{i-1}(\omega) - \omega r_{i-1}(\gamma)}{\gamma - \omega}. \end{aligned} \tag{3.4}$$

Here the third equality follows from the substitution $y = t + u$. As observed in the Introduction, for $\rho < 1$ the waiting times $\mathbf{W}_i$ have a proper limiting distribution for $i \to \infty$. Denote by $\mathbf{W}$ a stochastic variable with this distribution. Let $w(\omega) := E(e^{-\omega \mathbf{W}})$ for $Re\,\omega \geq 0$. From (3.4),

$$w(\omega) = \frac{\gamma r(\omega) - \omega r(\gamma)}{\gamma - \omega}, \qquad Re\,\omega \geq 0. \tag{3.5}$$

From (3.5) or immediately from (3.3), using (2.7),

$$Pr\{\mathbf{W} = 0\} = 1 - \lambda\beta,$$

a result which may also be obtained by applying the PASTA property.
From (3.5), using (1.9) and (2.7),

$$E\mathbf{W} = E\mathbf{R} - E\boldsymbol{\tau}, \tag{3.6}$$

as should be the case since $\mathbf{R}_i = \mathbf{W}_i + \boldsymbol{\tau}_i$ for $i = 1, 2, \ldots$. Combining (2.19) and (3.6),

$$E\mathbf{W} = E\mathbf{W}_{M/G/1} + \frac{\lambda\beta}{1 - \lambda\beta} \int\limits_{t=0}^{\infty} t e^{-\gamma t} dR(t). \tag{3.7}$$

The following Lemma gives some bounds for the integral occuring in (3.7).

Lemma 3.1

(i). $$\int_{t=0}^{\infty} te^{-\gamma t}dR(t) \leq \min\left\{\frac{e^{-1}}{\gamma}(1-\frac{\gamma}{\gamma+\lambda}r(\gamma+\lambda)), \frac{\lambda\beta}{\gamma}\right\}.$$

(ii). $$\int_{t=0}^{\infty} te^{-\gamma t}dR(t) \geq \frac{\lambda\beta}{\gamma}\frac{\lambda(1-\beta(\gamma))}{\gamma-\lambda(1-\beta(\gamma))}\frac{\gamma+\lambda(1-\beta(\gamma))+2\lambda\gamma\beta'(\gamma)}{\gamma+\lambda(1-\beta(\gamma))}.$$

Proof

See appendix B.

□

Lemma 3.1 (i) implies that $\lim_{\gamma\to\infty} E\mathbf{W} = E\mathbf{W}_{M/G/1}$. Because of the PASTA property the mean *amount of work* at the server also approaches the mean amount of work in the corresponding $M/G/1$ queue without collection, which is obvious as for $\gamma \to \infty$ the individual customers are collected instantaneously.

Lemma 3.1 (ii) implies that $\lim_{\gamma\downarrow 0} E\mathbf{W} = \infty$. Because of the PASTA property the mean amount of work at the server also tends to infinity, which is obvious as the mean service times tend to infinity.

Next we compare the mean waiting time of a batch customer with the mean waiting time in an ordinary $M/G/1$ queue with identical traffic characteristics, but without dependence. Denote by $\mathbf{W}_I$ the waiting time in an ordinary $M/G/1$ queue with arrival rate γ and service time distribution having LST $\frac{\gamma}{\gamma+\lambda(1-\beta(\omega))}$, cf. (1.6). In the sequel we refer to this queue as the corresponding $M/G/1$ queue without dependence. Using (1.9),

$$E\mathbf{W}_I = \frac{\gamma(\frac{\lambda\beta^{(2)}}{\gamma}+2(\frac{\lambda\beta}{\gamma})^2)}{2(1-\lambda\beta)} = E\mathbf{W}_{M/G/1} + \frac{\lambda^2\beta^2}{\gamma(1-\lambda\beta)}. \tag{3.8}$$

Combining (3.7) and (3.8),

$$E\mathbf{W} = E\mathbf{W}_I + \frac{\lambda\beta}{1-\lambda\beta}[\int_{t=0}^{\infty} te^{-\gamma t}dR(t) - \frac{\lambda\beta}{\gamma}]. \tag{3.9}$$

From Lemma 3.1 (i) we find that $E\mathbf{W} \leq E\mathbf{W}_I + \frac{\lambda\beta}{1-\lambda\beta}\min\left\{\frac{e^{-1}-\lambda\beta}{\gamma} - \frac{e^{-1}}{\gamma+\lambda}r(\gamma+\lambda), 0\right\}$.

The following argument explains why $E\mathbf{W} \leq E\mathbf{W}_I$. From Wolff [1989] p. 279, we have:

$$E\mathbf{V} = \gamma E[\mathbf{W}\boldsymbol{\tau}] + \gamma E[\boldsymbol{\tau}^2]/2, \tag{3.10}$$

which is an application of the generalized form of Little's law. Using (1.9), (2.19), (2.21), (3.8) and (3.10),

$$E\mathbf{W} = E\mathbf{W}_I + \frac{\gamma}{1-\lambda\beta}Cov(\mathbf{W}, \boldsymbol{\tau}). \tag{3.11}$$

A batch customer having a relatively short/long interarrival time is likely to have a relatively long/short waiting time, but also a relatively short/long service time, due to the dependence. So $Cov(\mathbf{W}, \boldsymbol{\tau}) < 0$.

We now study the waiting time of a non-empty batch-customer. Similarly to (2.28), using (2.27) and (3.5),

$$\begin{aligned} E(e^{-\omega \mathbf{W}} \,|\, \mathbf{K} > 0) &= \frac{E(e^{-\omega \mathbf{W}}) - E(e^{-\omega \mathbf{R}} I_{\{\mathbf{K}>0\}})}{Pr\{\mathbf{K} > 0\}} \qquad (3.12) \\ &= \frac{-\gamma\lambda(\gamma+\lambda)r(\omega) - \gamma\omega(\gamma-\omega)r(\gamma+\lambda) + \omega(\gamma+\lambda)(\gamma+\lambda-\omega)r(\gamma)}{\lambda(\gamma-\omega)(\gamma+\lambda-\omega)}. \end{aligned}$$

From (3.12) we obtain, using (1.9) and (2.7),

$$\begin{aligned} E(\mathbf{W} \,|\, \mathbf{K} > 0) &= E\mathbf{R} + \frac{\gamma}{\lambda(\gamma+\lambda)}[1 - r(\gamma+\lambda)] - \frac{\beta(\gamma+\lambda)}{\gamma} \\ &= E\mathbf{W} + \frac{1}{\lambda}[r(\gamma) - \frac{\lambda}{\gamma+\lambda}r(0) - \frac{\gamma}{\gamma+\lambda}r(\gamma+\lambda)]. \qquad (3.13) \end{aligned}$$

As $r(\cdot)$ is a convex function, (3.13) confirms that a non-empty batch customer is likely to have a relatively short waiting time.

We finally study the waiting time of an *individual* customer. Denote by $\tilde{\mathbf{W}}$ the waiting time of an individual customer. Let $\tilde{w}(\omega) := E(e^{-\omega \tilde{\mathbf{W}}})$ for $Re\,\omega \geq 0$. From (2.34),

$$\tilde{w}(\omega) = E(e^{-\omega \mathbf{W}_{M/G/1}}) \frac{\gamma}{\gamma+\omega} \frac{r(\gamma+\omega)}{r(\gamma)}, \qquad Re\,\omega \geq 0, \qquad (3.14)$$

since the waiting time and the subsequent service time of an *individual* customer are independent. From (3.14) using (2.7), or immediately from (2.35),

$$E\tilde{\mathbf{W}} = E\mathbf{W}_{M/G/1} + \frac{1}{\gamma} + \frac{1}{1-\lambda\beta} \int_{t=0}^{\infty} t e^{-\gamma t} dR(t), \qquad (3.15)$$

and

$$\begin{aligned} Var(\tilde{\mathbf{W}}) &= Var(\mathbf{W}_{M/G/1}) + \frac{1}{\gamma^2} + \qquad (3.16) \\ & \frac{1}{1-\lambda\beta} \int_{t=0}^{\infty} t^2 e^{-\gamma t} dR(t) - (\frac{1}{1-\lambda\beta} \int_{t=0}^{\infty} t e^{-\gamma t} dR(t))^2. \end{aligned}$$

From (3.7) and (3.15), $E\mathbf{W}_{M/G/1} \leq E\mathbf{W} \leq E\tilde{\mathbf{W}}$, as expected.
An alternative derivation of (3.15) proceeds as follows. The waiting time of an individual customer is the sum of three terms, (different from the terms occuring in (3.14)), viz.:
(i). the time from its arrival until its collection, $\tilde{\mathbf{W}}^{(i)}$;
(ii). the waiting time of the batch customer to which it belongs, $\tilde{\mathbf{W}}^{(ii)}$;
(iii). the time from its admission until its service, $\tilde{\mathbf{W}}^{(iii)}$.
These three terms are dependent, however

$$E\tilde{\mathbf{W}} = E\tilde{\mathbf{W}}^{(i)} + E\tilde{\mathbf{W}}^{(ii)} + E\tilde{\mathbf{W}}^{(iii)}. \tag{3.17}$$

$\tilde{\mathbf{W}}^{(i)}$ is the length of a residual, negative exponentially distributed, collect interval. So

$$E\tilde{\mathbf{W}}^{(i)} = \frac{1}{\gamma}. \tag{3.18}$$

$E\tilde{\mathbf{W}}^{(ii)}$ does not equal $E\mathbf{W}$ because the batch customer containing a tagged individual customer is not typical but is likely to have a long interarrival time and hence a short waiting time. However, applying Little's formula,

$$E\tilde{\mathbf{W}}^{(ii)} = \frac{E\tilde{\mathbf{N}}^{(ii)}}{\lambda}, \tag{3.19}$$

where $\tilde{\mathbf{N}}^{(ii)}$ denotes the number of individual customers belonging to waiting batch customers. Furthermore

$$E\tilde{\mathbf{N}}^{(ii)} = \frac{E\mathbf{V}^{(ii)}}{\beta}, \tag{3.20}$$

where $\mathbf{V}^{(ii)}$ denotes the amount of work associated with individual customers belonging to waiting batch customers. From Wolff [1989], p. 279 we have:

$$E\mathbf{V}^{(ii)} = \gamma E[\mathbf{W}\boldsymbol{\tau}]. \tag{3.21}$$

Combining (2.19), (2.21), (3.10), (3.19), (3.20) and (3.21),

$$E\tilde{\mathbf{W}}^{(ii)} = E\mathbf{W}_{M/G/1} - \frac{\lambda\beta}{\gamma} + \frac{1}{1-\lambda\beta}\int_{t=0}^{\infty} te^{-\gamma t}dR(t). \tag{3.22}$$

$\tilde{\mathbf{W}}^{(iii)}$ is the amount of work that arrived during a past negative exponentially distributed collect interval. So

$$E\tilde{\mathbf{W}}^{(iii)} = \frac{\lambda\beta}{\gamma}. \tag{3.23}$$

Substituting (3.18), (3.22) and (3.23) in (3.17) yields (3.15).

4 The number of customers

In this section we focus our attention on the number of batch and individual customers in the system. Using the distributional form of Little's law, cf. Keilson & Servi [1990], it is easily seen that the generating function of $\mathbf{N}$, the number of batch customers at an arbitrary time, and also at arrival epochs (PASTA), is given by $E\{z^{\mathbf{N}}\} = E\{e^{-\gamma(1-z)\mathbf{R}}\}$.

The remainder of this section is devoted to the number of individual customers in the system. At time t this quantity is $\mathbf{Y}(t) + \mathbf{Z}(t)$, with $\mathbf{Y}(t)$ the number of individual customers which have arrived but not yet been collected at time t and $\mathbf{Z}(t)$ the number of individual customers which have been collected but have not yet departed at time t. Also define $\mathbf{Z}_n$ and $\mathbf{Y}_n$, the values of $\mathbf{Z}(t)$ and $\mathbf{Y}(t)$ immediately after the departure of the n-th individual customer. Notice that $\{\mathbf{Z}_n, \mathbf{Y}_n\}_{n\geq 1}$ is a two-dimensional Markov chain, whereas $\{\mathbf{Z}_n\}_{n\geq 1}$ and $\{\mathbf{Y}_n\}_{n\geq 1}$ are not Markov chains. Let $\{\mathbf{Z}, \mathbf{Y}\}$ be a vector with distribution the steady state distribution of this Markov chain. By letting $n \to \infty$ in the generating functions of $\{\mathbf{Z}_n, \mathbf{Y}_n\}$ we will derive the generating function of $\{\mathbf{Z}, \mathbf{Y}\}$.

For our analysis we need to define:

$\hat{\tau}_n$:= Service time of individual customer n.

$\hat{\sigma}_n$:= Interarrival time of the first batch customer after and counted from the start of service of individual customer n. If $\hat{\tau}_n < \hat{\sigma}_n$ no batch customers arrive during this service, otherwise at least one batch customer arrives.

μ_n := Number of individual customers which arrive during the service of individual customer n and are collected.

ν_n := Number of individual customers which arrive after the last batch arrival during the service of individual customer n. If no batch arrival occurs, then ν_n is the total number of individual customers which arrive during the service of individual customer n.

ζ_n := Number of individual customers which arrive between the end of service of individual customer $n-1$ and the start of service of individual customer n. If $\mathbf{Z}_{n-1} > 0$, then $\zeta_n = 0$.

To illustrate these definitions, an example of the arrival and departure processes of customers is presented in figure 1.

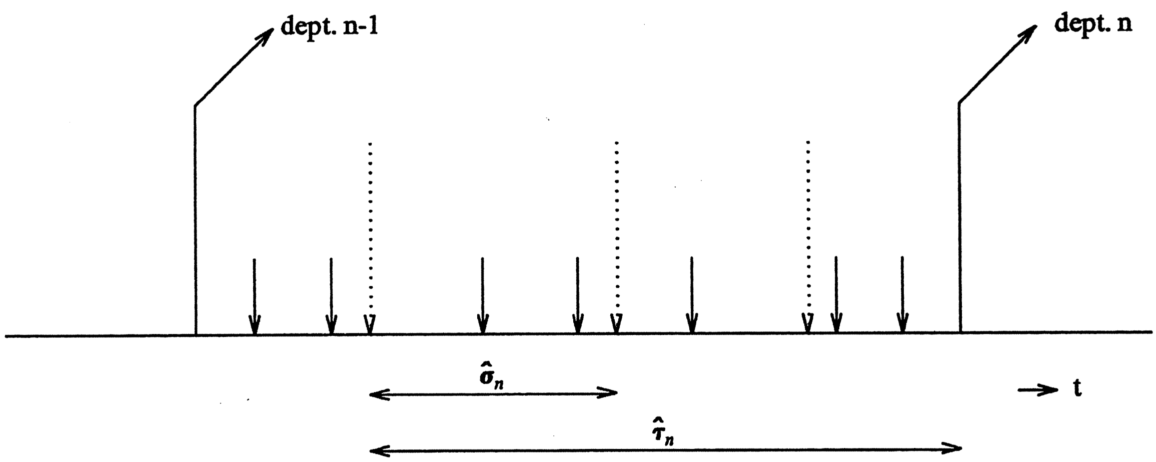

Figure 1: Arrivals and departures of individual customers, and arrivals of batch customers (....). In this example, $\zeta_n = 2$, $\mu_n = 3$, $\nu_n = 2$.

With this notation, it is readily seen that the transition equations of $\{\mathbf{Z}_n, \mathbf{Y}_n\}_{n\geq 1}$ are given by

$$\{\mathbf{Z}_{n+1}, \mathbf{Y}_{n+1}\} = \begin{cases} \{\mathbf{Z}_n - 1, \mathbf{Y}_n + \nu_{n+1}\} & \text{if } \mathbf{Z}_n \geq 1 \text{ and } \hat{\tau}_{n+1} < \hat{\sigma}_{n+1} \\ \{\mathbf{Z}_n - 1 + \mathbf{Y}_n + \mu_{n+1}, \nu_{n+1}\} & \text{if } \mathbf{Z}_n \geq 1 \text{ and } \hat{\tau}_{n+1} \geq \hat{\sigma}_{n+1} \\ \{\mathbf{Y}_n + \zeta_{n+1} - 1 + \mu_{n+1}, \nu_{n+1}\} & \text{if } \mathbf{Z}_n = 0 \end{cases} \quad (4.1)$$

Now define for $|r| \leq 1, |q| \leq 1, n = 1, 2, \ldots$, the generating functions $\Phi_n(r,q) := E\{r^{\mathbf{Z}_n} q^{\mathbf{Y}_n}\}$. Then

$$
\begin{aligned}
E\{r^{\mathbf{Z}_{n+1}} q^{\mathbf{Y}_{n+1}}\} \;=\;& E\{r^{\mathbf{Z}_{n+1}} q^{\mathbf{Y}_{n+1}} I_{\{\mathbf{Z}_n \geq 1\}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} < \hat{\boldsymbol{\sigma}}_{n+1}\}}\} + \\
& E\{r^{\mathbf{Z}_{n+1}} q^{\mathbf{Y}_{n+1}} I_{\{\mathbf{Z}_n \geq 1\}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} \geq \hat{\boldsymbol{\sigma}}_{n+1}\}}\} + \\
& E\{r^{\mathbf{Z}_{n+1}} q^{\mathbf{Y}_{n+1}} I_{\{\mathbf{Z}_n = 0\}}\}.
\end{aligned} \tag{4.2}
$$

From (4.1) and (4.2) it follows

$$
\begin{aligned}
E\{r^{\mathbf{Z}_{n+1}} q^{\mathbf{Y}_{n+1}}\} \;=\;& E\{r^{\mathbf{Z}_n - 1} q^{\mathbf{Y}_n + \boldsymbol{\nu}_{n+1}} I_{\{\mathbf{Z}_n \geq 1\}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} < \hat{\boldsymbol{\sigma}}_{n+1}\}}\} + \\
& E\{r^{\mathbf{Z}_n - 1 + \mathbf{Y}_n + \boldsymbol{\mu}_{n+1}} q^{\boldsymbol{\nu}_{n+1}} I_{\{\mathbf{Z}_n \geq 1\}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} \geq \hat{\boldsymbol{\sigma}}_{n+1}\}}\} + \\
& E\{r^{\mathbf{Y}_n + \boldsymbol{\zeta}_{n+1} + \boldsymbol{\mu}_{n+1} - 1} q^{\boldsymbol{\nu}_{n+1}} I_{\{\mathbf{Z}_n = 0\}}\}.
\end{aligned} \tag{4.3}
$$

Using that $\mathbf{Z}_n, \mathbf{Y}_n, \boldsymbol{\zeta}_{n+1}, I_{\{\mathbf{Z}_n \geq 1\}}$ and $I_{\{\mathbf{Z}_n = 0\}}$ are independent of $\boldsymbol{\nu}_{n+1}, \boldsymbol{\mu}_{n+1}, I_{\{\hat{\boldsymbol{\tau}}_{n+1} \geq \hat{\boldsymbol{\sigma}}_{n+1}\}}$ and $I_{\{\hat{\boldsymbol{\tau}}_{n+1} < \hat{\boldsymbol{\sigma}}_{n+1}\}}$, it follows for $|r| \leq 1$, $|q| \leq 1$ that

$$
\begin{aligned}
\Phi_{n+1}(r,q) \;=\;& \frac{1}{r} E\{r^{\mathbf{Z}_n} q^{\mathbf{Y}_n} I_{\{\mathbf{Z}_n \geq 1\}}\} E\{q^{\boldsymbol{\nu}_{n+1}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} < \hat{\boldsymbol{\sigma}}_{n+1}\}}\} + \\
& \frac{1}{r} E\{r^{\mathbf{Z}_n + \mathbf{Y}_n} I_{\{\mathbf{Z}_n \geq 1\}}\} E\{r^{\boldsymbol{\mu}_{n+1}} q^{\boldsymbol{\nu}_{n+1}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} \geq \hat{\boldsymbol{\sigma}}_{n+1}\}}\} + \\
& \frac{1}{r} E\{r^{\mathbf{Y}_n + \boldsymbol{\zeta}_{n+1}} I_{\{\mathbf{Z}_n = 0\}}\} E\{r^{\boldsymbol{\mu}_{n+1}} q^{\boldsymbol{\nu}_{n+1}}\}.
\end{aligned} \tag{4.4}
$$

We consider each of the three terms in the right-hand side of (4.4) in turn.

$$
E\{q^{\boldsymbol{\nu}_{n+1}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} < \hat{\boldsymbol{\sigma}}_{n+1}\}}\} = \int_{\tau=0}^{\infty} dB(\tau)\, e^{-\gamma\tau} \sum_{k=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} q^k = \beta(\gamma + \lambda(1-q)), \tag{4.5}
$$

$$
\begin{aligned}
& E\{r^{\boldsymbol{\mu}_{n+1}} q^{\boldsymbol{\nu}_{n+1}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} \geq \hat{\boldsymbol{\sigma}}_{n+1}\}}\} = \\
& \int_{\tau=0}^{\infty} dB(\tau) \int_{u=0+}^{\tau} E\{r^{\boldsymbol{\mu}_{n+1}} q^{\boldsymbol{\nu}_{n+1}} \mid \textit{last batch arr. at } u\} dP\{\textit{last batch arr. before } u\} = \\
& \int_{\tau=0}^{\infty} dB(\tau) \int_{u=0+}^{\tau} \sum_{k=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^k}{k!} r^k \sum_{j=0}^{\infty} e^{-\lambda(\tau-u)} \frac{(\lambda(\tau-u))^j}{j!} q^j \, d_u\{(1 - e^{-\gamma u})\, e^{-\gamma(\tau-u)}\} = \\
& \int_{\tau=0}^{\infty} dB(\tau) \int_{u=0+}^{\tau} e^{-\lambda(1-r)u} e^{-\lambda(1-q)(\tau-u)} e^{-\gamma(\tau-u)} \gamma du = \\
& \frac{\gamma}{\gamma + \lambda(r-q)} [\beta(\lambda(1-r)) - \beta(\gamma + \lambda(1-q))],
\end{aligned} \tag{4.6}
$$

$$E\{r^{\boldsymbol{\mu}_{n+1}} q^{\boldsymbol{\nu}_{n+1}}\} =$$

$$E\{r^{\boldsymbol{\mu}_{n+1}} q^{\boldsymbol{\nu}_{n+1}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} \geq \hat{\boldsymbol{\sigma}}_{n+1}\}}\} + E\{q^{\boldsymbol{\nu}_{n+1}} I_{\{\hat{\boldsymbol{\tau}}_{n+1} < \hat{\boldsymbol{\sigma}}_{n+1}\}}\} =$$
$$\frac{\gamma}{\gamma+\lambda(r-q)}[\beta(\lambda(1-r)) - \beta(\gamma+\lambda(1-q))] + \beta(\gamma+\lambda(1-q)) =$$
$$\frac{\gamma}{\gamma+\lambda(r-q)}\beta(\lambda(1-r)) + \frac{\lambda(r-q)}{\gamma+\lambda(r-q)}\beta(\gamma+\lambda(1-q)). \tag{4.7}$$

If $\mathbf{Z}_n = 0$ and $\mathbf{Y}_n > 0$ then the service of individual customer $n+1$ starts immediately after the arrival of the first batch customer after the completion of service of individual customer n. Then, $\boldsymbol{\zeta}_{n+1}$ is the number of individual customers which arrive during a residual arrival interval of a batch customer.
If $\mathbf{Z}_n = 0$ and $\mathbf{Y}_n = 0$ then the service of individual customer $n+1$ starts immediately after the arrival of the first non-empty batch customer after the completion of service of individual customer n. In this case, $\boldsymbol{\zeta}_{n+1} = 1+$ {the number of individual customers which arrive in the residual arrival interval of the first batch customer after the arrival of individual customer $n+1$}. Since the batch arrival intervals are negative exponentially distributed it follows

$$E\{r^{\mathbf{Y}_n+\boldsymbol{\zeta}_{n+1}} I_{\{\mathbf{Z}_n=0\}}\} =$$
$$E\{r^{\mathbf{Y}_n+\boldsymbol{\zeta}_{n+1}} I_{\{\mathbf{Z}_n=0\}} I_{\{\mathbf{Y}_n>0\}}\} + E\{r^{\mathbf{Y}_n+\boldsymbol{\zeta}_{n+1}} I_{\{\mathbf{Z}_n=0\}} I_{\{\mathbf{Y}_n=0\}}\} =$$
$$\frac{\gamma}{\gamma+\lambda(1-r)} E\{r^{\mathbf{Y}_n} I_{\{\mathbf{Z}_n=0\}} I_{\{\mathbf{Y}_n>0\}}\} + \frac{\gamma r}{\gamma+\lambda(1-r)} E\{I_{\{\mathbf{Z}_n=0\}} I_{\{\mathbf{Y}_n=0\}}\}. \tag{4.8}$$

We next let $n \to \infty$; note that $\{\mathbf{Z}_n, \mathbf{Y}_n\}_{n\geq 1}$ is an irreducible, aperiodic Markov chain, so that $\Phi(r,q) := \lim_{n\to\infty} \Phi_n(r,q)$ and related limits exist. Using (4.4)-(4.8) together with

$$\begin{aligned}
E\{r^{\mathbf{Z}_n+\mathbf{Y}_n} I_{\{\mathbf{Z}_n\geq 1\}}\} &= \Phi_n(r,r) - \Phi_n(0,r), \\
E\{r^{\mathbf{Z}_n} q^{\mathbf{Y}_n} I_{\{\mathbf{Z}_n\geq 1\}}\} &= \Phi_n(r,q) - \Phi_n(0,q), \\
E\{I_{\{\mathbf{Z}_n=0\}} I_{\{\mathbf{Y}_n=0\}}\} &= \Phi_n(0,0), \\
E\{r^{\mathbf{Y}_n} I_{\{\mathbf{Z}_n=0\}} I_{\{\mathbf{Y}_n>0\}}\} &= \Phi_n(0,r) - \Phi_n(0,0),
\end{aligned}$$

we obtain the functional equation

$$\begin{aligned}
& \Phi(r,q)[r - \beta(\gamma+\lambda(1-q))] = \\
- \;& \Phi(0,q)\beta(\gamma+\lambda(1-q)) \\
+ \;& \Phi(r,r)\frac{\gamma}{\gamma+\lambda(r-q)}[\beta(\lambda(1-r)) - \beta(\gamma+\lambda(1-q))] \\
- \;& \Phi(0,0)\frac{\gamma}{\gamma+\lambda(r-q)}\frac{1-r}{\gamma+\lambda(1-r)}[\gamma\beta(\lambda(1-r)) + \lambda(r-q)\beta(\gamma+\lambda(1-q))] + \\
+ \;& \Phi(0,r)\frac{\gamma}{\gamma+\lambda(r-q)}\frac{1}{\gamma+\lambda(1-r)}[(\gamma+\lambda(1-q))\beta(\gamma+\lambda(1-q)) - \lambda(1-r)\beta(\lambda(1-r))].
\end{aligned} \tag{4.9}$$

We now solve this functional equation. First we express $\Phi(r,r)$ into $\Phi(0,r)$ and $\Phi(0,0)$, and subsequently we determine $\Phi(0,r)$, $|\, r \,| \leq 1$. Take $q = r$ in (4.9). Then for $|\, r \,| \leq 1$:

$$\Phi(r,r) = \frac{-1}{\gamma+\lambda(1-r)} \frac{(1-r)\beta(\lambda(1-r))}{r-\beta(\lambda(1-r))}[\lambda\Phi(0,r)+\gamma\Phi(0,0)]. \tag{4.10}$$

Substituting (4.10) in (4.9), we obtain

$$\begin{aligned}
& \Phi(r,q)[r-\beta(\gamma+\lambda(1-q))] = \\
- \;& \Phi(0,q)\beta(\gamma+\lambda(1-q)) \\
- \;& \Phi(0,0)\frac{\gamma}{\gamma+\lambda(r-q)}\frac{1-r}{\gamma+\lambda(1-r)} \times \\
& [\gamma\beta(\lambda(1-r))+\lambda(r-q)\beta(\gamma+\lambda(1-q)) \\
& +\frac{\gamma\beta(\lambda(1-r))}{r-\beta(\lambda(1-r))}[\beta(\lambda(1-r))-\beta(\gamma+\lambda(1-q))]] \\
+ \;& \Phi(0,r)\frac{\gamma}{\gamma+\lambda(r-q)}\frac{1}{\gamma+\lambda(1-r)} \times \\
& [-\lambda(1-r)\beta(\lambda(1-r))+(\gamma+\lambda(1-q))\beta(\gamma+\lambda(1-q)) - \\
& -\frac{(1-r)\lambda\beta(\lambda(1-r))}{r-\beta(\lambda(1-r))}[\beta(\lambda(1-r))-\beta(\gamma+\lambda(1-q))]].
\end{aligned} \tag{4.11}$$

Formula (4.11) shows us that $\Phi(r,q)$ can be expressed in terms of $\Phi(0,q)$. We therefore solve $\Phi(0,q)$.
Define for $|\, q \,| \leq 1$:

$$\begin{aligned}
\delta(q) &:= \beta(\gamma+\lambda(1-q)), \\
\delta^{(0)}(q) &:= q, \\
\delta^{(i)}(q) &:= \delta(\delta^{(i-1)}(q)), \qquad i \geq 1.
\end{aligned} \tag{4.12}$$

Also introduce

$$\Psi(q) := \Phi(0,q). \tag{4.13}$$

The system is stable for $\lambda\beta < 1$, cf. the introduction. So for $|\, r \,| \leq 1, |\, q \,| \leq 1$ and $\lambda\beta < 1$, $\Phi(r,q)$ should be bounded and analytic. Because the left-hand side of (4.11) becomes zero for $r = \delta(q)$ (note that $|\, \delta(q) \,| \leq 1$), the same must hold for its right-hand side. Hence we obtain, for $|\, q \,| \leq 1$, and using definitions (4.12) and (4.13);

$$\begin{aligned}
\Psi(q) &= \frac{\gamma\,(\gamma+\lambda(1-q))}{(\gamma+\lambda(\delta(q)-q))(\gamma+\lambda(1-\delta(q)))}\,\Psi(\delta(q)) \\
&\quad -\frac{\lambda\,\gamma\,(\delta(q)-q)(1-\delta(q))}{(\gamma+\lambda(\delta(q)-q))(\gamma+\lambda(1-\delta(q)))}\,\Psi(0).
\end{aligned} \tag{4.14}$$

To simplify notations introduce

$$\phi(q) := \frac{\gamma(\gamma+\lambda(1-q))}{(\gamma+\lambda(\delta(q)-q))(\gamma+\lambda(1-\delta(q)))}, \qquad |\, q \,| \leq 1, \tag{4.15}$$

$$\chi(q) := -\frac{\lambda\,\gamma(\delta(q)-q)(1-\delta(q))}{(\gamma+\lambda(\delta(q)-q))(\gamma+\lambda(1-\delta(q)))}, \qquad |\, q \,| \leq 1. \tag{4.16}$$

Then (4.14) becomes

$$\Psi(q) = \phi(q)\Psi(\delta(q)) + \chi(q)\Psi(0), \qquad | q | \leq 1. \tag{4.17}$$

Iterating (4.17) we obtain for $| q | \leq 1$, and $M \geq 1$

$$\Psi(q) = \prod_{h=0}^{M} \phi(\delta^{(h)}(q))\Psi(\delta^{(M+1)}(q)) + \sum_{h=0}^{M} \chi(\delta^{(h)}(q)) \prod_{k=0}^{h-1} \phi(\delta^{(k)}(q))\Psi(0). \tag{4.18}$$

Lemma 4.1

(i). The equation $\delta(q) = q, | q | \leq 1$, has a unique solution q^*. q^* is real.

(ii). $\lim_{M\to\infty} \delta^{(M)}(q) = q^*$ for all q with $| q | \leq 1$.

(iii). $\prod_{h=0}^{\infty} \phi(\delta^{(h)}(q))$ and $\sum_{h=0}^{\infty} \chi(\delta^{(h)}(q)) \prod_{k=0}^{h-1} \phi(\delta^{(k)}(q))$ converge for all q with $| q | \leq 1$.

Proof

See appendix C.

□

Letting $M \to \infty$ in (4.18), then Lemma 4.1 leads to the following expression for $\Psi(q)$:

$$\Psi(q) = \prod_{h=0}^{\infty} \phi(\delta^{(h)}(q))\Psi(q^*) + \sum_{h=0}^{\infty} \chi(\delta^{(h)}(q)) \prod_{k=0}^{h-1} \phi(\delta^{(k)}(q))\Psi(0), \qquad | q | \leq 1. \tag{4.19}$$

Again, to simplify notations introduce for $| q | \leq 1$,

$$\begin{aligned} \alpha(q) &:= \prod_{h=0}^{\infty} \phi(\delta^{(h)}(q)), \\ \eta(q) &:= \sum_{h=0}^{\infty} \chi(\delta^{(h)}(q)) \prod_{k=0}^{h-1} \phi(\delta^{(k)}(q)). \end{aligned}$$

So (4.19) becomes

$$\Psi(q) = \alpha(q)\, \Psi(q^*) + \eta(q)\, \Psi(0). \tag{4.20}$$

Letting $q = 0$ in (4.20), we obtain

$$\Psi(q^*) = \frac{1 - \eta(0)}{\alpha(0)}\, \Psi(0). \tag{4.21}$$

Substituting (4.21) in (4.20) leads to

$$\Psi(q) = \left[\frac{1 - \eta(0)}{\alpha(0)}\, \alpha(q) + \eta(q)\right] \Psi(0). \tag{4.22}$$

Letting $q = 1$ in (4.22), we obtain

$$\Psi(1) = \left[\frac{1 - \eta(0)}{\alpha(0)}\, \alpha(1) + \eta(1)\right] \Psi(0). \tag{4.23}$$

Also, letting $r = 1$ in (4.10), we obtain

$$\Psi(1) = \frac{\gamma}{\lambda}(1-\lambda\beta) - \frac{\gamma}{\lambda}\Psi(0). \tag{4.24}$$

Solving $\Psi(0)$ from (4.23) and (4.24), and substituting $\Psi(0)$ in (4.22), we finally derive the following expression for $\Phi(0,q) = \Psi(q)$, which immediately yields an expression for $\Phi(r,q)$, cf. (4.11):

Theorem 4.1

$$\Phi(0,q) = E\{q^{\mathbf{Y}} I_{\{\mathbf{Z}=0\}}\} = \Psi(q) = \frac{\frac{1-\eta(0)}{\alpha(0)}\alpha(q) + \eta(q)}{\frac{1-\eta(0)}{\alpha(0)}\alpha(1) + \eta(1) + \frac{\gamma}{\lambda}} \frac{\gamma}{\lambda}[1-\lambda\beta], \quad |\, q \,| \leq 1. \tag{4.25}$$

Next we consider $\Phi(r,r) = E\{r^{\mathbf{Z}+\mathbf{Y}}\}$ in more detail. Since individual customers arrive and depart one by one, we can use a level crossing argument to show that the number of individual customers immediately before the arrival of an individual customer in the system and the number of individual customers immediately after the departure of an individual customer are equally distributed. Using PASTA it is seen that $\Phi(r,r)$ is the generating function of the number of individual customers at an arbitrary time. This function we obtained earlier using a decomposition argument (see section 2). Rewriting the functional equation (4.10) for $\Phi(r,r)$ into

$$\Phi(r,r) = \frac{(1-\lambda\beta)(1-r)\beta(\lambda(1-r))}{\beta(\lambda(1-r)) - r} \frac{\gamma}{\gamma+\lambda(1-r)} \frac{\frac{\lambda}{\gamma}\Phi(0,r) + \Phi(0,0)}{1-\lambda\beta}, \tag{4.26}$$

and comparing (4.26) with (2.30)- (2.33) gives

$$\frac{\lambda}{\gamma}\Phi(0,z) + \Phi(0,0) = r(\gamma + \lambda(1-z)). \tag{4.27}$$

One can verify (4.27) algebraically, but a more intuitive proof is given below.
Equation (4.27) shows a relation between the number of individual customers in the system at times when an individual customer leaves the server idle, and at times when a batch customer leaves no batch customers behind (see section 2).
Denote by $\mathbf{X}$ the number of individual customers immediately after the departure of a batch customer. Denote by $\mathbf{R} < \boldsymbol{\sigma}$ the event that the sojourn time of a batch customer is smaller than the interarrival time of the next batch customer. Consequently such a batch customer leaves no batch customers behind.
We have (cf. (2.31))

$$\begin{aligned} r(\gamma+\lambda(1-z)) &= E\{z^{\mathbf{X}} I_{\{\mathbf{R}<\boldsymbol{\sigma}\}}\} \\ &= E\{z^{\mathbf{X}} I_{\{\mathbf{R}<\boldsymbol{\sigma}\}} I_{\{\mathbf{R}=0\}}\} \\ &+ E\{z^{\mathbf{X}} I_{\{\mathbf{R}<\boldsymbol{\sigma}\}} I_{\{\mathbf{R}>0\}}\}. \end{aligned} \tag{4.28}$$

To a batch customer having $\mathbf{R} < \boldsymbol{\sigma}$ and $\mathbf{R} > 0$ corresponds a unique individual customer having $\mathbf{Z} = 0$, namely the individual customer leaving at the same moment as the batch customer, and vice versa. Using this one-to-one correspondence we have

$$\begin{aligned} E\{z^{\mathbf{X}}\, I_{\{\mathbf{R}<\boldsymbol{\sigma}\}}\, I_{\{\mathbf{R}>0\}}\} &= E\{z^{\mathbf{X}} \mid \mathbf{R}<\boldsymbol{\sigma}\,,\, \mathbf{R}>0\}\, Pr\{\mathbf{R}<\boldsymbol{\sigma}\,,\, \mathbf{R}>0\} \\ &= E\{z^{\mathbf{Y}+\mathbf{Z}} \mid \mathbf{Z}=0\}\, Pr\{\mathbf{R}<\boldsymbol{\sigma}\,,\, \mathbf{R}>0\} \\ &= \Phi(0,z)\, \frac{Pr\{\mathbf{R}<\boldsymbol{\sigma}\,,\, \mathbf{R}>0\}}{Pr\{\mathbf{Z}=0\}}. \end{aligned} \tag{4.29}$$

Since at an arbitrary time t the number of batch customers until t having $\mathbf{R} < \boldsymbol{\sigma}$ and $\mathbf{R} > 0$ equals the number of *individual* customers until t having $\mathbf{Z} = 0$, we obtain as a limiting result

$$\frac{Pr\{\mathbf{R}<\boldsymbol{\sigma}\,,\, \mathbf{R}>0\}}{Pr\{\mathbf{Z}=0\}} = \frac{\lambda}{\gamma}. \tag{4.30}$$

Using a level crossing argument and the PASTA property leads to

$$E\{r^{\mathbf{X}}\, I_{\{\mathbf{R}<\boldsymbol{\sigma}\}}\, I_{\{\mathbf{R}=0\}}\} = \Phi(0,0). \tag{4.31}$$

Combining (4.28) - (4.31) yields (4.27).

5 The Busy Period

For the sojourn time of a batch customer we were able to derive the LST, starting from recurrence relation (2.1). This relation is a typical starting point in $M/G/1$ analysis concerning waiting and sojourn processes. A similar starting point in the ordinary $M/G/1$ queue for busy period analysis is the branching argument, cf. Cohen [1982] p. 249. Unfortunately, this argument does not apply to our model. This is due to:

- The distribution of the service time of a batch customer initiating a busy period is hard to determine because the interarrival time of this customer is atypical. The fact that the previous busy period has ended during its interarrival interval suggests that this interarrival interval is relatively large.
- The length of a (sub)busy period initiated by a batch customer arriving during the service of the first batch customer in a busy period depends on the number of batch customers arriving during that service.

While deriving the LST of the busy period seems a difficult problem, the average length of a busy period is easily obtained using a balancing argument.
With $E\mathbf{B}$ the mean busy period length, $E\mathbf{I} = \frac{1}{\gamma}$ the mean idle period length and using

$$\frac{E\mathbf{B}}{E\mathbf{B} + E\mathbf{I}} = \lambda\beta = \rho\,, \tag{5.1}$$

the mean busy period length is given by

$$E\mathbf{B} = \frac{\lambda\beta/\gamma}{1-\lambda\beta}\,, \tag{5.2}$$

which is the same as $E\mathbf{B}_I$, the mean busy period length in the corresponding $M/G/1$ queue without dependence. Note that a busy period can have length zero.
Of more interest are 'real' busy periods, viz. busy periods initiated by a batch customer with positive service time. This conditioned mean busy period length is obviously larger than $E\mathbf{B}$ and $E\mathbf{B}_I$.
Define:

$$\begin{array}{lll} \boldsymbol{\tau}^1 & := & \text{service time of a customer initiating a new busy period.} \\ \tilde{\mathbf{B}} & := & \text{the length of a non-zero busy period.} \end{array}$$

Then,

$$E\mathbf{B} = E\tilde{\mathbf{B}} \cdot Pr\{\boldsymbol{\tau}^1 > 0\}. \tag{5.3}$$

Using

$$\begin{aligned} Pr\{\boldsymbol{\tau}^1 > 0\} &= Pr\{\boldsymbol{\tau} > 0 \mid \mathbf{W} = 0\} = 1 - Pr\{\boldsymbol{\tau} = 0 \mid \mathbf{W} = 0\} \\ &= 1 - \frac{Pr\{\boldsymbol{\tau} = 0, \mathbf{W} = 0\}}{Pr\{\mathbf{W} = 0\}} = 1 - \frac{Pr\{\mathbf{R} = 0\}}{1 - \lambda\beta} \end{aligned}$$

and

$$Pr\{\mathbf{R} = 0\} = \frac{\gamma}{\gamma + \lambda} r(\gamma + \lambda) \quad (\textit{see Remark } 2.1), \tag{5.4}$$

we obtain

$$E\tilde{\mathbf{B}} = \frac{\lambda\beta}{\gamma} \frac{1}{1 - \lambda\beta - \frac{\gamma}{\gamma+\lambda} r(\gamma + \lambda)}. \tag{5.5}$$

Remark 5.1 Letting $\gamma \to \infty$ and using Remark 2.2 gives as expected $E\tilde{\mathbf{B}} \to \frac{\beta}{1 - \lambda\beta}$, the mean of $\mathbf{B}_{M/G/1}$, the busy period length in an ordinary $M/G/1$ queue.

□

Remark 5.2 Letting $\omega = \gamma$ in (2.5) and using the convexity of $r(\cdot)$ gives $E\tilde{\mathbf{B}} > E\mathbf{B}_{M/G/1}$.

□

Remark 5.3 Although $E\tilde{\mathbf{B}}$ is larger than $E\mathbf{B}_{M/G/1}$ and $E\mathbf{B}_I$, we expect $cv(\tilde{\mathbf{B}})$, the coefficient of variation of the length of a non-zero busy period, to be smaller than $cv(\mathbf{B}_{M/G/1})$ and $cv(\mathbf{B}_I)$. Due to the positive correlation between service and interarrival time of a batch customer the injection of workload is more regulated than in an ordinary $M/G/1$ queue. This regulation has a stabilizing effect on the busy period length. Moreover, the stronger the correlation, the smaller we expect $cv(\tilde{\mathbf{B}})$ to be. Our conjectures, also stated by Hadidi [1981] for a similar model, are supported by simulation results which we present in the next section.

□

6 Numerical results

In this section we present some numerical results to see the quantitative effects of arrival and traffic intensities in our model. We consider the influence of different service time distributions and we also consider which effect higher moments of the service time distribution have. Finally, we support our claims about the coefficient of variation of the busy period length. The results have mostly been obtained by numerical evaluation of the infinite product (2.12), its derivative (2.15) and well known formulas for the ordinary $M/G/1$ queue. The infinite product (2.12) and the infinite sum (2.15) converge very fast unless ρ is close to one; it is easily verified that the difference between the k-th term in (2.12) and 1 is of order $O(\rho^k)$ and the k-th term in (2.15) is of order $O(\rho^k)$ for $k = 1, 2, \ldots$. We have only taken recourse to simulation for determining the coefficient of variation of the busy period length.

Waiting times of batch customers and individual customers.
In tables 1a, 1b and 1c we compare the mean waiting times of batch customers, $E\mathbf{W}$, customers in the corresponding $M/G/1$ queue without dependence, $E\mathbf{W}_I$, and individual customers, $E\tilde{\mathbf{W}}$. We also compare these mean waiting times with the mean waiting times of customers in the corresponding $M/G/1$ queue without collection, $E\mathbf{W}_{M/G/1}$ (in the tables represented by $\gamma = \infty$). Here and in the rest of this section λ is fixed with value 1.

Table 1
Comparison of the mean waiting times with those in the corresponding $M/G/1$ queue without dependence.

1a Exponential service								
	$\rho = 0.5$				$\rho = 0.9$			
γ	$correl(\sigma,\tau)$	$E\mathbf{W}$	$E\mathbf{W}_I$	$E\tilde{\mathbf{W}}$	$correl(\sigma,\tau)$	$E\mathbf{W}$	$E\mathbf{W}_I$	$E\tilde{\mathbf{W}}$
0.1	0.9128	3.392	5.500	16.292	0.9128	23.197	89.100	34.875
0.5	0.7071	0.987	1.500	3.474	0.7071	9.951	24.300	12.157
1	0.5773	0.701	1.000	1.902	0.5773	8.729	16.200	9.798
2	0.4472	0.574	0.750	1.148	0.4472	8.290	12.150	8.811
4	0.3333	0.524	0.625	0.797	0.3333	8.152	10.125	8.408
10	0.2180	0.504	0.550	0.609	0.2180	8.109	8.910	8.210
∞	0	0.5	0.5	0.5	0	8.1	8.1	8.1

1b Deterministic service								
	$\rho = 0.5$				$\rho = 0.9$			
γ	$correl(\sigma,\tau)$	$E\mathbf{W}$	$E\mathbf{W}_I$	$E\tilde{\mathbf{W}}$	$correl(\sigma,\tau)$	$E\mathbf{W}$	$E\mathbf{W}_I$	$E\tilde{\mathbf{W}}$
0.1	0.9534	3.213	5.250	16.176	0.9534	20.494	85.050	32.321
0.5	0.8164	0.790	1.250	3.330	0.8164	6.452	20.250	8.718
1	0.7071	0.491	0.750	1.731	0.7071	4.937	12.150	6.036
2	0.5733	0.346	0.500	0.943	0.5733	4.327	8.100	4.859
4	0.4472	0.282	0.375	0.565	0.4472	4.121	6.075	4.378
10	0.3015	0.255	0.300	0.361	0.3015	4.060	4.860	4.161
∞	0	0.25	0.25	0.25	0	4.05	4.05	4.05

1c Hyper-2 exponential service								
	$\rho = 0.5$				$\rho = 0.9$			
γ	$correl(\sigma, \tau)$	$E\mathbf{W}$	$E\mathbf{W}_I$	$E\tilde{\mathbf{W}}$	$correl(\sigma, \tau)$	$E\mathbf{W}$	$E\mathbf{W}_I$	$E\tilde{\mathbf{W}}$
0.1	0.9393	3.521	5.666	16.374	0.8980	25.160	91.800	36.756
0.5	0.7745	1.128	1.666	3.589	0.6742	12.463	27.000	14.647
1	0.6546	0.853	1.166	2.040	0.5423	11.364	18.900	12.426
2	0.5222	0.734	0.916	1.302	0.4152	10.974	14.850	11.494
4	0.3973	0.689	0.791	0.961	0.3071	10.850	12.825	11.105
10	0.2641	0.671	0.671	0.775	0.2000	10.809	11.610	10.910
∞	0	0.666	0.666	0.666	0	10.8	10.8	10.8

In table 1c the service time of a message is with probability $\frac{1}{4}$ exponentially distributed with parameter μ_1 and with probability $\frac{3}{4}$ exponentially distributed with parameter μ_2. $\mu_1 = 1$ and $\mu_2 = 3$ for $\rho = 0.5$; $\mu_1 = \frac{5}{9}$ and $\mu_2 = \frac{5}{3}$ for $\rho = 0.9$.
Comparing $E\mathbf{W}$ and $E\mathbf{W}_I$ in 1a-c we conclude that the positive correlation between interarrival and service times leads to a reduction of mean waiting times. The reduction is particularly strong in heavy traffic. A similar observation has been made by Hadidi [1981] for the dependence structure displayed in formula (1.13). Tables 1a, 1b and 1c also show that the influence of the service time distribution of individual customers decreases when γ approaches 0. In fact, it can be seen from (1.6) that for any service time distribution $B(\cdot)$, $\lim_{\gamma \downarrow 0} E\{e^{-\gamma\omega\boldsymbol{\tau}_i}\} = (1 + \lambda\beta\omega)^{-1}$, i.e. the distribution of the scaled service time of a batch customer converges to the negative exponential distribution with mean $\lambda\beta$.
Finally we see that $E\tilde{\mathbf{W}}$ converges slower towards $E\mathbf{W}_{M/G/1}$ than $E\mathbf{W}$ for $\gamma \to \infty$, but most of the difference is due to the remaining collect interval.

The influence of higher moments

In an ordinary $M/G/1$ queue, the influence of the service time distribution on mean waiting time is limited to its first and second moment. The question arises whether that is the case in our model. Formula (2.19) shows that $r'(\gamma)$ contributes to the mean waiting time and therefore we would suspect that the whole service time distribution plays a role. Tables 2a and 2b indicate that this conjecture is correct but also that the influence of higher moments is almost negligible. In these tables we consider mixtures of exponential distributions for the service time of individual customers.

Table 2
The influence of higher service time moments on the mean waiting time.

2a $\rho = 0.5$		
γ	$E\mathbf{W}1$	$E\mathbf{W}2$
0.1	10.548	10.596
0.5	4.482	4.530
1	3.815	3.845
2	3.506	3.845
4	3.359	3.364
10	3.275	3.276

	β	$\beta^{(2)}$	$\beta^{(3)}$
Mix.1	0.5	3.222	38.407
Mix.2	0.5	3.222	52.345

2b $\rho = 0.9$		
γ	$E\mathbf{W}1$	$E\mathbf{W}2$
0.1	80.597	81.293
0.5	66.456	66.743
1	65.191	65.326
2	64.623	64.676
4	64.358	64.374
10	64.206	64.209

	β	$\beta^{(2)}$	$\beta^{(3)}$
Mix.1	0.9	12.822	307.207
Mix.2	0.9	12.822	423.554

In table 2 $E\mathbf{W}1$ and $E\mathbf{W}2$ are the mean waiting times for a batch customer composed of individual customers with service time distribution mixtures 1 and 2, respectively.

The busy period
In section 5 we suggested that in our model the coefficient of variation for real busy periods, $cv(\tilde{\mathbf{B}})$, would be smaller than $cv(\mathbf{B}_{M/G/1})$ and $cv(\mathbf{B}_I)$. As explained there, analytical and numerical results are not available, so to obtain column $cv(\tilde{\mathbf{B}})$ in tables 3a, 3b and 3c below we have used a simulation. The simulation was performed with the queueing simulation software package Q+, running the proces for 10^6 time units. $E\tilde{\mathbf{B}}$ has been obtained using formula (5.5).

Table 3
The busy period length, mean and coefficient of variation.

3a Exponential service								
	$\rho = 0.5$				$\rho = 0.9$			
γ	$E\tilde{\mathbf{B}}$	$cv(\tilde{\mathbf{B}})$	$E\mathbf{B}_I$	$cv(\mathbf{B}_I)$	$E\tilde{\mathbf{B}}$	$cv(\tilde{\mathbf{B}})$	$E\mathbf{B}_I$	$cv(\mathbf{B}_I)$
0.1	10.076	0.954	10.000	1.844	90.057	1.473	90.000	4.583
0.5	2.439	1.230	2.000	2.236	20.423	2.763	18.000	5.385
1	1.603	1.407	1.000	2.646	13.356	3.546	9.000	6.245
2	1.233	1.557	0.500	3.317	10.517	3.991	4.500	7.681
4	1.080	1.652	0.250	4.359	9.465	4.123	2.250	9.950
10	1.016	1.711	0.100	6.557	9.089	4.266	0.900	14.799
∞	1	1.732	0	∞	9	4.359	0	∞

3b Deterministic service								
	$\rho = 0.5$				$\rho = 0.9$			
γ	$E\tilde{\mathbf{B}}$	$cv(\tilde{\mathbf{B}})$	$E\mathbf{B}_I$	$cv(\mathbf{B}_I)$	$E\tilde{\mathbf{B}}$	$cv(\tilde{\mathbf{B}})$	$E\mathbf{B}_I$	$cv(\mathbf{B}_I)$
0.1	10.053	0.894	10.000	1.789	90.012	1.168	90.000	4.472
0.5	2.374	0.983	2.000	2.000	19.525	2.043	18.000	4.899
1	1.534	0.995	1.000	2.236	12.324	2.540	9.000	5.385
2	1.167	0.994	0.500	2.646	9.702	2.856	4.500	6.245
4	1.032	1.002	0.250	3.317	9.061	3.002	2.250	7.811
10	1.001	0.999	0.100	4.796	9.000	2.971	0.900	10.909
∞	1	1	0	∞	9	3	0	∞

3c Hyper-2 Exponential service (same as in table 1c)								
	$\rho = 0.5$				$\rho = 0.9$			
γ	$E\tilde{\mathbf{B}}$	$cv(\tilde{\mathbf{B}})$	$E\mathbf{B}_I$	$cv(\mathbf{B}_I)$	$E\tilde{\mathbf{B}}$	$cv(\tilde{\mathbf{B}})$	$E\mathbf{B}_I$	$cv(\mathbf{B}_I)$
0.1	10.086	1.010	10.000	1.880	90.090	1.591	90.000	4.655
0.5	2.463	1.390	2.000	2.380	20.720	3.284	18.000	5.686
1	1.625	1.645	1.000	2.887	13.614	4.068	9.000	6.758
2	1.249	1.863	0.500	3.697	10.675	4.492	4.500	8.505
4	1.089	1.973	0.250	4.933	9.548	4.831	2.250	11.210
10	1.019	2.052	0.100	7.506	9.107	4.852	0.900	16.902
∞	1	2.082	0	∞	9	5.066	0	∞

In tables 3a, 3b, 3c $cv(\tilde{\mathbf{B}})$ is smaller than $cv(\mathbf{B}_{M/G/1})$ and $cv(\mathbf{B}_I)$, supporting the conjectures made in Remark 5.3.

7 Conclusions and suggestions for further research

A detailed exact analysis has been presented of a variant of an $M/G/1$ queue in which a positive correlation exists between interarrival and service times. Explicit expressions are derived for (transforms of) sojourn time, waiting time and queue length distributions. The mean waiting time is proven to be smaller than the mean waiting time in the uncorrelated situation. Numerical results show that a strong positive correlation between interarrival and service times may lead to a very strong reduction of mean waiting times, in particular in heavy traffic. The latter observation agrees with results obtained in Boxma [1979] for a somewhat similar model. That paper studies two queues in series with *identical*, generally distributed, service times of any customer at the two queues. Obviously interarrival and service times at the second queue are now also identical when the first queue is not empty. Numerical and asymptotic results in Boxma [1979] expose the significant reduction in mean and variance of the sojourn time at the second queue, that occurs in heavy traffic.

We have not yet succeeded in analysing the distribution of the busy period in the $M/G/1$ queue under consideration. This is left as a topic for further research. Another research direction is the decision problem of determining the collection frequency, γ, so as to minimize some function based on both grade of service and customer handling costs.

Finally, in the context of customer collection it seems rather natural that the intervals between collector arrivals at the service facility have a coefficient of variation between 0 (constant intervals) and 1 (negative exponentially distributed intervals); cf. our reference in the Introduction to the CRMA protocol with backpressure. A useful approximate approach to the resulting queue with correlated interarrival and service times might be to interpolate in some way between the $D/G/1$ queue and the queue studied in the present paper, as indicated in the Introduction.

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APPENDICES

A PROOF OF LEMMA 2.1

(i) & (ii). We prove that $\mid g(\omega_1)-g(\omega_2) \mid\leq \lambda\beta \mid \omega_1-\omega_2 \mid$ for all ω_1,ω_2 with $Re\,\omega_1, Re\,\omega_2 \geq 0$. Distinguish two cases.
a. $Re\,\omega_1 \geq Re\,\omega_2$. Using (2.9),

$$\begin{aligned} & \mid g(\omega_1) - g(\omega_2) \mid \\ = \ & \lambda \mid \beta(\omega_1) - \beta(\omega_2) \mid \\ = \ & \lambda \mid \int\limits_{t=0}^{\infty} e^{-\omega_2 t}[1 - e^{-(\omega_1-\omega_2)t}]dB(t) \mid \\ \leq \ & \lambda\beta \mid \omega_1 - \omega_2 \mid, \end{aligned}$$

since $\mid e^{-\omega} \mid \leq 1$ and $\mid 1 - e^{-\omega} \mid \leq \mid \omega \mid$ for all ω with $Re\,\omega \geq 0$.
b. $Re\,\omega_1 \leq Re\,\omega_2$.
This case proceeds similarly.
Moreover $Re\,g(\omega) \geq 0$ for all ω with $Re\,\omega \geq 0$. As we assumed $\rho = \lambda\beta < 1$, we conclude from the fixed-point theorem for contractions, cf. Apostol [1974], that the equation $g(\omega) = \omega, Re\,\omega \geq 0$ has a unique solution ω^* and that $\lim_{M\to\infty} g^{(M)}(\omega) = \omega^*$ for all ω with $Re\,\omega \geq 0$. Since $g(\gamma) = \gamma + \lambda(1-\beta(\gamma)) \geq \gamma$ and $g(\gamma+\lambda) = \gamma + \lambda(1-\beta(\gamma+\lambda)) \leq \gamma + \lambda$, ω^* is real and $\gamma \leq \omega^* \leq \gamma + \lambda$.
(iii). From the theory of infinite products, cf. Titchmarsh [1939] p. 18,

$$\prod_{h=0}^{\infty} \frac{f(g^{(h)}(\omega))}{g^{(h+1)}(\omega)}$$

converges iff

$$\sum_{h=0}^{\infty}[1 - \frac{f(g^{(h)}(\omega))}{g^{(h+1)}(\omega)}]$$

converges. Using (2.8) and (2.9),

$$1 - \frac{f(g^{(h)}(\omega))}{g^{(h+1)}(\omega)} = \frac{\gamma(1 - \frac{g^{(h)}(\omega)}{g^{(h+1)}(\omega)}) + g^{(h)}(\omega) - g^{(h+1)}(\omega)}{\gamma + g^{(h)}(\omega) - g^{(h+1)}(\omega)}.$$

Since $| g^{(h+1)}(\omega) - g^{(h)}(\omega) | \leq \lambda\beta | g^{(h)}(\omega) - g^{(h-1)}(\omega) |$, cf. the proof of (i) & (ii),

$$\sum_{h=0}^{\infty} \frac{\gamma(1 - \frac{g^{(h)}(\omega)}{g^{(h+1)}(\omega)}) + g^{(h)}(\omega) - g^{(h+1)}(\omega)}{\gamma + g^{(h)}(\omega) - g^{(h+1)}(\omega)}$$

converges and so

$$\prod_{h=0}^{\infty} \frac{f(g^{(h)}(\omega))}{g^{(h+1)}(\omega)}$$

converges.

□

B Proof of Lemma 3.1

(i). We successively prove both parts of the inequality.

$$\frac{d}{dt}[te^{-\gamma t}] = (1 - \gamma t)e^{-\gamma t}. \tag{B.1}$$

Using (2.6) and (B.1),

$$\int_{t=0}^{\infty} te^{-\gamma t} dR(t) = \int_{t=0+}^{\infty} te^{-\gamma t} dR(t) \leq \sup_{t \in (0,\infty)} [te^{-\gamma t}] \int_{t=0+}^{\infty} dR(t) = \frac{e^{-1}}{\gamma}(1 - \frac{\gamma}{\gamma + \lambda} r(\gamma + \lambda)).$$

Using (2.7),

$$\int_{t=0}^{\infty} te^{-\gamma t} dR(t) - \frac{\lambda\beta}{\gamma} = \int_{t=0}^{\infty} [te^{-\gamma t} + \frac{1}{\gamma}e^{-\gamma t} - \frac{1}{\gamma}] dR(t).$$

$$\frac{d}{dt}[te^{-\gamma t} + \frac{1}{\gamma}e^{-\gamma t} - \frac{1}{\gamma}] = -\gamma te^{-\gamma t} \leq 0, \qquad t \geq 0. \tag{B.2}$$

Using (B.2),

$$te^{-\gamma t} + \frac{1}{\gamma}e^{-\gamma t} - \frac{1}{\gamma} \leq [te^{-\gamma t} + \frac{1}{\gamma}e^{-\gamma t} - \frac{1}{\gamma}]_{|t=0} = 0, \qquad t \geq 0.$$

(ii). Recalling Remark 2.3,

$$\frac{\lambda\beta}{1 - \lambda\beta} \int_{t=0}^{\infty} te^{-\gamma t} dR(t) = \sum_{h=1}^{\infty} \frac{g^{(h)\prime}(0)(f(g^{(h)}(0))g'(g^{(h)}(0)) - f'(g^{(h)}(0))g(g^{(h)}(0)))}{f(g^{(h)}(0))g(g^{(h)}(0))}.$$

Using (2.8) and (2.9), we find that the first term in the sum equals

$$\frac{\lambda\beta}{\gamma} \frac{\lambda(1 - \beta(\gamma))}{\gamma - \lambda(1 - \beta(\gamma))} \frac{\gamma + \lambda(1 - \beta(\gamma)) + 2\lambda\gamma\beta'(\gamma)}{\gamma + \lambda(1 - \beta(\gamma))}.$$

Moreover it is straightforward to verify by induction that all terms in the sum are strictly positive.

□

C Proof of Lemma 4.1

(i) & (ii). Using (2.9) and (4.12) we find that $\delta(q) = q$ iff $g(\gamma + \lambda(1-q)) = \gamma + \lambda(1-q)$ and that $\delta^{(h)}(q) = \dfrac{\gamma + \lambda - g^{(h)}(\gamma + \lambda(1-q))}{\lambda}$ for $\mid q \mid \leq 1, h = 1, 2, \ldots$. From the proof of Lemma 2.1 we know that the equation $g(\omega) = \omega, Re\,\omega \geq 0$ has a unique solution ω^*, that ω^* is real, that $\gamma \leq \omega^* \leq \gamma + \lambda$ and that $\lim_{M\to\infty} g^{(M)}(\omega) = \omega^*$ for all ω with $Re\,\omega \geq 0$. As $Re\,(\gamma + \lambda(1-q)) \geq 0$ for all q with $\mid q \mid \leq 1$ we conclude that the equation $\delta(q) = q$, $\mid q \mid \leq 1$ has a unique solution $q^* = \dfrac{\gamma + \lambda - \omega^*}{\lambda}$, q^* is real, $0 \leq q^* \leq 1$ and that $\lim_{M\to\infty} \delta^{(M)}(q) = q^*$ for all q with $\mid q \mid \leq 1$.

(iii). Using (4.15),

$$\prod_{h=0}^{M} \phi(\delta^{(h)}(q)) = \frac{\gamma + \lambda(1-q)}{\gamma + \lambda(1 - \delta^{(M+1)}(q))} \prod_{h=0}^{M} \frac{\gamma}{\gamma + \lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}.$$

So

$$\prod_{h=0}^{\infty} \phi(\delta^{(h)}(q)) = \frac{\gamma + \lambda(1-q)}{\gamma + \lambda(1 - q^*)} \prod_{h=0}^{\infty} \frac{\gamma}{\gamma + \lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}.$$

From the theory of infinite products, cf. Titchmarsh [1939] p.18,

$$\prod_{h=0}^{\infty} \frac{\gamma}{\gamma + \lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}$$

converges iff

$$\sum_{h=0}^{\infty} [1 - \frac{\gamma}{\gamma + \lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}]$$

converges. Rewrite

$$1 - \frac{\gamma}{\gamma + \lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))} = \frac{\lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}{\gamma + \lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}.$$

Since $\mid \delta^{(h+1)}(q) - \delta^{(h)}(q) \mid \leq \lambda\beta \mid \delta^{(h)}(q) - \delta^{(h-1)}(q) \mid$,

$$\sum_{h=0}^{\infty} \frac{\lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}{\gamma + \lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}$$

converges and so

$$\prod_{h=0}^{\infty} \phi(\delta^{(h)}(q))$$

converges. Using (4.15) and (4.16),

$$
\begin{aligned}
&\sum_{h=0}^{\infty} \chi(\delta^{(h)}(q)) \prod_{k=0}^{h-1} \phi(\delta^{(k)}(q)) \\
= &\sum_{h=0}^{\infty} \frac{\lambda(\delta^{(h)}(q) - \delta^{(h+1)}(q))(1 - \delta^{(h)}(q))}{\gamma + \lambda(1 - \delta^{(h+1)}(q))} \frac{\gamma + \lambda(1-q)}{\gamma + \lambda(1 - \delta^{(h)}(q))} \times \\
&\prod_{k=0}^{h} \frac{\gamma}{\gamma + \lambda(\delta^{(k+1)}(q) - \delta^{(k)}(q))}.
\end{aligned}
$$

Since $\mid \delta^{(h+1)}(q) - \delta^{(h)}(q) \mid \leq \lambda\beta \mid \delta^{(h)}(q) - \delta^{(h-1)}(q) \mid$ and

$$
\prod_{h=0}^{\infty} \frac{\gamma}{\gamma + \lambda(\delta^{(h+1)}(q) - \delta^{(h)}(q))}
$$

converges,

$$
\sum_{h=0}^{\infty} \frac{\lambda(\delta^{(h)}(q) - \delta^{(h+1)}(q))(1 - \delta^{(h)}(q))}{\gamma + \lambda(1 - \delta^{(h+1)}(q))} \frac{\gamma + \lambda(1-q)}{\gamma + \lambda(1 - \delta^{(h)}(q))} \prod_{k=0}^{h} \frac{\gamma}{\gamma + \lambda(\delta^{(k+1)}(q) - \delta^{(k)}(q))}
$$

converges and so

$$
\sum_{h=0}^{\infty} \chi(\delta^{(h)}(q)) \prod_{k=0}^{h-1} \phi(\delta^{(k)}(q))
$$

converges.

□