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On Constructing Kernel Polynomials of a Spectral Function: Application to ARMA Models

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Useful formulas are presented for constructing kernel polynomials associated with a certain class of spectral functions, including spectral functions with rational densities which characterize ARMA models. As applied to the last subclass, this formula expresses kernel polynomials of an ARMA model in terms of kernel polynomials of an AR part which are easily constructed.

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1. Introduction

1.1. Let F(x) be a distribution function of the infinite type, absolutely continuous with respect to another distribution function, say $F_0(x)$, $x \in [-\pi, \pi]$. It is assumed throughout this paper that the density $dF(x) / dF_0(x)$ is a non-negative trigonometric polynomial in x of degree r, say

$$\frac{dF}{dF_0}(x) = a_0 + 2\sum_{k=1}^{r} (a_k \cos kx + b_k \sin kx).$$

According to the Fejér-Riesz theorem (see, e.g. Grenander and Szegö (1958), section 1.12) this density can be represented as the square of the modulus of a polynomial in z of equal degree where z is on the unit circle, $z = e^{ix}$. That is, there exists a polynomial, say $\alpha_r(z) = \alpha_0 + \alpha_1 z + ... + \alpha_r z^r$, such that

$$\frac{dF}{dF_0}(x) = |\alpha_r(z)|^2, \quad z = e^{ix}. \tag{1.1.1}$$

This representation is unique under the following additional conditions:

(i) the polynomial $\alpha_r(z)$ has no zeros inside the unit circle, i.e. $\alpha_r(z) \neq 0$ for $|z| < 1$,

(ii) $\alpha_r(0)$ is real and positive.

We always will assume the conditions (i) and (ii). Moreover, without loosing generality we will assume $\alpha_r(0) = 1$.

1.2. The results of the present paper, concerning distribution functions F(x) specified

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in section 1.1, are easily applied to the special situation typical in time series analysis where F(x) is interpreted as a spectral distribution function, absolutely continuous relative to the Lebesgue measure, with the spectral density $f(x) = dF(x) / dx$ which is a rational function

$$(1.2.1) \qquad f(x) = f_{q,r}(x) = \frac{\sigma^2}{2\pi}\frac{|\alpha_r(z)|^2}{|\beta_q(z)|^2}, \quad \sigma^2 > 0, \quad z = e^{ix},$$

where the polynomials $\alpha_r(z) = 1 + \alpha_1 z + \ldots + \alpha_r z^r$ and $\beta_q(z) = 1 + \beta_1 z + \ldots + \beta_q z^q$ have no common roots, $\alpha_r(z)$ has no zeros inside the unit circle, i.e. $\alpha_r(z) \neq 0$ for $|z| < 1$, and $\beta_q(z)$ has no zeros inside or on the unit circle, i.e. $\beta_q(z) \neq 0$ for $|z| \leq 1$. The function $f_{q,r}(x)$ is the spectral density of a mixed autoregressive - moving average time series (ARMA(q,r)). In particular, $f_{q,0}(x)$ characterizes an AR(q) process, $f_{0,r}(x)$ a MA(q) process, and finally $f_{0,0}(x) = \sigma^2/2\pi$ a white noise process with intensity $\sigma^2 > 0$.

1.3. In applications of the classical least squares theory (such as forcasting, interpolating and filtering time series, or drawing statistical inference in regression and time series analysis) one usually needs explicite expressions for the determinants and the inverses of the covariance matrices of the observed time series - Hermitian positive definite matrices of Toeplitz type associated with the spectral function F(x) of the observed time series as follows:

$$(1.3.1) \qquad \mathbb{T}_n(F) = \int_{-\pi}^{\pi} z_n z_n^* \, dF(x), \quad z_n = \text{col}\,\{1, z, \ldots, z^n\}, \quad z = e^{ix};$$

z_n^* is the conjugate transpose of z_n. We use the following notation for the Toeplitz determinants:

$$D_n(F) = \det \mathbb{T}_n(F) = |\mathbb{T}_n(F)| > 0.$$

For any n+1 dimensional column vectors $z_n = \text{col}\,\{1, z, \ldots, z^n\}$ and $\zeta_n = \text{col}\,\{1, \zeta, \ldots, \zeta^n\}$ we define

$$(1.3.2) \qquad s_n(\zeta, z; F) = \zeta_n^* \, \mathbb{T}_n(F)^{-1} \, z_n,$$

which for a fixed ζ are polynomials in z of degree n, called the kernel polynomials; cf. (3.2.1) below for the alternative definition. They possess the important *reproducing* property indicated in Proposition 3.2.1; cf. Grenander and Szegö (1958), theorem 2.2(c). Knowing $s_n(\zeta, z; F)$, one can compute the desired entries of $\mathbb{T}_n(F)^{-1}$: indeed, the entry at the jth row and kth column is $\frac{1}{j!}\frac{\partial^j}{\partial\bar{\zeta}^j}\frac{1}{k!}\frac{\partial^k}{\partial z^k} s_n(\zeta, z; F)|_{z,\zeta=0}$.

1.4. Assuming (1.1.1), we present in this paper useful formulas expressing the Toeplitz determinants $D_n(F)$ and kernel polynomials $s_n(\zeta, z; F)$, associated with a distribution function F(x), in terms of the Toeplitz determinants $D_n(F_0)$ and kernel polynomials $s_n(\zeta, z; F_0)$, associated with the dominating distribution function $F_0(x)$; see theorems 3.1.1 and 3.4.1. These formulas are primarily intended for applications to the problems in time series analysis mentioned in section 1.3; see the forthcoming report, Dzhaparidze (1991), on asymptotic analysis of the behaviour of $D_n(F)$ and $s_n(\zeta, z; F)$ as $n \to \infty$. In section 2 we reproduce the well-known results concerning the relatively easy autoregressive case, since they are also needed in the forthcoming report just mentioned. The proofs of the assertions presented in sections 2 and 3 are given in sections 4 and 5 respectively. Finally, in section 6 an explicite expression is given for the inverse of the Toeplitz matrix associated with a distribution function of type (1.1.1); see theorem 6.4.1. We refer to Grenander and Szegö (1958), for more details on the spectral domain methods used here, and the notions introduced (cf. also the books by Grenander and Rosenblatt (1957), Grenander (1981), Dzhaparidze (1986) and the papers by Parzen (1961), Hajék (1962) and Pham Dinh Tuan (1987) where different approaches to the related problems can be found; they may serve as a guidance for further references).

1.5. For easy reference we reproduce here the following well-known formula for computing determinants of block matrices (see, e.g. Lancaster and Tismenetsky (1985), p. 46): under the obvious invertibility assumptions

(1.5.1) $$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D|\,|A - B\,D^{-1}\,C| = |A|\,|D - C\,A^{-1}\,B|.$$

2. Autoregression

2.1. We consider first the relatively simple case of the distribution function F(x) with the density $f_{q,0}(x)$, the spectral density of an AR(q) process; cf. (1.2.1) with r = 0. We have

(2.1.1) $$f_{q,0}(x) = \frac{\sigma^2}{2\pi}|\beta_q(z)|^{-2} = \frac{\sigma^2}{2\pi}|\beta_q{}^*(z)|^{-2},\ \sigma^2 > 0,\ z = e^{ix},$$

where $\beta_q{}^*(z) = z^q\,\overline{\beta}_q(z^{-1})$ is the so-called reciprocal polynomial.

Theorem 2.1.1. *Let* F(x) *and* $F_0(x)$ *be the distribution functions, absolutely continuous relative to the Lebesgue measure, with the densities* $f_{q,0}(x)$ *and* $f_{0,0}(x) = \sigma^2/2\pi$, *respectively* (cf. (2.1.1) or (1.2.1) with r = 0 and q = r = 0). *Then*

(i) *The kernel polynomials* $s_n(\zeta, z; F)$ *and*

(2.1.2) $$s_n(\zeta, z; F_0) = \frac{1}{\sigma^2}\frac{1 - (\overline{\zeta}z)^{n+1}}{1 - \overline{\zeta}z}$$

are related as follows:

(2.1.3) $$s_{n+q}(\zeta, z; F) = s_{q-1}(\zeta, z; F) + \overline{\beta_q^*(\zeta)}\, \beta_q^*(z)\, s_n(\zeta, z; F_0), \quad n \geq 0,$$

with

(2.1.4) $$(1 - \bar{\zeta}z)\, \sigma^2\, s_{q-1}(\zeta, z; F) = \overline{\beta_q(\zeta)}\, \beta_q(z) - \overline{\beta_q^*(\zeta)}\, \beta_q^*(z).$$

(ii) *For* $n \geq q$ *the Toeplitz determinants* $D_n(F)$ *and*

(2.1.5) $$D_n(F_0) = \sigma^{2(n+1)}$$

are related as follows:

(2.1.6) $$D_n(F) = D_n(F_0) \exp \{\frac{1}{\pi}\iint \left|\frac{\beta_q'(z)}{\beta_q(z)}\right|^2 ds\}$$

where the integration is extended over the unit disc $|z| \leq 1$.

2.2. The integral on the left hand side of (2.1.6) can be expressed in terms of the roots of the polynomial $\beta_q(z)$; see (2.2.4) below. Let the polynomial $\beta_q(z)$ have m distinct roots, $m \leq q$, of multiplicity $q_1, \ldots, q_m$ respectively, with $q_1 + \ldots + q_m = q$. Denote by $\xi_1, \ldots, \xi_m$ the corresponding roots of the reciprocal polynomial $\beta_q^*(z)$. Then $\bar{\xi}_1^{-1}, \ldots, \bar{\xi}_m^{-1}$ coincide with the roots of $\beta_q(z)$, and therefore $|\xi_j| < 1$ by assumption. Obviously

(2.2.1) $$\beta_q(z) = (1 - \bar{\xi}_1 z)^{q_1} \ldots (1 - \bar{\xi}_m z)^{q_m} \text{ and } \beta_q^*(z) = (z - \xi_1)^{q_1} \ldots (z - \xi_m)^{q_m},$$

also

(2.2.2) $$\nabla_{jz}\beta_q(z)|_{z=\bar{\xi}_j^{-1}} = 0 \text{ and } \nabla_{jz}\beta_q^*(z)|_{z=\xi_j} = 0$$

where

(2.2.3) $$\nabla_{jz} = \text{col}\, \{\frac{1}{k!}\frac{\partial^k}{\partial z^k}, 0 \leq k < q_j\}.$$

As is shown in section 4.2 below, we have

(2.2.4) $$\exp \{\frac{1}{\pi}\iint \left|\frac{\beta_q'(z)}{\beta_q(z)}\right|^2 ds\} = \prod_{1\leq j,k\leq m} |1 - \bar{\xi}_j\xi_k|^{-q_k q_j}.$$

2.3. Using the notation (2.2.3), we define a column vector $\nabla(\xi_k, z)$ of dimension q_k and a $q_k \times q_j$ matrix $\nabla(\xi_k, \xi_j)$ by

$$\nabla(\xi_k, z) = \nabla_{k\bar{\zeta}}\, (1 - \bar{\zeta} z)^{-1}|_{\zeta=\xi_k} \text{ and } \nabla(\xi_j, \xi_k) = \nabla_{j\bar{\zeta}}\nabla_{k\bar{z}}^*\, (1 - \bar{\zeta} z)^{-1}|_{\zeta=\xi_j,\, z=\xi_k}$$

respectively (as usual, the sign * indicates the conjugate transposition), and then the q dimensional column vector $\nabla(\xi, z)$ and the q dimensional matrix $\nabla(\xi, \xi)$ by

$$\nabla(\xi, z) = \text{col}\, \{\nabla(\xi_k, z), 1 \leq k \leq m\} \text{ and } \nabla(\xi, \xi) = [\nabla(\xi_k, \xi_j), 1 \leq k,j \leq m]$$

respectively. Notice that $\nabla(\xi, z)^* = \nabla(z, \xi)$ by definition.

In assertion (ii) of the following corollary the expression (2.3.3) for Toeplitz determinants is derived by applying (2.1.5) and (2.2.4) to (2.1.6), while in assertion (i), proved in section 4.3, the desired inverse of the Toeplitz matrix $\mathbb{T}_{n+q}(F)$ is expressed in terms of the inverse of $\nabla(\xi, \xi)$ and two other matrices B and $\mathbb{B}$ of dimensionality q and n+q+1, respectively, defined as follows (for convenience, we assume $\beta_k = 0$ for $k < 0$ or $k > q$; as usual β_k for $0 \le k \le q$ are the coefficients in the polynomial $\beta_q(z)$). If the matrix B of dimension q is presented as a column B = col $\{B_1, ..., B_m\}$ of $q_j \times q$ block matrices B_j, then the ℓth row ($1 \le \ell \le q_j$) of B_j consists of the coefficients at $z^{k-\ell}$ in the polynomial $\beta_q(z) / (1 - \bar{\xi}_j z)^\ell$, for $k = 1, ..., q$ successively. As for the matrix $\mathbb{B}$ of dimensionality n+q+1, its entry at the jth row and kth column is $\beta_{q-j}\,\bar{\beta}_{q-k} + ... + \beta_{n+q-j}\,\bar{\beta}_{n+q-k}$ (cf. remarks 2.3.2 and 2.3.3 below).

Corollary 2.3.1. *Under the conditions of theorem 2.1.1 the following assertions hold true.*

(i) *For each* $n \ge 0$

(2.3.1) $$\sigma^2\, \mathbb{T}_{n+q}(F)^{-1} = \begin{bmatrix} \sigma^2\, \mathbb{T}_{q-1}(F)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \mathbb{B}$$

with

(2.3.2) $$\sigma^2\, \mathbb{T}_{q-1}(F)^{-1} = B^*\, \nabla(\xi, \xi)^{-1}\, B.$$

(ii) *For* $n \ge q$ *the Toeplitz determinants* $D_n(F)$ *are given by*

(2.3.3) $$D_n(F) = \sigma^{2(n+1)} \prod_{1 \le j,k \le m} |1 - \bar{\xi}_j \xi_k|^{-q_k q_j}.$$

Remark 2.3.2. It is easily seen by taking into consideration (2.2.1) and (2.2.3) that the matrix B introduced above transforms the q dimensional column vector z_{q-1} = col $\{1, z, ..., z^{q-1}\}$ into the column vector $\beta_q(z)\, \nabla(\xi, z)$ of the same dimension:

(2.3.4) $$B\, z_{q-1} = \beta_q(z)\, \nabla(\xi, z)$$

(this transformation is non-singular; see lemma 4.3.1(i) below). Therefore by definition (1.3.2) we get from (2.3.2) the following representaton for $s_{q-1}(\zeta, z; F)$, alternative to (2.1.4):

(2.3.5) $$\sigma^2\, s_{q-1}(\zeta, z; F) = \overline{\beta_q(\zeta)}\, \beta_q(z)\, \nabla(\zeta, \xi)\, \nabla(\xi, \xi)^{-1}\, \nabla(\xi, z).$$

Notice the useful identity

$$\overline{\beta_q{}^*(\zeta)}\, \beta_q{}^*(z) = \overline{\beta_q(\zeta)}\, \beta_q(z) \begin{vmatrix} (1 - \bar{\zeta} z)^{-1} & \nabla(\zeta, \xi) \\ \nabla(\xi, z) & \nabla(\xi, \xi) \end{vmatrix} \Big/ \begin{vmatrix} (1 - \bar{\zeta} z)^{-1} & 0 \\ 0 & \nabla(\xi, \xi) \end{vmatrix}$$

verified by comparing (2.1.4) and (2.3.3) and using (1.5.1) (it is used, in particular, for asymptotic analysis in Dzhaparidze (1991)). Finally, it seems worthy of mentioning that since for each k = 0, 1, ..., q

$$\frac{1}{k!}\frac{\partial^k}{\partial z^k}(1-\bar{\zeta}z)^{-1}\,\beta_q(z)|_{z=0}=\overline{\beta_k{}^*(\zeta)}$$

where $\beta_k(z) = 1 + \beta_1 z + \ldots + \beta_k z^k$, we have

$$\frac{1}{k!}\frac{\partial^k}{\partial z^k}\nabla(\xi, z)\,\beta_q(z)|_{z=0} = \text{col}\,\{\nabla_{j\bar{\zeta}}\,\overline{\beta_k{}^*(\zeta)}|_{\zeta=\xi_j},\ 1 \le j \le m\}$$

that is, according to (2.3.4), the k^{th} column of B.

Remark 2.3.3. Concerning the matrix $\mathbb{B}$ of dimensionality n+q+1, in virtue of (1.3.2), (2.1.3) and (2.3.1) we have

$$\text{(2.3.6)}\qquad \zeta_{n+q}{}^*\mathbb{B}\ z_{n+q} = \overline{\beta_q{}^*(\zeta)}\,\beta_q{}^*(z)\,\sigma^2\,s_n(\zeta, z; F_0)$$

$$=\sum_{k=0}^{n}(\beta_q\bar{\zeta}^k + \beta_{q-1}\bar{\zeta}^{k+1} + \ldots + \bar{\zeta}^{k+q})\,(\bar{\beta}_q z^k + \bar{\beta}_{q-1}z^{k+1} + \ldots + z^{k+q}).$$

Indeed, the first identity is proved in section 4.3 below, and the second is an easy consequence of (2.1.2)).

3. Distributions of type (1.1.1)

3.1. Consider now the situation described in section 1.1 where a distribution function F(x) has the density (1.1.1) with respect to a certain dominating distribution $F_0(x)$ (not necessarily such as in section 2), and the polynomial $\alpha_r(z)$ satisfies conditions (i) and (ii) in section 1.1. Condition (ii) means in particular that if $\zeta_1, \ldots, \zeta_s$ are the distinct roots ($s \le r$) of the reciprocal polynomial $\alpha_r{}^*(z)$, of multiplicity $r_1, \ldots, r_s$ respectively, with $r_1 + \ldots + r_s = r$, then $|\zeta_j| \le 1$; cf. section 2.2.

In theorem 3.1.1 below the kernel polynomial $s_n(\zeta, z; F)$ is expressed in terms of $s_n(\zeta, z; F_0)$, evaluated at points ζ_j. Consider, namely, the following column vector and the matrix of dimensionality r

$$S_n(\zeta, z; F_0) = \text{col}\,\{\nabla_{k\bar{\zeta}}\ s_n(\zeta, z; F_0)|_{\zeta=\zeta_k},\ 1 \le k \le s\},$$

(3.1.1)

$$S_n(\zeta, \zeta; F_0) = \text{col}\,\{\nabla_{k\bar{\zeta}}\ S_n(\zeta, \zeta; F_0)|_{\zeta=\zeta_k},\ 1 \le k \le s\}$$
$$= [\nabla_{k\bar{\zeta}}\nabla_{j\bar{z}}{}^*\ s_n(\zeta, z; F_0)|_{\zeta=\zeta_k,\ z=\zeta_j},\ 1 \le k,j \le s],$$

respectively (with $S_n(\zeta, z; F_0)^* = S_n(z, \zeta; F_0)$), where ∇_{jz} is the same vector valued differentiation operator as in (2.2.3), but with r_j instead of q_j.

Theorem 3.1.1. *Under the conditions of section 1.1*

(i) *The kernel polynomials* $s_n(\zeta, z; F)$ *are given by*

$$(3.1.2)\quad \overline{\alpha_r^*(\zeta)}\,\alpha_r^*(z)\, s_n(\zeta, z; F) = \begin{vmatrix} s_{n+r}(\zeta, z; F_0) & S_{n+r}(\zeta, \zeta; F_0) \\ S_{n+r}(\zeta, z; F_0) & S_{n+r}(\zeta, \zeta; F_0) \end{vmatrix} \Big/ |S_{n+r}(\zeta, \zeta; F_0)|$$

(ii) *The Toeplitz determinants* $D_n(F)$ *are given by*

$$(3.1.3)\qquad D_n(F) = D_{n+r}(F_0)\, |S_{n+r}(\zeta, \zeta; F_0)|\, |V(\alpha_r^*)|^{-2},$$

where $V(\alpha_r^*)$ *is the (generalized) Vandermonde matrix of dimension* r, *associated with the polynomial* $\alpha_r^*(z)$, *hence*

$$(3.1.4)\qquad |V(\alpha_r^*)| = \prod_{k>j} (\zeta_k - \zeta_j)^{r_k r_j}$$

(cf. remark 3.1.3 below).

Remark 3.1.2. Using (1.5.1) we may rewrite (3.1.2) in the following alternative form:

$$(3.1.2^*)\qquad \overline{\alpha_r^*(\zeta)}\,\alpha_r^*(z)\, s_n(\zeta, z; F) = s_{n+r}(\zeta, z; F_0) - S_{n+r}(\zeta, \zeta; F_0)\, S_{n+r}(\zeta, \zeta; F_0)^{-1}\, S_{n+r}(\zeta, z; F_0).$$

Remark 3.1.3. Recall that by definition (see, e.g. Lancaster and Tismenetsky (1985), p. 70) the Vandermonde matrix $V(\alpha_r^*) = [V_1, \ldots, V_s]$ of dimension r, associated with the polynomial $\alpha_r^*(z)$, is such that each $r \times r_j$ submatrix V_j is defined as follows:

$$(3.1.5)\qquad V_j^* = \nabla_{j\bar{z}}\, z_{r-1}^*|_{z=\zeta_j};$$

(3.1.4) is indeed the Vandermonde determinant. For asymptotic analysis (as $n \to \infty$) carried out in Dzhaparidze (1991), it is useful to rewrite (3.1.3) as follows:

$$(3.1.3^*)\qquad D_n(F) = D_{n+r}(F_0)\, |\mathbb{S}_{n+r}(\zeta, \zeta; F_0)| \exp\left\{\frac{1}{\pi}\iint \left|\frac{\alpha_r'(z)}{\alpha_r(z)}\right|^2 ds\right\}$$

where the integration is extended over the unit disc $|z| \le 1$ and

$$|\mathbb{S}_{n+r}(\zeta, \zeta; F_0)| = |S_{n+r}(\zeta, \zeta; F_0)| \,/\, |\nabla(\zeta, \zeta)|.$$

Here the matrix $\nabla(\zeta, \zeta)$ is defined as in section 2.3, and its determinant is calculated according to lemma 4.3.1(ii) below (with the obvious substitution of ξ by ζ) which tells us namely that (cf. (2.2.4) and (3.1.4)).

$$|\nabla(\zeta, \zeta)| = |V(\alpha_r^*)|^2 \exp\left\{\frac{1}{\pi}\iint \left|\frac{\alpha_r'(z)}{\alpha_r(z)}\right|^2 ds\right\}.$$

3.2. Let $\{\phi_n(z; F)\}_{n=0,1,\ldots}$ be a system of polynomials of a complex variable z, orthonormal on the unit circle $|z| = 1$, $z = e^{ix}$, with respect to the weight $dF(x)$. This system is uniquely determined by the following conditions (for more details see Grenander and Szegö (1958), chapter 2):

(i) $\phi_n(z)$ is a polynomial of degree n in which the coefficient of z^n is real and positive;

(ii)
$$\int_{-\pi}^{\pi} \phi_n(z; F)\, \overline{\phi_m(z; F)}\, dF(x) = \delta_{nm}\,, \quad z = e^{ix}.$$

Note that $\phi_0(z; F) = c_0^{-1/2}$ where $c_0 = F(\pi) - F(-\pi)$, and the reciprocal polynomials are $\phi_n^*(z; F) = z^n \,\overline{\phi}_n(z^{-1}; F)$.

We form now the kernel polynomials $s_n(\zeta, z; F)$ defined for complex variables ζ and z as follows:

(3.2.1)
$$s_n(\zeta, z; F) = \sum_{k=0}^{n} \overline{\phi_k(\zeta; F)}\, \phi_k(z; F).$$

They possess the following properties, proved in section 4.1 below.

Proposition 3.2.1. (i) *The kernel polynomials defined by* (1.3.2) *and* (3.2.1) *coincide.*

(ii) *The kernel polynomials* $s_n(\zeta, z; F)$ *are the only polynomials possessing the following reproducing property: for any polynomial* $p_n(z)$ *of degree n and each fixed complex number* ζ

(3.2.2)
$$\int_{-\pi}^{\pi} s_n(\zeta, z; F)\, \overline{p_n(z)}\, dF(x) = \overline{p_n(\zeta)}, \quad z = e^{ix}.$$

Remark 3.2.2. In the special case of the autoregression treated in section 2, when the density of F(x) has the form (2.1.1), the orthonormal polynomials are easily constructed, at least for $n \ge q$. We have

(3.2.3) $\qquad \sigma\, \phi_{n+q}(z; F) = z^n \beta_q^*(z)$ and $\sigma\, \phi_{n+q}^*(z; F) = \beta_q(z), \ n \ge 0,$

as the properties (i) and (ii), mentioned in the beginning of the present section, are easily checked (alternatively, check the relationship (4.1.1) below by using (2.1.2)-(2.1.6)). We make use of formulas (3.2.3) in section 4 where the results asserted in section 2 are proved, concerning the autoregressive case.

Remark 3.2.3. In view of proposition 3.2.1(i), the orthonormal polynomials $\phi_k(z; F)$, $0 \le k \le n$ determine not only the kernel polynomial $s_n(\zeta, z; F)$ and hence the inverse $\mathbb{T}_n(F)^{-1}$, but also the *square root* $T_n(F)$ of $\mathbb{T}_n(F)^{-1}$, that is uniquely defined as the lower triangular matrix with real and positive entries along the main diagonal (equal to $\phi_k^*(0; F)$, $0 \le k \le n$) such that

(3.2.4)
$$\mathbb{T}_n(F)^{-1} = T_n(F)^*\, T_n(F).$$

(Thus $D_n(F)^{-1} = \phi_0^*(0; F)^2 \ldots \phi_n^*(0; F)^2$; cf. the second formula in (4.1.1) below). Indeed, according to (1.3.2), (3.2.1) and (3.2.4),

(3.2.5)
$$T_n(F)\, z_n = \mathrm{col}\,\{\phi_k(z; F),\ 0 \le k \le n\}.$$

So, the entry in $T_n(F)$ at the jth row and kth column is $\frac{1}{k!} \frac{\partial^k}{\partial z^k} \phi_j(z; F)|_{z=0}$.

3.3. Turning back to the case of our main interest, described in section 1.1 and treated in section 3.1, we may now make use of (3.2.1) to rewrite (3.1.1) in the following form:

$$S_n(\zeta, z; F_0) = \sum_{k=0}^{n} \Phi_k(\bar{\zeta}; F_0)\, \phi_k(z; F_0) \text{ and } S_n(\zeta, \zeta; F_0) = \sum_{k=0}^{n} \Phi_k(\bar{\zeta}; F_0)\, \Phi_k(\bar{\zeta}; F_0)^*$$

where the q dimensional column vector $\Phi_n(\bar{\zeta}; F_0)$ is associated with $\phi_n(z; F_0)$ as follows:

(3.3.1) $$\Phi_n(\bar{\zeta}; F_0) = \text{col}\,\{\nabla_{k\bar{\zeta}}\, \overline{\phi_n(\zeta; F_0)}|_{\zeta=\zeta_k},\ 1 \le k \le s\}.$$

According to proposition 3.2.1(ii), for any polynomial $p_n(z)$ of degree n with the similarly associated r dimensional column vector $P_n(\bar{\zeta})$, we have then

(3.3.2) $$\int_{-\pi}^{\pi} S_n(\zeta, z; F_0)\, \overline{p_n(z)}\, dF_0(x) = P_n(\bar{\zeta}),\quad z = e^{ix}.$$

These observations yield

Lemma 3.3.1. *For* $n \ge r$ *let* $\varphi_n(z)$ *be polynomials of degree* n *defined as follows:*

(3.3.3) $$\varphi_n(z) = \phi_n(z; F_0) - \Phi_n(\bar{\zeta}; F_0)^*\, S_n(\zeta, \zeta; F_0)^{-1}\, S_n(\zeta, z; F_0).$$

Then

(i) *The set of the roots of (3.3.3) includes all roots of the polynomial* $\alpha_r^*(z)$.

(ii) $$\frac{\varphi_n^*(0)}{\phi_n^*(0, F_0)} = \frac{|S_{n-1}(\zeta, \zeta; F_0)|}{|S_n(\zeta, \zeta; F_0)|}.$$

(iii) *The system of polynomials* $\{\varphi_n(z)\}_{n=r,r+1,\dots}$ *is orthogonal on the unit circle* $|z| = 1$ *relative to the weight* $dF_0(x)$, *with* ($n \ge r$ and $\|\varphi_n\|_0^2 = \int_{-\pi}^{\pi} |\varphi_n(z)|^2\, dF_0(x)$, $z = e^{ix}$)

(3.3.4) $$\|\varphi_n\|_0^2 = \frac{\varphi_n^*(0)}{\phi_n^*(0, F_0)} = \frac{|S_{n-1}(\zeta, \zeta; F_0)|}{|S_n(\zeta, \zeta; F_0)|}.$$

3.4. The following theorem is in fact a direct consequence of lemma 3.3.1.

Theorem 3.4.1. *Under the circumstances described in section 1.1 the following assertions are true.*

(i) *The system of polynomials* $\{\psi_n(z; F)\}_{n=0,1,\dots}$ *is orthogonal on the unit circle* $|z| = 1$, $z = e^{ix}$, *with respect to the weight* $dF(x)$, *where* $\psi_n(z; F) = \varphi_{n+r}(z) / \alpha_r^*(z)$ *are polynomials of degree* n *in which the coefficient of* z^n *is real and positive, namely*

$$\psi_n^*(0; F) = \varphi_{n+r}^*(0) = \phi_{n+r}^*(0; F_0)\, \|\varphi_{n+r}\|_0^2 \text{ and } |\alpha_r|^2\, \|\psi_n\|^2 = \|\varphi_{n+r}\|_0^2$$

(here $\|\psi_n\|^2 = \int_{-\pi}^{\pi} |\psi_n(z; F)|^2 \, dF(x)$, $z = e^{ix}$; cf. (3.3.4)).

(ii) *The kernel polynomials* $s_n(\zeta, z; F)$, *associated with a distribution function* $F(x)$ *of type (1.1.1), can be presented in the following form, alternative to (3.1.2) and (3.1.2*):*

$$(3.4.1) \qquad s_n(\zeta, z; F) = \sum_{k=0}^{n} \frac{\overline{\psi_k(\zeta; F)}\, \psi_k(z; F)}{\|\psi_k\|^2} = \sum_{k=r}^{n+r} \frac{\overline{\varphi_k(\zeta)}}{\overline{\alpha_r(\zeta)}} \frac{\varphi_k(z)}{\alpha_r(z)} \frac{|S_k(\zeta, \zeta; F_0)|}{|S_{k-1}(\zeta, \zeta; F_0)|}.$$

(iii) *In particular*

$$(3.4.2) \qquad s_n(0, 0; F) = s_{n+r}(0, 0; F_0) \frac{|S_{n+r-1}(\zeta, \zeta; F_0)|}{|S_{n+r}(\zeta, \zeta; F_0)|}.$$

Remark 3.4.2. In view of the second identity in (4.1.1) below, the relationship (3.4.2) is equivalent to

$$(3.4.3) \qquad \frac{D_{n-1}(F)}{D_n(F)} = \frac{D_{n+r-1}(F_0)}{D_{n+r}(F_0)} \frac{|S_{n+r-1}(\zeta, \zeta; F_0)|}{|S_{n+r}(\zeta, \zeta; F_0)|}, \quad n > 0.$$

The last relationship is directly seen also from (3.1.3).

4. Proofs: autoregression

4.1. This section 4 will be devoted to proving the assertions in section 2, except the present subsection 4.1 in which the assertions of proposition 3.2.1 will be proved concerning general properties of the kernel polynomials.

Proof of proposition 3.2.1. We prove first (3.2.2) for $s_n(\zeta, z; F)$ defined by (1.3.2). It suffices to verify (3.2.2) for every z^k, $k = 0, \ldots, n$. Indeed

$$\int_{-\pi}^{\pi} s_n(\zeta, z; F)\, \bar{z}^k \, dF(x) = \zeta_n^* \, \mathbb{T}_n(F)^{-1} \int_{-\pi}^{\pi} \bar{z}^k z_n \, dF(x)$$

$$= \zeta_n^* \, \mathbb{T}_n(F)^{-1} \, \mathbb{T}_n(F)\, e_k = \bar{\zeta}^k, \quad z = e^{ix},$$

where e_k is the k^{th} coordinate column vector of dimension $n+1$.

Obviously, $s_n(\zeta, z; F)$ is the only polynomial of degree n in z satisfying condition (3.2.2), so that statement (i) is true since $s_n(\zeta, z; F)$ defined by (3.2.1) also satisfies (3.2.2): this is easily seen by presenting an arbitrary polynomial $p_n(z)$ of degree n as a sum $v_0 \phi_0(z; F) + \ldots + v_n \phi_n(z; F)$ and then taking the integral in (3.2.2) by using the orthonormality of the system of polynomials $\{\phi_n(z; F)\}_{n=0,1,\ldots}$. ◊

Remark 4.1.1. Notice the following identity (see Grenander and Szegö (1958), sections 2.2 and 2.3):

$$s_n(\zeta, z; F) = (\bar{\zeta}z)^n s_n(\bar{z}^{-1}, \bar{\zeta}^{-1}; F) = \sum_{k=0}^{n} (\bar{\zeta}z)^{n-k} \overline{\phi_k^*(\zeta; F)} \phi_k^*(z; F),$$

which yields

(4.1.1) $s_n(0, z; F) = \phi_n^*(0; F) \phi_n^*(z; F)$, $s_n(0, 0; F) = \phi_n^*(0; F)^2 = D_{n-1}(F) / D_n(F)$
where $\phi_n^*(0; F)$, the coefficient of z^n in the polynomial $\phi_n(z; F)$, is real by assumption.

4.2. *Proof of theorem 2.1.1.* (i) By definition (3.2.1)

$$s_{n+q}(\zeta, z; F) = s_{q-1}(\zeta, z; F) + \sum_{k=0}^{n} \overline{\phi_{k+q}(\zeta; F)} \phi_{k+q}(z; F).$$

Hence (2.1.3) follows from (2.1.2) and the first of relations (3.2.3). To verify (2.1.4), use the identity

(4.2.1) $$(1 - \bar{\zeta}z) s_n(\zeta, z; F) = \overline{\phi_{n+1}^*(\zeta; F)} \phi_{n+1}^*(z; F) - \overline{\phi_{n+1}(\zeta; F)} \phi_{n+1}(z; F);$$

see Grenander and Szegö (1958), section 2.3(b)), which for $n = q - 1$ yields (2.1.4) due to (3.2.3) for $n = 0$.

(ii) By using the representation (2.2.1) for the polynomial $\beta_q(z)$, we shall prove here the relation (2.2.4) which yields the desired assertion, provided formula (2.3.3) is indeed true (see section 4.3 for the proof of the last formula). Since

$$\beta_q'(z) / \beta_q(z) = -\sum_{k=1}^{m} q_k \bar{\xi}_k / (1 - \bar{\xi}_k z),$$

we get (2.2.4) by calculating the integral on the right hand side of the equation

$$\frac{1}{\pi}\iint |\beta_q'(z) / \beta_q(z)|^2 \, ds = \sum_{k,j=1}^{m} q_k q_j \, \bar{\xi}_j \xi_k \frac{1}{\pi}\iint (1 - \bar{\xi}_j z)^{-1} (1 - \xi_k \bar{z})^{-1} \, ds.$$

In fact

(4.2.2) $$\frac{1}{\pi}\iint (1 - \bar{\xi}_j z)^{-1} (1 - \xi_k \bar{z})^{-1} \, ds = \frac{1}{\pi}\int_0^1 r \, dr \int_{-\pi}^{\pi} (1 - \bar{\xi}_j re^{ix})^{-1} (1 - \xi_k e^{-ix})^{-1} \, dx$$

$$= \int_0^1 r \, dr \frac{1}{\pi i} \int_{|z|=1} (1 - \bar{\xi}_j rz)^{-1} (z - \xi_k r)^{-1} \, dz = 2\int_0^1 (1 - \bar{\xi}_j \xi_k r^2)^{-1} r \, dr = -(\bar{\xi}_j \xi_k)^{-1} \ln (1 - \bar{\xi}_j \xi_k);$$

cf. Grenander and Szegö (1958), section 5.5(d). ◊

Remark 4.2.1. Alternatively, (2.1.3) - (2.1.4) can be derived directly from (4.2.1). Indeed, by (4.2.1) and the second of the relations (3.2.3)

$$s_{n+q-1}(\zeta, z; F) = [\overline{\phi_{n+q}^*(\zeta; F)} \phi_{n+q}^*(z; F) - \overline{\phi_{n+q}(\zeta; F)} \phi_{n+q}(z; F)] / (1 - \bar{\zeta}z)$$

$$= [\overline{\beta_q(\zeta)} \beta_q(z) - \overline{\beta_q^*(\zeta)} \beta_q^*(z) (\bar{\zeta}z)^n] / \sigma^2 (1 - \bar{\zeta}z),$$

so that it suffices to take into consideration (2.1.2) and (2.1.4).

4.3. *Proof of corollary 2.3.1.* (i) By (1.3.1) and (2.3.4)

$$(4.3.1) \qquad B\,\mathbb{T}_{q-1}(F)\,B^* = \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} \nabla(\xi, z)\,\nabla(\xi, z)^*\,dx = \sigma^2\,\nabla(\xi, \xi), \quad z = e^{ix}$$

(the integral is taken in the same way as in (4.2.3); cf. Grenander and Szegö (1958), section 5.5(c)). Since B is non-singular, (4.3.1) is equivalent to (2.3.2), which in turn is equivalent to (2.3.5) by definition (1.3.2), as B satisfies (2.3.4). To prove (2.3.1), it remains then to verify the first equality in (2.3.6), that is the relationship

$$\overline{\beta_q{}^*(\zeta)}\,\beta_q{}^*(z)\sum_{k=0}^{n}(\bar{\zeta}z)^k = \sum_{j,k=0}^{n+q}\bar{\zeta}^j z^k \sum_{\ell=0}^{n}\beta_{\ell+q-j}\,\bar{\beta}_{\ell+q-k}.$$

Changing the summation order on the right hand side we indeed get the right hand side by the following considerations (use the definition of the reciprocal polynomials):

$$\sum_{\ell=0}^{n}\sum_{j=0}^{n+q}\beta_{\ell+q-j}\,\bar{\zeta}^j\sum_{k=0}^{n+q}\bar{\beta}_{\ell+q-k}\,z^k = \sum_{\ell=0}^{n}\sum_{j=\ell-n}^{\ell+q}\beta_j\,\bar{\zeta}^{\ell+q-j}\sum_{k=\ell-n}^{\ell+q}\bar{\beta}_k\,z^{\ell+q-k}$$

$$= \sum_{\ell=0}^{n}(\bar{\zeta}z)^{\ell+q}\sum_{j=\ell-n}^{\ell+q}\beta_j\,\bar{\zeta}^{-j}\sum_{k=\ell-n}^{\ell+q}\bar{\beta}_k\,z^{-k} = \sum_{\ell=0}^{n}(\bar{\zeta}z)^{\ell+q}\sum_{j=0}^{q}\beta_j\,\bar{\zeta}^{-j}\sum_{k=0}^{q}\bar{\beta}_k\,z^{-k}.$$

(ii) According to Grenander and Szegö (1958), section 5.3(d), in the present special case $D_n(F) / D_n(F_0) = D_{q-1}(F) / D_{q-1}(F_0)$ for $n \geq q$. But by (4.3.1) we also have $D_{q-1}(F) / D_{q-1}(F_0) = \det \nabla(\xi, \xi) / |\det B|^2$. Hence the desired assertion follows from these two identities and lemma 4.3.1 below. ◊

Lemma 4.3.1.

(i) *The transformation* (2.3.4) *is non-singular, and* $|B| = \prod_{k>j}(\bar{\xi}_k - \bar{\xi}_j)^{q_k q_j}$.

(ii) $$|\nabla(\xi, \xi)| = \prod_{k>j}|\xi_k - \xi_j|^{2q_k q_j}\prod_{1\leq j,k\leq m}|1 - \bar{\xi}_j\xi_k|^{-q_k q_j}.$$

Proof. (i) Let $V(\beta_q) = [V_1(\beta_q), \ldots, V_m(\beta_q)]$ be the q dimensional Vandermonde matrix, associated with the polynomial $\beta_q(z)$. Its determinant is

$$(4.3.2) \qquad |V(\beta_q)| = \prod_{k>j}|\bar{\xi}_k^{-1} - \bar{\xi}_j^{-1}|^{q_k q_j};$$

cf. remark 3.1.3. Due to (2.2.2), (2.3.4) and (3.1.5), it is easily verified that $B_k V_j(\beta_q) = 0$ if $k \neq j$ and $B_j V_j(\beta_q)$ are upper triangular matrices with entries along the main diagonal so that

$$(4.3.3)\qquad |BV(\beta_q)| = |B|\,|V(\beta_q)| = \prod_{k\neq j}(1 - \bar{\xi}_k/\bar{\xi}_j)^{q_k q_j}.$$

The desired assertion follows from (4.3.2) and (4.3.3).
(ii) See Grenander and Szegö (1958), section 5.5(c), formula (16), for the case of simple roots; the general case is treated analogously. ◊

5. Proofs of the assertions in section 3

5.1. In this section 5 the proofs are provided of the assertions in section 3, except proposition 3.2.1 which is already proved in section 4.1. It seems then natural to begin with varifying the assertions of corollary 3.3.1 to this proposition, and to continue with deducing lemma 3.3.2. This lemma will be used in section 5.4 for proving the results asserted in section 3.4, which in turn will imply assertion (i) of theorem 3.1.1; see section 5.5. We will use the following abridged notations:

(5.1.1) $S_n = S_n(\zeta, \zeta; F_0)$, $S_n(z) = S_n(\zeta, z; F_0)$ and $\Phi_n = \Phi_n(\bar{\zeta}; F_0)$, $\phi_n(z) = \phi_n(z; F_0)$.

5.2. *Proof of lemma 3.3.1.* (i) It suffices to verify col $\{\nabla_{k\bar{z}}\, \overline{\varphi_n(z)},\ 1 \le k \le s\} = 0$ for $\varphi_n(z)$ defined in (3.3.3) by taking into consideration (3.1.1) and (3.3.1).
(ii) Since

$$(5.2.1)\qquad \varphi_n(z) = \phi_n(z) - \sum_{k=0}^{n} m_{nk}\, \phi_k(z) \ \text{ and } \ m_{nk} = \Phi_n^*\, S_n^{-1}\, \Phi_k,$$

we have

$$\varphi_n^*(z) = \phi_n^*(z) - z^n \sum_{k=0}^{n} m_{kn}\, \bar{\phi}_k(z^{-1}) \ \text{ and } \ \varphi_n^*(0) = \phi_n^*(0)\,(1 - m_{nn}).$$

Hence, by definition (3.3.3) the desired assertion is reduced to the identity

$$(5.2.2)\qquad 1 - m_{nn} = 1 - \Phi_n^*\, S_n^{-1}\, \Phi_n = |S_{n-1}| \,/\, |S_n|$$

which is true due to (1.5.1), since $S_n - S_{n-1} = \Phi_n\, \Phi_n^*$. Observe, by the way, that due to the second identity (5.4.1) below

$$(5.2.3)\qquad (1 - m_{nn})^{-1} = (1 - \Phi_n^*\, S_n^{-1}\, \Phi_n)^{-1} = 1 + \Phi_n^*\, S_{n-1}^{-1}\, \Phi_n.$$

(iii) It is easily seen that for $n,m \ge r$ we indeed have

$$\int_{-\pi}^{\pi} \varphi_n(z)\, \overline{\varphi_m(z)}\; dF_0(x) = \delta_{nm}\,(1 - \Phi_n^*\, S_n^{-1}\, \Phi_n),\quad z = e^{ix},$$

since by the reproducing property (3.3.2) of S_n

$$\int_{-\pi}^{\pi} S_n(z)\, S_m(z)^*\; dF_0(x) = S_m \ \text{ if } \ m \le n,\quad z = e^{ix}$$

and

$$\int_{-\pi}^{\pi} S_n(z)\,\overline{\phi_m(z)}\, dF_0(x) = \Phi_m \text{ if } m \le n,$$

$$= 0 \quad \text{if } m > n,\ z = e^{ix}.\ \Diamond$$

5.3. *Proof of theorem 3.4.1.* (i) The desired properties of the system of polynomials $\{\psi_n(z; F)\}_{n=0,1,...}$ is easily verified by taking into consideration the corresponding properties of the polynomials $\varphi_n(z)$, proved in section 5.2.
(ii) Since the system of polynomials $\{\psi_n(z; F)\}_{n=0,1,...}$ is orthogonal on the unit circle $|z| = 1$, $z = e^{ix}$, relative to the weight $dF(x)$, with the known norms given in assertion (i), we can easily construct the corresponding system of orthonormal polynomials in orther to confirm (3.4.1) by using definition (3.2.1).
(iii) See remark 3.4.2. ◊

5.4. *Proof of theorem 3.3.1.* (i) With a sequence $\{x_n\}_{n=0,1,...}$ we associate the sequence of differences defined as usual : $\Delta x_0 = x_0$, $\Delta x_n = x_n - x_{n-1}$, $n > 0$. We will need below the following identities:
(5.4.1) $\Delta(x_n y_n) = \Delta x_n y_{n-1} + x_n \Delta y_n$ and $\Delta x_n^{-1} = - x_{n-1}^{-1} \Delta x_n x_n^{-1}$,
assuming the required invertibility, of course.

It will be proved below that the expression (3.1.2) for the kernel polynomials $s_n(\zeta, z; F)$ (or (3.1.2*), equivalently) coincides with (3.4.1), which obviously will imply the desired assertion. To this end, it suffices then to show that for $n \ge r$

(5.4.2) $$\Delta\{s_{n-r}(\zeta, z) - S_{n-r}(\zeta)^* S_{n-r}^{-1} S_{n-r}(z)\} = \overline{\varphi_n(\zeta)}\, \varphi_n(z)\, |S_n| / |S_{n-1}|$$
$$= \overline{\varphi_n(\zeta)}\, \varphi_n(z)\, (1 + \Phi_n^* S_{n-1}^{-1} \Phi_n)$$

(cf. (5.2.2) and (5.2.3); again, the notations (5.1.1) are used, as well as the similar abbreviation $s_n(\zeta, z) = s_n(\zeta, z; F_0)$). Use (5.4.1) to verify that the expresion on the left hand side of (5.4.2) equals to

$$\Delta s_n(\zeta, z) - S_{n-1}(\zeta)^* S_{n-1}^{-1} \Delta S_n(z) - \Delta S_n(\zeta)^* S_n^{-1} S_n(z) - S_n(\zeta)^* \Delta S_n^{-1} S_n(z).$$

Due to (3.2.1) and the versions in section 3.3 of $S_n(z)$ and S_n, the last expression equals to

$$\{\phi_n(\zeta) - \Phi_n^* S_{n-1}^{-1} S_{n-1}(\zeta)\}^* \{\phi_n(z) - \Phi_n^* S_n^{-1} S_n(z)\},$$

which indeed is easily reduced to the expression on the right hand side of (5.4.2) by using definition (3.3.3).
(ii) See section 6.5 and remark 5.4.1 below. ◊

Remark 5.4.1. Concerning assertion (ii) of theorem 3.3.1, observe that in view of (3.4.3) it suffices to prove (3.1.3) for $n = 0$, say. It seems instructive, however, to provide the direct proof based on the algebraic considerations of the next section 6; cf. Dzhaparidze (1986), section I.1.2.

6. Inverse of $\mathbb{T}_n(F)$ with F of type (1.1.1)

6.1. In order to shed more light upon the results presented in section 3, we utilize in this section simple algebraic arguments (see, e.g. Lancaster and Tismenetsky (1985) for more details) leading to the formulas in theorem 6.4.1 below for the inverse of the Toeplitz matrix $\mathbb{T}_n(F)$ and its root $T_n(F)$, defined by (3.2.4); cf. Dzhaparidze (1986), section 1.1. For simplicity we will retain the abbreviations (5.5.1), writing also $\mathbb{T}_n$, T_n and D_n instead of $\mathbb{T}_n(F_0)$, $T_n(F_0)$ and $D_n(F_0)$ respectively. Besides, we denote

(6.1.1) $\tau_n(z) = \text{col}\,\{\phi_k(z), 0 \le k \le n\}$ and $W_n = \text{col}\,\{\Phi_k^*, 0 \le k \le n\}$

which are a n+1 dimensional vector and an n+1×r matrix, respectively. According to (3.2.5) and (3.3.1), we have

(6.1.2) $$\tau_n(z) = T_n\, z_n \text{ and } W_n = T_n\, \mathbb{V}_n$$

where $\mathbb{V}_n$ is a n+1×r matrix constructed similarly to the Vandermonde matrix $V(\alpha_r^*)$ introduced in theorem 3.1.1(ii), with the only difference that the column vector z_{r-1} in (3.1.5) is substituted by z_n. Of course $\mathbb{V}_{r-1} = V(\alpha_r^*)$. Then, along with the two equivalent presentations of $S_n(z)$ and S_n in sections 3.1 and 3.3, we also have

(6.1.3) $S_n(z) = \mathbb{V}_n^*\, \mathbb{T}_n^{-1}\, z_n = W_n^*\, \tau_n(z)$ and $S_n = \mathbb{V}_n^*\, \mathbb{T}_n^{-1}\, \mathbb{V}_n = W_n^*\, W_n$.

6.2. In these notations the representation (3.1.2*) can be rewritten in the following manner:

(6.2.1) $$\overline{\alpha_r^*(\zeta)}\, \alpha_r^*(z)\, s_n(\zeta, z; F) = \tau_{n+r}(\zeta)^*\, P_{n+r}\, \tau_{n+r}(z) = \zeta_{n+r}^*\, \Pi_{n+r}\, z_{n+r}$$

where $P_n = I_n - W_n\,(W_n^*\, W_n)^{-1}\, W_n^*$ and $\Pi_n = T_n^*\, P_n\, T_n$, hence

$$\Pi_n = \mathbb{T}_n^{-1} - \mathbb{T}_n^{-1}\, \mathbb{V}_n\, (\mathbb{V}_n^*\, \mathbb{T}_n^{-1}\, \mathbb{V}_n)^{-1}\, \mathbb{V}_n^*\, \mathbb{T}_n^{-1};$$

cf. remark 6.2.1 below. On the other hand, the representation (3.4.1) can also be rewritten in a matrix form. In fact, consider the n+r+1 dimensional lower triangular matrix with the non-zero entry $\delta_{jk} - m_{jk}$ at the jth row and kth column, where $j \ge k$; see (5.2.1) for definition of m_{jk}. Denote by M its lower n+1×n+r+1 submatrix

$$M = \begin{bmatrix} -m_{r0} & \dots\ 1-m_{rr} & \\ \dots & \dots & \\ -m_{n+r,0} & \dots\ -m_{n+r,r} & \dots\ 1-m_{n+r,n+r} \end{bmatrix}.$$

Let d_n be the n+1 dimensional diagonal matrix

$$d_n = \text{diag}\,\{|S_{k-1}|\,/\,|S_k|, r \le k \le n+r\} = \text{diag}\,\{1 - m_{jk}, r \le k \le n+r\};$$

cf. (5.2.2). Then

(6.2.2) $$\overline{\alpha_r^*(\zeta)}\, \alpha_r^*(z)\, s_n(\zeta, z; F) = \tau_{n+r}(\zeta)^*\, M^*\, d_n^{-1}\, M\, \tau_{n+r}(z),$$

since by definitions (5.2.1) and (6.1.1)

$$\mathrm{col}\,\{\varphi_k(z),\, r \le k \le n+r\} = M\, \tau_{n+r}(z).$$

As will be shown in the next remark

(6.2.3) $$P_{n+r} = M^*\, d_n^{-1}\, M,$$

that confirms the equivalence of the representations (6.2.1) and (6.2.2), stated in theorem 3.3.1 and proved in section 5.4.

Remark 6.2.1. The matrix P_n in (6.2.1) is an *orthogonal projector*, since it is Hermitian and indepotent, and Π_n (= $\Pi_n\, \mathbb{T}_n\, \Pi_n$) is Hermitian matrix of dimension n+1 whose columns belong to the *kernel* of $\mathbb{V}_n$ as $\Pi_n\, \mathbb{V}_n = 0$. As for the matrix M of full rank (= n+1), involved in (6.2.2), it consists of mutually orthogonal rows (this is easily seen by using the definitions of $S_n(z)$, S_n and m_{jk}; cf. (5.2.1) and (6.1.3)) which span the kernel of W_n defined in (6.1.1), since $M\, M^* = d_n$ and $M\, W_n = 0$. These properties of M yield the representation (6.2.3).

6.3. The n+r+1xr matrix $\mathbb{V}_{n+r}$, defined in section 6.1, is of full rank (= r), since if it is subdivided into two blocks V_1 and V_2 so that $\mathbb{V}_{n+r} = \mathrm{col}\,\{V_1, V_2\}$ and V_1 is of dimension r, then $V_1 = V(\alpha_r^*)$, that is the Vandermonde matrix introduced above. It will be shown below (see (6.3.4)) that the columns of $\mathbb{V}_{n+r}$ span the *kernel* of the transformation of the n+r+1 dimensional column vector z_{n+r} into the n+1 dimensional column vector $\alpha_r^*(z)\, z_n$, defined by

(6.3.1) $$C^*\, z_{n+r} = \alpha_r^*(z)\, z_n$$

where C is a n+r+1xn+1 matrix whose entry at the j^{th} row and k^{th} column is α_{r+k-j} (to see this apply the differential operator $(\partial^k/\partial z^k)\, /\, k!$, $0 \le k \le n+r$ to the both sides of (6.3.1) and evaluate the result at z = 0. For convenience, it is assumed that $\alpha_k = 0$ for $k < 0$ or $k > r$; as usual α_k for $0 \le k \le r$ are the coefficients in the polynomial $\alpha_r(z)$). Notice that due to (1.1.1), (1.3.1) and (6.3.1)

(6.3.2) $$\mathbb{T}_n(F) = C^*\, \mathbb{T}_{n+r}(F_0)\, C.$$

The matrix C is of full rank (= n+1), and if $C = \mathrm{col}\,\{C_1, C_2\}$ where C_2 is a matrix of dimension n+1, then C_2^* is lower triangular with the entries 1 along the main diagonal, i.e. $|C_2| = 1$. In corollary 6.4.1 below we will use the following notation

(6.3.3) $$\Sigma = \mathrm{col}\,\{0, C_2^{*-1}\},$$

that is a n+r+1xn+1 matrix (for more details see remark 6.3.1 below). To see that the columns of $\mathbb{V}_{n+r}$ span the *kernel* of the transformation (6.3.1), note that $\alpha_r^*(z)$ is in the same relation to its roots as $\beta_q^*(z)$ in (2.2.2), and therefore (6.3.1) yields

(6.3.4) $$C^*\, \mathbb{V}_{n+r} = C_1^*\, V_1 + C_2^*\, V_2 = 0.$$

Remark 6.3.1.The submatrix C_2^* of the matrix C^*, determining the transformation (6.3.1), may be expressed in terms of the roots $z_1, \ldots, z_r$ of the reciprocal polynomial

$\alpha_r^*(z)$ as the product $C_2^* = C(z_1)...C(z_r)$ where $C(z)$ is a lower triangular matrix of dimension n+1, with the non-zero entry at the j^{th} row and k^{th} column equal to 1 or -z when $j = k$ or $j - k = 1$, respectively (so far we always have indicated the multiplicity of the roots, but this is irrelevant here as the considered matrix product is commutative). Due to this representation, the inverse of C_2^* (and hence the $n+1 \times n+r+1$ matrix Σ defined by (6.3.3)) is easily calculated, since $C(z)^{-1}$ is also a lower triangular matrix, with the non-zero entry at the j^{th} row and k^{th} column equal to z^{j-k} when $j \geq k$. For any column vector $X = \text{col}\,\{X_0, ..., X_n\}$, namely, the k_0th entry ($0 \leq k_0 \leq n$) in the column $C_2^{*-1} X$ is

$$\sum_{k_0 \geq ... \geq k_r \geq 0} z_1^{k_0 - k_1} ... z_r^{k_{r-1} - k_r} X_{k_r}.$$

6.4. Using the notations introduced above we may formulate the following statement.

Theorem 6.4.1. *Under the conditions of section 1.1*

(6.4.1) $$\mathbb{T}_n(F)^{-1} = \Sigma^* \Pi_{n+r} \Sigma = (M\, T_{n+r}\, \Sigma)^*\, d_n^{-1}\, M\, T_{n+r}\, \Sigma,$$

therefore

(6.4.2) $$T_n(F) = d_n^{-1/2}\, M\, T_{n+r}\, \Sigma.$$

Proof. By the first identity in (6.1.2), the second equality in (6.4.1) is a consequence of (6.2.1) and (6.2.2). It suffices, therefore, to prove the first equality in (6.4.1), since (6.4.1) will then imply (6.4.2). As was noted in remark 6.2.1, $\Pi_{n+r}\, \mathbb{V}_{n+r} = 0$. Thus by definition (1.3.2), the first equality in (6.4.1) is deduced from (6.2.1), provided the following identity is true:

(6.4.2) $$z_{n+r} = \alpha_r^*(z)\, \Sigma\, z_n + \mathbb{V}_{n+r}^*\, V_1^{-1}\, z_{r-1},$$

To complete the proof, use (6.3.3) and (6.3.4) to rewrite (6.4.2) in the following equivalent form

$$z^r z_n = \alpha_r^*(z)\, C_2^{*-1} z_n - C_2^{*-1} C_1^* z_{r-1} \text{ or } C_1^* z_{r-1} + z^r C_2^* z_n = \alpha_r^*(z)\, z_n,$$

and then verify that the last identity is in turn equivalent to (6.3.1). ◊

6.5. *Proof of theorem 3.3.1(ii).* By (6.3.4)

$$Q = [-V_2\, V_1^{-1}, I_n] = [(C_1\, C_2^{-1})^*, I_n] = (C\, C_2^{-1})^*$$

which, due to (6.3.2), yields

$$Q\, \mathbb{T}_{n+r}\, Q^* = C_2^{-1*}\, \mathbb{T}_n(F)\, C_2^{-1}.$$

Hence (recall that $|C_2| = 1$)

(6.5.1) $$|\mathbb{T}_n(F)| = |Q\, \mathbb{T}_{n+r}\, Q^*|.$$

It will be shown now that

(6.5.2) $$|Q\, \mathbb{T}_{n+r}\, Q^*| = (-1)^r \begin{vmatrix} 0 & (\mathbb{V}_{n+r} V_1^{-1})^* \\ \mathbb{V}_{n+r} V_1^{-1} & \mathbb{T}_{n+r} \end{vmatrix}$$

which, in view of (6.5.1), yields the desired assertion, since by (1.5.1) and (6.1.3) the expression on the right hand side of (6.5.2) equals to

$$D_{n+r}\,|(\mathbb{V}_{n+r}\,V_1^{-1})^*\,\mathbb{T}_{n+r}\,\mathbb{V}_{n+r}\,V_1^{-1}| = D_{n+r}\,|V_1^{-1*}\,S_{n+r}\,V_1^{-1}|$$

and therefore coincides with the expression on the right hand side of (3.1.3). Observe first that according to (1.5.1) the multiplier $(-1)^r$ is in fact the determinant of the matrix

$$\begin{bmatrix} 0 & I_r \\ I_r & \mathbb{T}_{r-1} \end{bmatrix} = \begin{bmatrix} -\mathbb{T}_{r-1} & I_r \\ I_r & 0 \end{bmatrix}^{-1},$$

which appears at the upper right hand side corner when subdividing appropriately (after the 2rth row and the 2rth column) the matrix whose determinant is written on the left hand side of (6.5.2). Hence, we may apply again (1.5.1) to see that the right hand side of (6.5.2) equals to the determinant of the matrix

$$\mathbb{T}_n - \mathbb{V}_{n+r}\,V_1^{-1}\,\mathbb{T}_{n+r}\,(\mathbb{V}_{n+r}\,V_1^{-1})^* + \mathbb{V}_{n+r}\,V_1^{-1}\,T_{12} + T_{21}\,(\mathbb{V}_{n+r}\,V_1^{-1})^*,$$

which easily can be rearanged in $Q\,\mathbb{T}_{n+r}\,Q^*$, since T_{12} and T_{21} denote the upper right hand side corner submatrix and the lower left hand side submatrix, respectively, in the appropriate subdivision of $\mathbb{T}_{n+r}$. ◊

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