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Normal Forms in Real Time Process Algebra

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Abstract

This paper is based on the extension of Basic Process Algebra with real time that has been defined in [Klu91]. The theory of this extension, called BPA$\rho\delta$I, contains an axiom that can only be verified by an uncountable number of checkings. So at first sight equality between process terms is undecidable. In this paper an explicit construction is given for reducing process terms to a normal form. Furthermore, we prove that if BPA$\rho\delta$I $\vdash p = q$, then p and q have the same normal form. Thus it is decidable for two process terms if they are equal or not.

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1 Introduction

In recent years many process algebras have been extended with a notion of time. For instance, [MT90] and [Wan90] contain an extension of CCS, [Gro90] and [BB91a] for ACP, [RR88] for CSP and [NS90] for ATP, which is a combination of CCS and ACP.

Baeten and Bergstra have extended BPAδ with *real* time in [BB91a] and introduced the notion of *integration*, which describes that an action can be executed within an interval of time. Integration enables an uncountable number of substitutions of real numbers for a time variable. Klusener has modified their syntax and semantics of BPA$\rho\delta$I (BPAδ with real time and integration) in [Klu91]. He defined the notion of *prefixed* integration, which requires that every action has as time stamp some variable and is directly preceded by an integral binding this variable. Furthermore, integration is only allowed over an interval of which the bounds are linear expressions of variables. In this paper we use the syntax and semantics of [Klu91].

BPA$\rho\delta$I concerns *time-closed* process terms, i.e. process terms for which all occurring variables are bound by integrals. The theory of BPA$\rho\delta$I contains axioms that can only be verified by an uncountable number of checkings. Thus equality between process terms seems to be undecidable.

In this paper we will give an explicit construction for reducing time-closed process terms. Via intermediate terms, the so-called *conditional* terms, a time-closed process term is rewritten to a normal form, which is again a time-closed process term. We will prove this normal

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form to be unique modulo commutativity and associativity of the $+$. Furthermore, if the equality $p = q$ holds in the axiom system, then p and q have the same normal form. Thus for each pair of process terms p, q it is decidable whether the equality $p = q$ holds or not.

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2 Real-time process algebra

This chapter contains a review of real-time process algebra, based on [Klu91].

2.1 Bounds and intervals

TVar denotes an infinite, countable set of *time variables*. Let $t \in [0, \infty]$, $r \in \langle 0, \infty \rangle$ and $v \in TVar$. The set *Bound* of *bounds*, with typical element b, is defined by

$$b \ ::= \ t \ | \ v \ | \ b + b \ | \ b \dot{-} b \ | \ r \cdot b$$

where $\dot{-}$ denotes the monus function, i.e. if $t_0 \leq t_1$ then $t_0 \dot{-} t_1 = 0$. In the sequel $\triangleleft$ and $\triangleright$ are elements of $\{\langle, [\}$ and $\{\rangle,]\}$ respectively. An interval V is of the form $\triangleleft b, c \triangleright$ with b, c bounds.

2.2 Process terms

We start with a set of *atomic actions* A and a *deadlock* δ (with $\delta \notin A$). Let $a \in A \cup \{\delta\}$, $v \in TVar$, V an interval and b a bound. The set $\mathcal{T}$ of *process terms*, with typical element p, is defined by

$$p \ ::= \ \int_{v \in V} a(v) \ | \ \int_{v \in V} (a(v) \cdot p) \ | \ p + p \ | \ p \cdot p \ | \ b \gg p$$

The $+$ denotes alternative composition, the $\cdot$ stands for sequential composition and $\int_{v \in V}$ is alternative composition over all elements of V. The *time shift* $\gg$ ensures that the execution of a process is started after a certain moment in time; the process $t \gg p$ can only execute those initial actions of p that happen after t.

An expression of the form $\int_{v \in [b,b]} a(v)$ may be abbreviated to $a(b)$. In the sequel we will use a scope convention, saying that if we do not write scope brackets, then the scope of an integral is as large as possible. Thus we write $\int_{v \in V} a(v) \cdot p$ for $\int_{v \in V} (a(v) \cdot p)$.

2.3 Free time variables

For b a bound, the set of time variables occurring in b is denoted by $tvar(b)$. Of course $tvar(\triangleleft b, c \triangleright) = tvar(b) \cup tvar(c)$. The collection $FV(p)$ of time variables appearing in a process term p that are not bound by an integral sign, the so-called *free* variables, is defined inductively by

$$\begin{array}{lcll} FV(\int_{v\in V} a(v)) & = & tvar(V) & \\ FV(\int_{v\in V} a(v)\cdot p) & = & tvar(V)\cup(FV(p)\backslash\{v\}) & \\ FV(p\Box q) & = & FV(p)\cup FV(q) & \Box\in\{+,\cdot\} \\ FV(b\gg p) & = & tvar(b)\cup FV(p) & \end{array}$$

A process term p is called *time-closed* if $FV(p)=\emptyset$. The set of time-closed process terms is denoted by $\mathcal{T}^{time-closed}$.

2.4 Ultimate delay

For $p\in\mathcal{T}^{time-closed}$ we define inductively its *ultimate delay* $U(p)$, which is the latest moment in time before which p can 'idle' without performing an action. Let $sup(\emptyset)=inf(\emptyset)=0$. Let P be of the form $a(v)$ or $a(v)\cdot p$.

$$\begin{array}{lcl} U(\int_{v\in V} P) & = & sup(V) \\ U(p+q) & = & max\{U(p),U(q)\} \\ U(p\cdot q) & = & U(p) \\ U(t\gg p) & = & max\{t,U(p)\} \end{array}$$

2.5 Depth

The *depth* of a term is the longest chain of sequential compositions in the term. It is defined inductively as follows, where $a\in A_\delta$.

$$\begin{array}{lcl} depth(\int_{v\in V} a(v)) & = & 1 \\ depth(\int_{v\in V} a(v)\cdot p) & = & depth(p)+1 \\ depth(X+Y) & = & max\{depth(X),depth(Y)\} \\ depth(X\cdot Y) & = & depth(X)+depth(Y) \\ depth(t\gg X) & = & depth(X) \end{array}$$

2.6 Substitutions

A *substitution* is a mapping of $TVar$ to $Bound$. For σ a substitution and b a bound, $\sigma(b)$ denotes the bound that results from substituting $\sigma(v)$ for each occurrence of v in b for all $v\in TVar$. Of course $\sigma(\lhd b_1,b_2\rhd)=\lhd\sigma(b_1),\sigma(b_2)\rhd$.

Substitutions are extended to process terms by defining five inductive rules. The first four rules are easy:

$$\begin{array}{c} \sigma(p+q)=\sigma(p)+\sigma(q) \\ \sigma(p\cdot q)=\sigma(p)\cdot\sigma(q) \\ \sigma(b\gg p)=\sigma(b)\gg\sigma(p) \\ \sigma(\int_{v\in V} a(v))=\int_{v\in\sigma(V)} a(v) \end{array}$$

The fifth rule, defining $\sigma(\int_{v\in V} a(v)\cdot p)$, is more complicated. First of all, the free occurrences of v in p are bound by the integral sign $\int_{v\in V}$. So these occurrences of v are not to be substituted by $\sigma(v)$. Hence, if σ_v denotes the substitution that is equal to σ on $TVar\backslash\{v\}$ while $\sigma_v(v) = v$, then

$$\sigma(\int_{v\in V} a(v)\cdot p) = \int_{v\in\sigma(V)} a(v)\cdot \sigma_v(p)$$

But there is a second problem; if $w \in FV(p)\backslash\{v\}$, then after substituting $\sigma(w)$ for w in p, all occurrences of v in $\sigma(w)$ are suddenly bound by $\int_{v\in V}$. So this definition of $\sigma(\int_{v\in V} a(v)\cdot p)$ is only valid if we have

$$\forall w \in FV(p)\backslash\{v\} \qquad v \notin tvar(\sigma(w))$$

If this requirement does not hold, then the expression $\sigma(\int_{v\in V} a(v)\cdot p)$ is undefined.

If there is only one $v \in TVar$ such that $\sigma(v) \neq v$, then $\sigma(p)$ can be denoted by $p[\sigma(v)/v]$.

2.7 α-conversion

Process terms are considered modulo α-conversion: if $w \in TVar$ does not occur in $p \in \mathcal{T}$, then

$$\begin{array}{lcl} \int_{v\in V} a(v) & \cong & \int_{w\in V} a(w) \\ \int_{v\in V} a(v)\cdot p & \cong & \int_{w\in V} a(w)\cdot p[w/v] \end{array}$$

We take the transitive closure of this relation.

Note that by applying α-conversion we can always ensure that for $p \in \mathcal{T}$ and σ a substitution, the expression $\sigma(p)$ is well-defined.

2.8 The theory and semantics of BPA$\rho\delta$I

In Table 1 an axiom system for time-closed process terms is given. Let $p, q \in \mathcal{T}$ with $FV(p), FV(p') \subset \{v\}$ and $X, Y \in \mathcal{T}^{time-closed}$. Furthermore, P is of the form $a(v)$ or $a(v)\cdot p$.

Note that axioms A3 and A5 from BPA, $X + X = X$ and $(X\cdot Y)\cdot Z = X\cdot(Y\cdot Z)$, have been excluded. This is because A3 can be deduced from INT1 and A5 from A4 and INT2.

Klusener has also given a Transition System Specification for $\mathcal{T}^{time-closed}$. We only give the action rules concerning integral and deadlock. For the other rules we refer to [Klu91].

$$\int_{v\in V} a(v) \xrightarrow{a(t)} \surd \qquad t \in V$$

$$\int_{v\in V} a(v)\cdot p \xrightarrow{a(t)} t \gg p[t/v] \qquad t \in V$$

Let $L(p)$ be the supremum of $t \in \langle 0, \infty\rangle$ for which there is an $a \in A$ with $p \xrightarrow{a(t)} p'$ or $p \xrightarrow{a(t)} \surd$. Then

$$\frac{L(p) < U(p)}{p \xrightarrow{\delta(U(p))} \surd}$$

Definition 2.1 *Two time-closed process terms p, q are called* bisimilar, *notation $p \leftrightarrow q$, if there exists a symmetric relation $\mathcal{R} \subset \mathcal{T}^{time-closed} \times \mathcal{T}^{time-closed}$ such that*

1. $\mathcal{R}(p, q)$.
2. *if $p_0 \xrightarrow{a(t)} p_1$ and $\mathcal{R}(p_0, q_0)$, then $q_0 \xrightarrow{a(t)} q_1$ for some $q_1 \in \mathcal{T}^{time-closed}$ with $\mathcal{R}(p_1, q_1)$.*

A1		$X+Y = Y+X$
A2		$(X+Y)+Z = X+(Y+Z)$
A4		$(X+Y)\cdot Z = X\cdot Z+Y\cdot Z$
INT1	$V=V_0\cup V_1$	$\int_{v\in V_0} P + \int_{v\in V_1} P = \int_{v\in V} P$
INT2		$\int_{v\in\emptyset} P = \delta$
INT3a		$\int_{v\in V}(a(v))\cdot Y = \int_{v\in V} a(v)\cdot Y$
INT3b		$\int_{v\in V}(a(v)\cdot p)\cdot Y = \int_{v\in V} a(v)\cdot(p\cdot Y)$
INT4		$\forall t\in V\ p[t/v]=q[t/v] \implies \int_{v\in V} a(v)\cdot p = \int_{v\in V} a(v)\cdot q$
ATI1a		$a(0)=\delta$
ATI1b		$a(\infty)=\delta(\infty)$
ATI2a		$\int_{v\in V}\delta(v)=\delta(sup(V))$
ATI2b		$\int_{v\in V}\delta(v)\cdot p=\delta(sup(V))$
ATI3	$t\le sup(V)$	$\int_{v\in V} P+\delta(t)=\int_{v\in V} P$
ATI4		$\int_{v\in V} a(v)\cdot p = \int_{v\in V} a(v)\cdot(v\gg p)$
ATBI1		$t\gg\int_{v\in V} P = \int_{v\in V\cap<t,\infty]} a(v)+\delta(t)$
ATB2		$t\gg(X+Y)=(t\gg X)+(t\gg Y)$

Table 1: An axiom system for BPA$\rho\delta$I

3. *if* $p_0 \xrightarrow{a(t)} \surd$ *and* $\mathcal{R}(p_0,q_0)$, *then* $q_0 \xrightarrow{a(t)} \surd$.

Klusener has proven soundness and completeness:

Theorem 2.2 *Let* $p,q\in\mathcal{T}^{time-closed}$. *Then* $p=q \iff p \underline{\leftrightarrow} q$.

2.9 Conditions

The set $Cond^{at}$ of *atomic conditions* is defined by

$$Cond^{at} := \{b<c,\ b\le c,\ b>c,\ b\ge c,\ b=c \mid b,c\in Bound\}\cup\{tt,\mathit{ff}\}$$

where tt denotes 'true' and ff 'false'. Let $\alpha^{at}\in Cond^{at}$. The set $Cond$ of *conditions*, with typical element α, is defined by

$$\alpha ::= \alpha^{at} \mid \alpha\wedge\alpha \mid \alpha\vee\alpha \mid \neg\alpha$$

Let Σ denote the collection of substitutions $\sigma : TVar \to [0,\infty]$. For $\alpha\in Cond$ we define a subset $[\alpha]$ of Σ.

$$[\alpha] := \{\sigma\in\Sigma \mid \sigma(\alpha) \text{ is true}\}$$

Clearly $[tt] = \Sigma$ and $[ff] = \emptyset$.

A collection of conditions $\{\alpha_1, ..., \alpha_n\}$ is called a *partition* if for each $\sigma \in \Sigma$ there is exactly one i such that $\sigma \in [\alpha_i]$.

In the sequel we will often abbreviate conditions by expressions like $b \in V$, $V \cap W \neq \emptyset$, $sup(V) > b$, $inf(V) < b$, $U(p) > b$ etc. An ad hoc notation we introduce here is $V \sim W$ and $V \not\sim W$, denoting that $V \cup W$ is an interval resp. is not an interval. Some examples of abbreviations are:

- $b \in \langle b_0, b_1]$ abbreviates $b_0 < b \wedge b \leq b_1$
- $\langle b_0, b_1] \cap [c_0, c_1] = \emptyset$ abbreviates $b_0 \geq b_1 \vee c_0 > c_1 \vee b_1 < b_0 \vee b_0 \geq c_1$
- $sup(\langle b_0, b_1]) < b$ abbreviates $b_1 < b \;\vee\; (\langle b_0, b_1] = \emptyset \wedge b > 0)$
- $\langle b_0, b_1] \sim [c_0, c_1]$ abbreviates $b_0 \geq b_1 \vee c_0 > c_1 \vee b_0 \in [c_0, c_1] \vee b_1 \in [c_0, c_1]$

2.10 Conditional terms

Let $\alpha \in Cond$, $p \in \mathcal{T}$ and $b \in Bound$. The set $\mathcal{C}$ of *conditional* terms, with typical element p_c, is defined by

$$p_c \;::=\; p \mid \alpha :\rightarrow p_c \mid \int_{v \in V} a(v) \cdot p_c \mid p_c + p_c \mid p_c \cdot p_c \mid b \gg p_c$$

Conditional terms are considered modulo commutativity and associativity of the $+$.

Often conditional terms will be abbreviated. For example, let $V = [b_0, b_1\rangle$ and $W = \langle c_0, c_1\rangle$. Then

$$\{V \sim W :\rightarrow \int_{V \cup W} P\} \;+\; \{V \not\sim W :\rightarrow \int_{v \in V} P \;+\; \int_{v \in W} P\}$$

abbreviates

$$\left\{\begin{array}{lll} b_0 \geq b_1 \wedge c_0 \geq c_1 & :\rightarrow & \int_{v \in \emptyset} P \\ + & & \\ b_0 < b_1 \;\wedge\; (c_0 \geq c_1 \;\vee\; (b_0 \leq c_0 \wedge b_1 > c_1)) & :\rightarrow & \int_{v \in [b_0, b_1\rangle} P \\ + & & \\ c_0 < c_1 \;\wedge\; (b_0 \geq b_1 \;\vee\; (b_0 > c_0 \wedge b_1 \leq c_1)) & :\rightarrow & \int_{v \in \langle c_0, c_1\rangle} P \\ + & & \\ b_0 < b_1 \wedge c_0 < c_1 \wedge b_0 \leq c_0 \wedge b_1 \in \langle c_0, c_1] & :\rightarrow & \int_{v \in [b_0, c_1\rangle} P \\ + & & \\ b_0 < b_1 \wedge c_0 < c_1 \wedge b_0 > c_0 \wedge c_1 \in [b_0, b_1\rangle & :\rightarrow & \int_{v \in \langle c_0, b_1\rangle} P \\ + & & \\ b_0 < b_1 \;\wedge\; c_0 < c_1 \;\wedge\; (b_0 > c_1 \vee b_1 \leq c_0) & :\rightarrow & \int_{v \in [b_0, b_1\rangle} P + \int_{v \in \langle c_0, c_1\rangle} P \end{array}\right\}$$

2.11 An operational semantics for conditional terms

First we extend the definition of substitutions $\sigma \in \Sigma$ to conditional terms by putting

$$\sigma(\alpha :\rightarrow p_c) = \begin{cases} \sigma(p_c) & \text{if } \sigma \in [\alpha] \\ \delta & \text{if } \sigma \notin [\alpha] \end{cases}$$

The operational semantics of BPA$\rho\delta$I is extended to one for conditional terms by adding the following rules.

$$\frac{\sigma(p_c) \xrightarrow{a(t)} q_c \quad \sigma \in [\alpha]}{\{\alpha :\rightarrow p_c\} \xrightarrow{a(t)} q_c} \qquad \frac{\sigma(p_c) \xrightarrow{a(t)} \surd \quad \sigma \in [\alpha]}{\{\alpha :\rightarrow p_c\} \xrightarrow{a(t)} \surd}$$

$$\frac{\sigma(p_c) \xrightarrow{a(t)} q_c \quad \sigma \in \Sigma}{p_c \xrightarrow{a(t)} q_c} \qquad \frac{\sigma(p_c) \xrightarrow{a(t)} \surd \quad \sigma \in \Sigma}{p_c \xrightarrow{a(t)} \surd}$$

Note that if $p_c \xrightarrow{a(t)} q_c$, then $q_c \in \mathcal{T}^{time-closed}$.

Using Definition 2.1 we define a bisimulation relation $\underline{\leftrightarrow}_c$ on $\mathcal{C}$. This relation is an extension of the bisimulation relation $\underline{\leftrightarrow}$, i.e. for all $p, q \in \mathcal{T}^{time-closed}$ we have

$$p \underline{\leftrightarrow}_c q \iff p \underline{\leftrightarrow} q$$

2.12 Basic conditional terms

A process term can contain "redundant" parts. For example:

$$\begin{array}{lcl} \int_{v\in\langle 0,1\rangle} a(v) + \delta(1) & = & \int_{v\in\langle 0,1\rangle} a(v) \\ \int_{v\in\langle 0,2\rangle} a(v) \cdot \int_{w\in[1,3\rangle} b(w) & = & \int_{v\in\langle 0,1\rangle} a(v) \cdot \int_{w\in[1,3\rangle} b(w) + \int_{v\in[1,2\rangle} a(v) \cdot \int_{w\in\langle v,3\rangle} b(w) \end{array}$$

The terms on the left-hand side contain redundant information, while the terms on the right-hand side do not.

In [Klu91] the notion of a *basic* term is introduced, which is a time-closed process term not containing redundant parts; $p \in \mathcal{T}^{time-closed}$ is basic if it is of the form

$$\Sigma_i \int_{v\in V_i} a_i(v) \cdot p_i \; + \; \Sigma_j \int_{v\in W_j} b_j(v)$$

with $a_i \in A$, $b_j \in A \cup \{\delta\}$ and the V_i and W_j non-empty such that

1. If $t \in W_j$, then there is a transition $p \xrightarrow{b_j(t)} \surd$.
2. If $t \in V_i$, then $p_i[t/v]$ is a basic term and $t \gg p_i[t/v] \underline{\leftrightarrow} p_i[t/v]$.

Requirement 1 says that terms like $a(2)+\delta(1)$ are not basic, while 2 ensures that $\int_{v\in V_i} a_i(v) \cdot p_i$ does not contain redundant information.

Now we can define the notion of a *basic* conditional term as follows.

Definition 2.3 *A conditional term is called basic if it is of the form $\Sigma_i\{\alpha_i :\rightarrow p_i\}$ such that*

- *$\{\alpha_i\}$ is a partition.*
- *if $\sigma \in [\alpha_i]$, then $\sigma(p_i)$ is a basic term.*

In [Klu91] the following theorem is proven.

Theorem 2.4 *For all $p \in \mathcal{T}$ there is a basic conditional term p_c such that*

$$FV(p_c) \subseteq FV(p) \quad \wedge \quad p \underline{\leftrightarrow}_c p_c$$

The proof consists of an explicit construction of p_c.

3 Reducing Process Terms

In this section we introduce the machinery to reduce each process term p to a normal form $p\downarrow$. For a time-closed process term its normal form will again be a time-closed process term. And in the next section we will prove that if $p, q \in \mathcal{T}^{time-closed}$ with $p = q$, then $p\downarrow \cong q\downarrow$ (where process terms are considered modulo commutativity and associativity of the $+$).

Thus it is possible to check in a finite number of steps if two terms $p, q \in \mathcal{T}^{time-closed}$ are equal or not; first p and q are reduced to normal form, and then a finite computation is carried out to see if A1,2 $\vdash p\downarrow = q\downarrow$ or not.

In the following paragraphs a number of *rewrite rules* are defined, which will be expressions of the form $p_c \longrightarrow q_c$ with $p_c, q_c \in \mathcal{C}$. Paragraph 3.6 will contain an explicit description how to reduce a basic conditional term to a normal form using the rewrite rules. This normal form will again be a basic conditional term.

3.1 Reducing bounds

In order to reduce time-closed process terms to a unique normal form, it is necessary to reduce bounds. For example, the equation $2v \dot{-} 1 = (v \dot{-} 1) + v$ holds for $v \geq 1$, but is untrue for $0 \leq v < 1$. So the equality

$$\int_{v\in\langle t,\infty\rangle} a(v) \cdot \int_{w\in\langle 2v\dot{-}1,2v\rangle} b(w) = \int_{v\in\langle t,\infty\rangle} a(v) \cdot \int_{w\in\langle (v\dot{-}1)+v,2v\rangle} b(w)$$

holds for $t \geq 1$, but not for $0 \leq t < 1$.

In Appendix A the notion of a bound is generalized to that of a *conditional* bound. Furthermore, it is described how a bound can be reduced to a normal form, which is a conditional bound in which all monus signs have been replaced by minus signs.

Now we can give rewrite rule 1, which enables us to reduce the bounds occurring in a process term. Let b be a bound that, using the construction described in Appendix A, is reduced to $\Sigma_i\{\alpha_i :\rightarrow b_i\}$. Then

1.	$\int_{v\in \{\!\!\{ b,c \}\!\!\}} P \longrightarrow \Sigma_i\{\alpha_i :\rightarrow \int_{v\in \{\!\!\{ b_i,c \}\!\!\}} P\}$

We have a similar rewrite rule for if b has normal form $b\downarrow$ and symmetric rules for if c has normal form $\Sigma_i\{\alpha_i :\rightarrow c_i\}$ or $c\downarrow$.

3.2 Reducing conditions

We now define four rewrite rules that reduce conditions. The first three are

2. $\{\alpha :\rightarrow \Sigma_i\{\beta_i :\rightarrow p_i\}\} \longrightarrow \Sigma_i\{\alpha \wedge \beta_i :\rightarrow p_i\}$
3. If $[\alpha] \cap [\beta] \neq \emptyset$, then

   $\{\alpha :\rightarrow p\} + \{\beta :\rightarrow q\} \longrightarrow \{\alpha \wedge \beta :\rightarrow p + q\} + \{\alpha \wedge \neg\beta :\rightarrow p\} + \{\neg\alpha \wedge \beta :\rightarrow q\}$
4. $\{\alpha :\rightarrow p\} + q \longrightarrow \{\alpha :\rightarrow p + q\} + \{\neg\alpha :\rightarrow q\}$

The fourth rule is more complicated. Let $\{\alpha_i\}$ be a partition and $v \in TVar$. In [Klu91] a construction is given for reducing $\{\alpha_i\}$ to a partition $\{\beta_j \wedge v \in W_j\}$, where $[\beta_j \wedge v \in W_j] \subseteq [\alpha_{i(j)}]$ for some $i(j)$. Furthermore, $tvar(\beta_j) \cup tvar(W_j) \subseteq tvar(\alpha_{i(j)})$. We define

5. $\int_{v\in V} a(v) \cdot \Sigma_i\{\alpha_i :\rightarrow p_i\} \longrightarrow \Sigma_j\{\beta_j :\rightarrow \int_{v\in V\cap V_j} a(v) \cdot p_{i(j)}\}$

The collection $\{\beta_j\}$ is not necessarily a partition. But using rewrite rule 3 it can be reduced to a partition again.

3.3 Substituting auxiliary variables

A variable occurring in a process term can be *auxiliary* in the sense that only one value can be substituted for it. For example:

$$\int_{v\in[1,1]} a(v) \cdot \int_{w\in\langle v,v+1\rangle} b(w) = \int_{v\in[1,1]} a(v) \cdot \int_{w\in\langle 1,2\rangle} b(w)$$

So in order to reduce time-closed process terms to a unique normal form, it is necessary to substitute the only possible value for such an auxiliary variable.

The following rewrite rule reduces process terms of the form $\int_{v\in[b,b]} a(v) \cdot p$. Ensure by applying α-conversion that $v \notin tvar(b)$ and also none of the variables in $tvar(b)$ are bound by integral signs occurring in p. Then

6. $\int_{v\in[b,b]} a(v) \cdot p \longrightarrow \int_{v\in[b,b]} a(v) \cdot p[b/v]$

3.4 Reducing double terms

The main problem of reducing time-closed process terms to a unique normal form is getting rid of the 'double terms'. We first give one rewrite rule, based on axiom INT1, to deal with this problem.

Let P be an expression of the form $a(v)$ or $a(v) \cdot p$. Remember that '$V_0 \sim V_1$' denotes '$V_0 \cup V_1$ is an interval'.

7. $\int_{v \in V_0} P + \int_{v \in V_1} P \longrightarrow \{V_0 \sim V_1 :\rightarrow \int_{v \in V_0 \cup V_1} P\} + \{V_0 \not\sim V_1 :\rightarrow \int_{v \in V_0} P + \int_{v \in V_1} P\}$

However, this rule is not sufficient in all cases. Consider the following two examples.

Example 3.1

$$\int_{v \in \langle 0,1 \rangle} a(v) \cdot \int_{w \in \langle v, v+1 \rangle} b(w) \; + \; a(1) \cdot \int_{w \in \langle 1,2 \rangle} b(w) \; = \; \int_{v \in \langle 0,1]} a(v) \cdot \int_{w \in \langle v, v+1 \rangle} b(w)$$

Although these terms are equal, they can not be rewritten by rule 7.

Example 3.2

$$\int_{v \in \langle 0,1 \rangle} a(v) \cdot \int_{w \in \langle v, v+1 \rangle} b(w) \; + \; \int_{v \in [1,2 \rangle} a(v) \cdot \int_{w \in \langle v, 2 \rangle} b(w)$$

$$= \int_{v \in \langle 0,1]} a(v) \cdot \int_{w \in \langle v, v+1 \rangle} b(w) \; + \; \int_{v \in \langle 1,2 \rangle} a(v) \cdot \int_{w \in \langle v, 2 \rangle} b(w)$$

Again both terms can not be rewritten by rule 7.

A logical solution for avoiding such situations seems to be allowing only integration over open intervals and over intervals consisting of one point. However, the following example shows that this restraint does not work.

Example 3.3

$$\int_{v \in \langle 0,1 \rangle} a(v) \cdot \int_{w \in \langle v, v+1 \rangle} b(w) \; + \; a(1) \cdot \int_{w \in \langle 1,2 \rangle} b(w) \; + \; \int_{v \in \langle 1,2 \rangle} a(v) \cdot \int_{w \in \langle v, v+1 \rangle} b(w)$$

$$= \int_{v \in \langle 0,2 \rangle} a(v) \cdot \int_{w \in \langle v, v+1 \rangle} b(w)$$

Both terms are equal and satisfy the restraint on intervals, but they can not be rewritten by rule 7. We therefore introduce two rewrite rules to deal with Examples 3.1 and 3.2.

Let $p, q \in \mathcal{T}$, $b \in Bound$ and $v \in TVar$. Ensure by applying α-conversion that $v \notin tvar(b)$ and that none of the variables occurring in b are bound by integrals occurring in p and q. Example 3.1 shows that we need a reduction

$$\int_{v \in V} a(v) \cdot p + \int_{v \in [b,b]} a(v) \cdot q \; \longrightarrow \; \int_{v \in V \cup \{b\}} a(v) \cdot p$$

if the following three statements are true:

1. $V \cup \{b\}$ is an interval.
2. $p[b/v]$ and $q[b/v]$ are equal.
3. $p[b/v]$ is a basic term.

The first statement can be expressed by the condition $\{b\} \sim V$. In order to translate the other two statements into a condition we define for $p, q \in \mathcal{T}$

$$\gamma(p,q) := \{\sigma \in \Sigma \mid \sigma(p) \underline{\leftrightarrow} \sigma(q) \ \wedge \ \sigma(p) \text{ is a basic term}\}$$

It is proven in Appendix B that $\gamma(p,q)$ is a condition for all $p, q \in \mathcal{T}$.

Now let γ be an abbreviation of $\gamma(p[b/v], q[b/v])$. Then statements 1-3 are translated into the condition $\gamma \wedge \{b\} \sim V$. We can reduce Example 3.1 by the following rewrite rule.

$$8.\ \ \int_{v\in V} a(v)\cdot p \ + \ \int_{v\in[b,b]} a(v)\cdot q \ \longrightarrow \ \{\gamma \wedge \{b\} \sim V :\rightarrow \int_{v\in V\cup\{b\}} a(v)\cdot p\} + \{\neg\gamma \vee \{b\} \not\sim V :\rightarrow \int_{v\in V} a(v)\cdot p + \int_{v\in[b,b]} a(v)\cdot q\}$$

Similarly, we can reduce Example 3.2 by

$$9.\ \ \int_{v\in\langle b,c\rangle} a(v)\cdot p + \int_{v\in V} a(v)\cdot q \longrightarrow \{\gamma \wedge b \in V :\rightarrow \int_{v\in[b,c\rangle} a(v)\cdot p + \int_{v\in V} a(v)\cdot q\} + \{\neg\gamma \vee b \notin V :\rightarrow \int_{v\in\langle b,c\rangle} a(v)\cdot p + \int_{v\in V} a(v)\cdot q\}$$

We also have a symmetric version of this rewrite rule in order to reduce the process term $\int_{v\in\langle c,b\rangle} a(v)\cdot p + \int_{v\in V} a(v)\cdot q$.

3.5 Soundness of the rewrite rules

Proposition 3.4 *If* $p_c \longrightarrow q_c$, *then* $p_c \underline{\leftrightarrow}_c q_c$.

Proof. We need to prove that for each rewrite rule $p_c \longrightarrow q_c$ we have $\sigma(p_c) \underline{\leftrightarrow} \sigma(q_c)$ for all $\sigma \in \Sigma$. It is easy to see that this holds for the first seven rewrite rules. We show that is also the case for rules 8 and 9.

Let $p, q \in \mathcal{T}$, $v \in TVar$ and $b \in Bound$ (where $v \notin tvar(b)$ and the variables occurring in b are not bound by integrals occurring in p and q). The condition $\gamma(p[b/v], q[b/v])$ is abbreviated to γ.

Fix a $\sigma \in \Sigma$. We show that $\sigma(\int_{v\in V} a(v)\cdot p + \int_{v\in[b,b]} a(v)\cdot q)$ is bisimilar to

$$\sigma(\{\gamma \wedge \{b\} \sim V :\rightarrow \int_{v\in V\cup\{b\}} a(v)\cdot p\} + \{\neg\gamma \vee \{b\} \not\sim V :\rightarrow \int_{v\in V} a(v)\cdot p + \int_{v\in[b,b]} a(v)\cdot q\})$$

If $\sigma \in [\neg\gamma \vee \{b\} \not\sim V]$, then this is trivially true. So assume that $\sigma \in [\gamma \wedge \{b\} \sim V]$. Then

$$\begin{array}{lcl} \sigma(\int_{v\in[b,b]} a(v)\cdot q) & \xrightarrow{a(\sigma(b))} & \sigma(q[b/v]) \\ \sigma(\int_{v\in V\cup\{b\}} a(v)\cdot p) & \xrightarrow{a(\sigma(b))} & \sigma(p[b/v]) \end{array}$$

Since $\sigma \in [\gamma]$ we have $\sigma(p[b/v]) \underline{\leftrightarrow} \sigma(q[b/v])$. It follows that

$$\sigma(\int_{v\in V} a(v)\cdot p + \int_{v\in[b,b]} a(v)\cdot q) \underline{\leftrightarrow} \sigma(\int_{v\in V\cup\{b\}} a(v)\cdot p)$$

and we are done

The proof for rule 9 is similar to that of rule 8. □

3.6 Constructing normal forms

In [Klu91] it has been proven that every process term can be reduced to a basic conditional term. We now show how to reduce a basic conditional term to a normal form, which will again be a basic conditional term. We use induction to depth, where the notion of depth is extended to conditional terms by putting

$$depth(\alpha :\rightarrow p_c) = depth(p_c)$$

First, let $\Sigma_i\{\alpha_i :\rightarrow p_i\}$ be a basic conditional term of depth 1. Then each p_i is of the form $\Sigma_j \int_{v\in V_{ij}} a_{ij}(v)$. Fix an i. We reduce p_i as follows.

- Apply rewrite rule 1 in order to the bounds of the V_{ij} to normal form (see Appendix A). Using rules 2-4 the conditions are reduced. Thus a basic conditional term is constructed of the form
$$\Sigma_k\{\beta_k :\rightarrow \Sigma_l \int_{v\in W_{kl}} b_{kl}(v)\}$$
where the bounds occurring in the W_{kl} and in the β_k are in normal form.

- Now apply rewrite rule 7 to each pair $\int_{v\in W_{kl}} b_{kl}(v) + \int_{v\in W_{kl'}} b_{kl'}(v)$ for which $b_{kl} \equiv b_{kl'}$. Use rules 2-4 again to reduce the conditions.

Thus we have constructed the normal form of p_i. Replace the p_i in $\Sigma_i\{\alpha_i :\rightarrow p_i\}$ by their normal forms. Use rule 2 to reduce conditions. The result is the normal form of $\Sigma_i\{\alpha_i :\rightarrow p_i\}$.

Now suppose that we have already constructed the normal forms for depth $\leq n$. Let $\Sigma_i\{\alpha_i :\rightarrow p_i\}$ be a basic conditional term of depth $n+1$. Fix an i and assume that

$$p_i \;\cong\; \Sigma_j \int_{v\in V_j} a_j(v)\cdot q_j + p'$$

where the q_j have depth n and p' has depth $\leq n$. According to the induction hypothesis we have already constructed normal forms for the q_j and for p'. We reduce $\Sigma_j \int_{v\in V_j} a_j(v)\cdot q_j$ as follows.

- Apply rewrite rule 1 in order to reduce the bounds of the V_j to normal form.

- Replace the q_j by their normal forms. Using rule 5, the conditions that occur in the normal form of q_j are lifted over the integral sign $\int_{v\in V_j}$. Apply rules 2-4 to reduce the conditions. Thus we have constructed a basic conditional term of the form
$$\Sigma_k\{\beta_k :\rightarrow \Sigma_l \int_{v\in W_{kl}} a_{kl}(v)\cdot q_{kl}\}$$

- Substitute auxiliary variables; if W_{kl} is of the form $[b,b]$ and $v \in FV(q_{kl})$, then apply rule 6 to $\int_{v\in W_{kl}} a_{kl}(v)\cdot q_{kl}$. Replace $q_{kl}[b/v]$ by its normal form and lift the conditions that occur in this normal form over $\int_{v\in W_{kl}}$ using rule 5. Apply rules 2-4 to reduce the conditions.

- Apply rules 8 and 9 to each pair

$$\int_{v\in W_{kl}} a_{kl}(v)\cdot q_{kl} \;+\; \int_{v\in W_{kl'}} a_{kl'}(v)\cdot q_{kl'}$$

  for which $a_{kl} \equiv a_{kl'}$. Reduce the conditions using rules 2-4.

- Apply rule 7 to each pair

$$\int_{v\in W_{kl}} a_{kl}(v)\cdot q_{kl} \;+\; \int_{v\in W_{kl'}} a_{kl'}(v)\cdot q_{kl'}$$

  for which $a_{kl} \equiv a_{kl'}$ and $q_{kl} \cong q_{kl'}$. Reduce the conditions using rules 2-4.

Now we have constructed the normal form of $\Sigma_j \int_{v\in V_j} a_j(v)\cdot q_j$. Add the normal form of p' to this term and reduce the conditions to a partition by applying rule 3. The result is the normal form of p_i.

Replace the p_i in $\Sigma_i\{\alpha_i :\rightarrow p_i\}$ by their normal forms. Use rule 2 to reduce the conditions. Thus we have constructed the normal form of $\Sigma_i\{\alpha_i :\rightarrow p_i\}$.

4 Unique normal forms

Let $p \in \mathcal{T}^{time-closed}$ have normal form $\Sigma_i\{\alpha_i :\rightarrow p_i\}$. The construction of the normal form has been such that $tvar(\alpha_i) \subseteq FV(p)$, so each α_i is equal to either tt or $f\!f$. By applying the following two rewrite rules

$$\boxed{\begin{array}{lcl} \{tt :\rightarrow p\} & \longrightarrow & p \\ \{f\!f :\rightarrow p\} & \longrightarrow & \delta \end{array}}$$

the normal form of p becomes a time-closed process term $p\downarrow$. We prove that if $p, q \in \mathcal{T}^{time-closed}$ with $p = q$ (in BPA$\rho\delta$I), then $p\downarrow \cong q\downarrow$.

4.1 Two lemmas

Let b be a bound occurring in a normal form. Then b is of the form 0, ∞ or

$$(*) \qquad (r_1\cdot v_1 + \ldots + r_k\cdot v_k + v_{k+1} + \ldots + v_l(+t)) \;-\; (s_1\cdot w_1 + \ldots + s_m\cdot w_m + w_{m+1} + \ldots + w_n(+t'))$$

where $v_1, \ldots, v_l, w_1, \ldots, w_n \in TVar$ are all different, $r_i, s_i \in \langle 0,\infty\rangle\backslash\{1\}$ and either t, t' or both do not occur (see Appendix A). However, if σ is a substitution mapping a subset Σ to $[0,\infty]$, then $\sigma(b)$ need not be of this form. Applying steps 1,3,4 and 5 of Appendix A, we can reduce $\sigma(b)$ to the form $(*)$ again, denoted by $\sigma^*(b)$. We define σ^* on process terms by putting

$$\begin{array}{lcl} \sigma^*(p+q) & = & \sigma^*(p)+\sigma^*(q) \\ \sigma^*(\int_{v\in V} a(v)) & = & \int_{v\in\sigma^*(V)} a(v) \\ \sigma^*(\int_{v\in V} a(v)\cdot p) & = & \int_{v\in\sigma^*(V)} a(v)\cdot\sigma^*_v(p) \end{array}$$

where in the third equation it is required that for all $w \in FV(p)\backslash\{v\}$ we have $v \notin \sigma(w)$. If $\sigma : \{v\} \rightarrow [0,\infty]$, then $\sigma^*(p)$ is abbreviated to $p^*[\sigma(v)/v]$.

Let $\int_{v\in V} a(v)\cdot p \in \mathcal{T}^{time-closed}$ be a normal form and let b, c be bounds occurring in p with $b \not\cong c$. Since b, c are of the form $(*)$, it follows that there is at the most one $t \in V$ such that $b^*[t/v] \cong c^*[t/v]$. We now prove two lemmas.

Lemma 4.1 *Let $\int_{v\in V} a(v)\cdot p,\ \int_{v\in V} a(v)\cdot q \in \mathcal{T}^{time-closed}$ be normal forms. If p' and q' are subterms of p and q with $p' \not\cong q'$, then there is only a finite number of $t \in V$ for which $p'^*[t/v] \cong q'^*[t/v]$.*

Proof. We use induction to the depth of p' and q'. (Note that if p' and q' do not have the same depth, then $p'^*[t/v] \not\cong q'^*[t/v]$ for all $t \in V$). Let

$$p' \ \cong\ \Sigma_i \int_{w\in V_i} a_i(w)\cdot p_i \ +\ \Sigma_j \int_{w\in W_j} b_j(w)$$

$$q' \ \cong\ \Sigma_k \int_{w\in V'_k} a'_k(w)\cdot q_k \ +\ \Sigma_l \int_{w\in W'_l} b'_l(w)$$

(In the first induction step the sums over i and k are empty). Assume that $p' \not\cong q'$. We distinguish two cases.

1. There is a j such that for all l we have $W_j \not\cong W'_l$ or $b_j \not\equiv b'_l$.

   If $W_j \not\cong W'_l$, then there is no more than one $t \in V$ such that $W_j^*[t/v] \cong W_l'^*[t/v]$. Thus the collection of $t \in V$ for which $\int_{w\in W_j^*[t/v]} b_j(w)$ is isomorphic to $\int_{w\in W_l'^*[t/v]} b'_l(w)$ for some l is finite. It follows that $\{t \in V \mid p'^*[t/v] \cong q'^*[t/v]\}$ is finite.

2. There is an i such that for all k we have $V_i \not\cong V'_k$ or $a_i \not\equiv a'_k$ or $p_i \not\cong q_k$.

   If $V_i \not\cong V'_k$, then there is no more than one $t \in V$ such that $V_i^*[t/v] \cong V_k'^*[t/v]$. If $p_i \not\cong q_k$, then by the induction hypothesis there is only a finite number of $t \in V$ such that $p_i^*[t/v] \cong q_k^*[t/v]$. It follows that $\{t \in V \mid p'^*[t/v] \cong q'^*[t/v]\}$ is finite.

□

Lemma 4.2 *Let $\int_{v\in V} a(v)\cdot p \in \mathcal{T}^{time-closed}$ be a normal form. There is only a finite number of $t \in V$ for which $p^*[t/v]$ is not a normal form.*

Proof. Suppose that $p^*[t/v]$ is not a normal form. Then if we apply the reduction described in Paragraph 3.6 to $p[t/v]$ we end up with a time-closed process term that is not isomorphic to $p^*[t/v]$. It follows that one of the following two cases must hold.

1. p has a subterm of the form $\int_{w\in[b,c]} a'(w)\cdot p'$ with $b \not\cong c$ such that $b^*[t/v] \cong c^*[t/v]$.

   Since b, c are of the form $(*)$, the fact that $b \not\cong c$ implies that there is no more than one $t \in V$ such that $b^*[t/v] \cong c^*[t/v]$.

2. p has a subterm of the form $\int_{w\in W_0} a'(w)\cdot p_0 + \int_{w\in W_1} a'(w)\cdot p_1$ with $p_0 \not\cong p_1$ such that $p_0^*[t/v] \cong p_1^*[t/v]$.

   Since $p_0 \not\cong p_1$, Lemma 4.1 implies that there is only a finite number of $t \in V$ such that $p_0^*[t/v] \cong p_1^*[t/v]$.

Since p has only a finite number of subterms, we are done. □

4.2 Unique normal forms

Theorem 4.3 *Let $p, q \in \mathcal{T}^{time-closed}$. If $p \underline{\leftrightarrow} q$, then $p{\downarrow} \cong q{\downarrow}$.*

Proof. Assume that $p \underline{\leftrightarrow} q$. Proposition 3.4 implies $p \underline{\leftrightarrow} p{\downarrow}$ and $q \underline{\leftrightarrow} q{\downarrow}$, and so $p{\downarrow} \underline{\leftrightarrow} q{\downarrow}$. We prove that $p{\downarrow} \cong q{\downarrow}$, using induction to the depth of $p{\downarrow}$ and $q{\downarrow}$. First let

$$p{\downarrow} \cong \Sigma_{i\in I} \int_{v\in V_i} a_i(v) \qquad\qquad q{\downarrow} \cong \Sigma_{j\in J} \int_{v\in W_j} b_j(v)$$

Fix an $i \in I$. For $t \in V_i$ we have $p{\downarrow} \xrightarrow{a_i(t)} \surd$. Since $p{\downarrow} \underline{\leftrightarrow} q{\downarrow}$, it follows that for each $t \in V_i$ there is a $j(t) \in J$ with $a_i \equiv b_{j(t)}$ and $t \in W_{j(t)}$. Rewrite rule 7 has been applied, so then there is a unique $j \in J$ with $a_i \equiv b_j$ and $V_i \subseteq W_j$. Similarly for this j there is a unique $i(j) \in I$ with $b_j \equiv a_{i(j)}$ and $W_j \subseteq V_{i(j)}$. Then rewrite rule 7 tells us that $i(j) = i$. Thus $V_i = W_j$.

Now suppose that we have proven the theorem for depth $\leq n$. Let

$$p{\downarrow} \cong \Sigma_{i\in I} \int_{v\in V_i} a_i(v) \cdot p_i + p' \qquad\qquad q{\downarrow} \cong \Sigma_{j\in J} \int_{v\in W_j} b_j(v) \cdot q_j + q'$$

where the p_i and q_j have depth n and p' and q' have depth $\leq n$. Since $p{\downarrow} \underline{\leftrightarrow} q{\downarrow}$, it follows that $p' \underline{\leftrightarrow} q'$ and thus by the induction hypothesis $p' \cong q'$.

Fix an $i \in I$. Since $p{\downarrow} \underline{\leftrightarrow} q{\downarrow}$, it follows that for each $t \in V_i$ there is a $j(t) \in J$ with $a_i \equiv b_{j(t)}$ and $t \in W_{j(t)}$ and $p_i[t/v] \underline{\leftrightarrow} q_{j(t)}[t/v]$.

1. First assume that V_i contains more than one point. Let $J' \subseteq J$ be the collection of j for which $b_j \equiv a_i$ and $q_j \cong p_i$, and define $W_{J'} := \cup_{j\in J'} W_j$. Then $V_i \backslash W_{J'}$ is just a finite number of points.

   For suppose not. Clearly for each $t \in V_i \backslash W_{J'}$ we have $j(t) \notin J'$. Then there is an infinite subset S of $V_i \backslash W_{J'}$ such that $j(t) = j_0$ for all $t \in S$ and for some $j_0 \in J \backslash J'$. Lemma 4.2 implies that $p_i^*[t/v]$ and $q_{j_0}^*[t/v]$ are normal forms for an infinite number of $t \in S$. Since $p_i^*[t/v] \underline{\leftrightarrow} q_{j_0}^*[t/v]$, the induction hypothesis tells us that $p_i^*[t/v] \cong q_{j_0}^*[t/v]$ for this infinite number of $t \in S$. Then Lemma 4.1 implies that $p_i \cong q_{j_0}$. It follows that $j_0 \in J'$, so we have a contradiction.

   Suppose that $t \in V_i \backslash W_{J'}$. Since this set is finite (and since V_i contains more than one point), there is a $j \in J'$ such that W_j is of the form $\langle t, t' \rangle\!\!\!>$ or $<\!\!\!\langle t', t\rangle$. The fact that $q_j \cong p_i$ implies that $q_j[t/v] \underline{\leftrightarrow} q_{j(t)}[t/v]$ and $q_j[t/v]$ is a basic term. Since rewrite rule 9 has been applied to the pair

   $$\int_{v\in W_{j(t)}} a_i(v) \cdot q_{j(t)} + \int_{v\in W_j} a_i(v) \cdot q_j$$

   we have $t \in W_j$. This is a contradiction, so it follows that $V_i \subseteq W_{J'}$. Then rewrite rule 7 implies that there is a unique $j \in J'$ with $V_i \subseteq W_j$.

   Similarly for this j there is an $i(j) \in I$ with $b_j \equiv a_{i(j)}$ and $q_j \cong p_{i(j)}$ and $W_j \subseteq V_{i(j)}$. Rewrite rule 7 implies that $i(j) = i$. Thus $V_i = W_j$.

2. Now assume that $V_i = [t,t]$. If $W_{j(t)}$ contains more than one point, then we have just proven that there is an $i(t) \in I$ with $a_{i(t)} \equiv b_{j(t)}$ and $V_{i(t)} = W_{j(t)}$ and $p_{i(t)} \cong q_{j(t)}$. Since $V_{i(t)} = W_{j(t)}$ it follows that $t \in V_{i(t)}$. And $p_{i(t)} \cong q_{j(t)}$ implies that $p_{i(t)}[t/v] \,\underline{\leftrightarrow}\, p_i[t/v]$ and $p_{i(t)}$ is a basic term. Since rewrite rule 8 has been applied to the pair
$$\int_{v\in V_{i(t)}} a_i(v)\cdot p_{i(t)} \;+\; \int_{v\in V_i} a_i(v)\cdot p_i$$
the term $\int_{v\in V_i} a_i(v)\cdot p_i$ should not be there at all. This is a contradiction, so it follows that $W_{j(t)} = [t,t]$.
Since we have applied rewrite rule 6, the auxiliary variable v does not occur in p_i and $q_{j(t)}$. So $p_i \cong p_i[t/v] \,\underline{\leftrightarrow}\, q_{j(t)}[t/v] \cong q_{j(t)}$. From the construction in Paragraph 3.6 it follows that p_i and $q_{j(t)}$ are normal forms. Then the induction hypothesis implies $p_i \cong q_{j(t)}$.

□

4.3 An example

The normal form of a process term can be much greater than the term itself. Consider for instance
$$\int_{v\in\langle 0,5\rangle} a(v)\cdot\Big(\int_{w\in\langle 7-4v,5-v\rangle} b(w) \;+\; \int_{w\in\langle 3,\frac{17}{6}+\frac{1}{3}v\rangle} b(w)\Big)$$
Its normal form can be deduced from Figure 1. The lines that are drawn there intersect for $v \in \{\frac{1}{2},\frac{2}{3},\frac{25}{26},1,\frac{7}{5},\frac{13}{8},2,\frac{5}{2},3,\frac{17}{4}\}$. Thus we get the following normal form:

$$\begin{aligned}
&\textstyle\int_{v\in\langle 0,\frac{1}{2}]} a(v)\cdot\delta(v) \;+\; \int_{v\in\langle\frac{1}{2},\frac{2}{3}]} a(v)\cdot\int_{w\in\langle 3,\frac{17}{6}+\frac{1}{3}v\rangle} b(w)\\
+\;&\textstyle\int_{v\in\langle\frac{2}{3},\frac{25}{26}]} a(v)\cdot\big(\int_{w\in\langle 3,\frac{17}{6}+\frac{1}{3}v\rangle} b(w) + \int_{w\in\langle 7-4v,5-v\rangle} b(w)\big)\\
+\;&\textstyle\int_{v\in\langle\frac{25}{26},1]} a(v)\cdot\int_{w\in\langle 3,5-v\rangle} b(w) \;+\; \int_{v\in[1,\frac{7}{5}]} a(v)\cdot\int_{v\in\langle 7-4v,5-v\rangle} b(w)\\
+\;&\textstyle\int_{v\in[\frac{7}{5},\frac{13}{8}]} a(v)\cdot\int_{w\in\langle v,5-v\rangle} b(w) \;+\; \int_{v\in[\frac{13}{8},2\rangle} a(v)\cdot\int_{w\in\langle v,\frac{17}{6}+\frac{1}{3}v\rangle} b(w)\\
+\;&\textstyle\int_{v\in[2,\frac{5}{2}\rangle} a(v)\cdot\big(\int_{w\in\langle v,5-v\rangle} b(w) + \int_{w\in\langle 3,\frac{17}{6}+\frac{1}{3}v\rangle} b(w)\big)\\
+\;&\textstyle\int_{v\in[\frac{5}{2},3]} a(v)\cdot\int_{w\in\langle 3,\frac{17}{6}+\frac{1}{3}v\rangle} b(w) \;+\; \int_{v\in[3,\frac{17}{4}\rangle} a(v)\cdot\int_{w\in\langle v,\frac{17}{6}+\frac{1}{3}v\rangle} b(w)\\
+\;&\textstyle\int_{v\in[\frac{17}{4},5\rangle} a(v)\cdot\delta(v)
\end{aligned}$$

4.4 Corollaries

For $p,q \in \mathcal{T}^{time-closed}$ we can check in a finite number of steps if $p = q$ holds in the axiom system or not. First p and q are rewritten, by a finite reduction, to normal forms p_n resp. q_n. Then a finite computation is carried out to see if $A1,2 \vdash p = q$ or not.

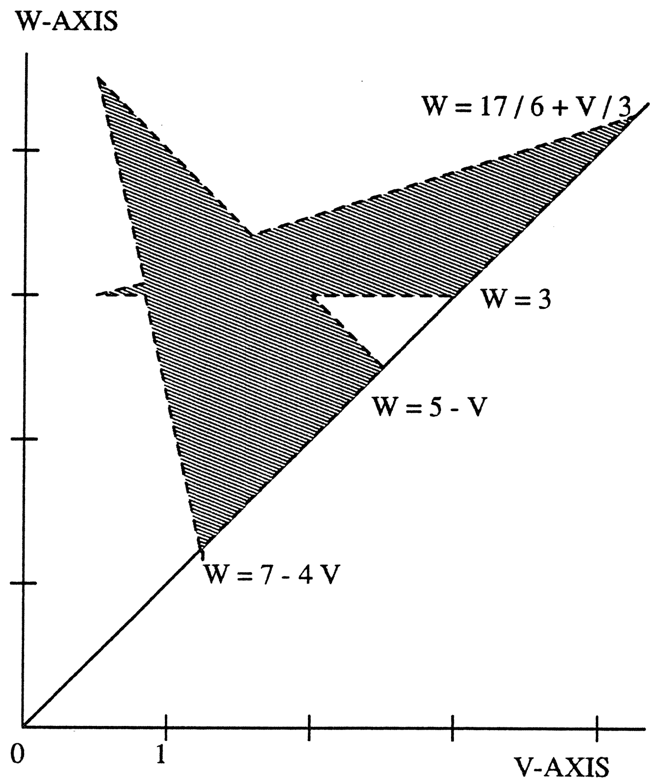

Figure 1: The graphical description of a process term

Baeten and Bergstra have defined the *state operator* λ_S ([BB91b]) and the *priority operator* Θ_H ([BB91c]). The operators were first defined on a subclass O of process terms. The definition was extended to all process terms by

$$p = q \implies \lambda_S(p) = \lambda_S(q),\ \Theta_H(p) = \Theta_H(q)$$

However, it was not proven that for $p, q \in O$ with $p = q$ we have $\lambda_S(p) = \lambda_S(q)$ and $\Theta_H(p) = \Theta_H(q)$. So it may be the case that the operators are not well-defined.

Using the theory of normal forms, the state and the priority operator can be defined as follows. First both operators are determined on the collection of normal forms. The definition is then extended to $\mathcal{T}^{time-closed}$ by $\lambda_S(p) := \lambda_S(p\downarrow)$ and $\Theta_H(p) := \Theta_H(p\downarrow)$.

We now prove an equality in BPA$\rho\delta$I which in [Klu91] was added to the axiom system.

Proposition 4.4 *Let X and $\int_{v\in V} a(v)\cdot p$ be time-closed process terms. If $X + a(t)\cdot p[t/v] = X$ for all $t \in V$, then*

$$X + \int_{v\in V} a(v)\cdot p = X$$

Proof. Let X and $\int_{v\in V} a(v)\cdot p$ have normal forms $X\downarrow$ and $\Sigma_i \int_{v\in V_i} a(v)\cdot p_i$ respectively. Fix an i. If V_i consists of one point, then clearly $X + \int_{v\in V_i} a(v)\cdot p_i = X$. And if V_i contains more than one point, then we can repeat the reasoning in point 1 of the proof of Theorem 4.3 to show that $X\downarrow$ has a subterm of the form $\int_{v\in W} a(v)\cdot p_i$ with $V_i \subseteq W$. □

A Reducing Bounds

The set of conditional bounds, with typical elements b_c, b'_c, is defined inductively as follows, where $b \in \mathit{Bound}$ and $\alpha \in \mathit{Cond}$.

$$b_c \ ::= \ b \ | \ \alpha :\!\longrightarrow b \ | \ b_c + b'_c \ | \ b_c \dot{-} b'_c$$

Conditional bounds are considered modulo commutativity and associativity of the $+$.

We now show how to reduce a bound to a normal form, which will be a conditional bound of the form $\Sigma_i\{\alpha_i :\!\longrightarrow b_i\}$. This construction is described in several steps, where each step consists of giving a collection of rewrite rules. Here, a rewrite rule is an expression of the form $b_c \longrightarrow b'_c$, where b_c, b'_c are conditional bounds.

Step 1

$$\begin{array}{|l|}\hline r \cdot (b + c) \ \longrightarrow \ r \cdot b + r \cdot c \\ r \cdot (b \dot{-} c) \ \longrightarrow \ r \cdot b \dot{-} r \cdot c \\ r \cdot t_0 \ \longrightarrow \ t_1 \text{ where } t_1 = r \cdot t_0 \\ r_0 \cdot (r_1 \cdot v) \ \longrightarrow \ r_2 \cdot v \text{ where } r_2 = r_0 \cdot r_1 \\ 1 \cdot v \ \longrightarrow \ v \\ \hline \end{array}$$

Using these rules, each bound can be reduced to a bound that can be defined as follows, where $t \in [0, \infty]$, $r \in \langle 0, \infty\rangle \backslash \{1\}$ and $v \in \mathit{TVar}$.

$$b \ ::= \ t \ | \ v \ | \ r \cdot v \ | \ b + c \ | \ b \dot{-} c$$

Step 2

Let $\{\alpha_i\}$ and $\{\beta_j\}$ be partitions and $\Box \in \{+, \dot{-}, -\}$. Then

$$\begin{array}{|l|}\hline b \dot{-} c \ \longrightarrow \ \{b > c :\!\longrightarrow b - c\} \ + \ \{b \leq c :\!\longrightarrow 0\} \\ \Sigma_i\{\alpha_i :\!\longrightarrow b_i\} \ \Box \ \Sigma_j\{\beta_j :\!\longrightarrow c_j\} \ \longrightarrow \ \Sigma_{(i,j)}\{\alpha_i \wedge \beta_j :\!\longrightarrow b_i \Box c_j\} \\ \Sigma_i\{\alpha_i :\!\longrightarrow b_i\} \ \Box \ c \ \longrightarrow \ \Sigma_i\{\alpha_i :\!\longrightarrow b_i \Box c\} \\ b \ \Box \ \Sigma_j\{\beta_j :\!\longrightarrow c_j\} \ \longrightarrow \ \Sigma_j\{\beta_j :\!\longrightarrow b \Box c_j\} \\ \Sigma_i\{\alpha_i :\!\longrightarrow \Sigma_j\{\beta_j :\!\longrightarrow b_{ij}\}\} \ \longrightarrow \ \Sigma_{(i,j)}\{\alpha_i \wedge \beta_j :\!\longrightarrow b_{ij}\} \\ \hline \end{array}$$

The first rewrite rule is applied to each monus sign once. The minus sign can be considered to be the monus sign together with some index, telling that the first rewrite rule has been applied to this monus sign.

The bounds occurring in the conditions have to be reduced too. For instance, if $b \longrightarrow b'$ then $b \leq c \longrightarrow b' \leq c$.

With these rewrite rules each bound can be reduced to the form $\Sigma_i\{\alpha_i :\!\longrightarrow b_i\}$, where the bounds b_i and the bounds occurring in the α_i do not contain monus signs.

Step 3

$b_0 + (b_1 - b_2)$	$\longrightarrow$	$(b_0 + b_1) - b_2$
$(b_0 - b_1) + b_2$	$\longrightarrow$	$(b_0 + b_2) - b_1$
$b_0 - (b_1 - b_2)$	$\longrightarrow$	$(b_0 + b_2) - b_1$
$(b_0 - b_1) - b_2$	$\longrightarrow$	$b_0 - (b_1 + b_2)$

Using these rewrite rules, each bound can be reduced to the form $\Sigma_i\{\alpha_i :\rightarrow b_i\}$, where the bounds b_i and the bounds that occur in the α_i are of the form

$$(r_1 \cdot v_1 + ... + r_i \cdot v_i + v_{i+1} + ... + v_j + t_1 + ... + t_k) - (s_1 \cdot w_1 + ... + s_l \cdot w_l + w_{l+1} + ... + w_m + t'_1 + ... + t'_n)$$

with $j + k \geq 1$ and $m + n \geq 0$.

Step 4

Let $\Box \in \{+, -\}$.

$t_0 + t_1 \longrightarrow t_2$ where $t_2 = t_0 + t_1$
$r_0 \cdot v + r_1 \cdot v \longrightarrow r_2 \cdot v$ where $r_2 = r_0 + r_1$
$r_0 \cdot v + v \longrightarrow r_1 \cdot v$ where $r_1 = r_0 + 1$
$v + v \longrightarrow 2 \cdot v$
$b \Box 0 \longrightarrow b$
$\infty \Box b \longrightarrow \infty$

With these rewrite rules we can reduce each bound to the form $\Sigma_i\{\alpha_i :\rightarrow b_i\}$, where the bounds b_i and the bounds that occur in the α_i are of the form 0, ∞ or

$$(*) \quad (r_1 \cdot v_1 + ... + r_i \cdot v_i + v_{i+1} + ... + v_j(+t)) - (s_1 \cdot w_1 + ... + s_l \cdot w_l + w_{l+1} + ... + w_m(+t'))$$

where $v_1, ..., v_j \in TVar$ are all different, $w_1, ..., w_m \in TVar$ are all different and $t, t' \in \langle 0, \infty \rangle$ are not necessarily there (that is why they have been put between brackets).

Step 5

Finally, by applying rewrite rules like

$(b + r_0 \cdot v) - (c + r_1 \cdot v) \longrightarrow (b + r_2 \cdot v) - c$ if $r_0 > r_1$ and $r_2 = r_0 - r_1$
$(b + r_0 \cdot v) - (c + r_1 \cdot v) \longrightarrow b - (c + r_2 \cdot v)$ if $r_0 < r_1$ and $r_2 = r_1 - r_0$
$(b + r \cdot v) - (c + r \cdot v) \longrightarrow b - c$

we can reduce bounds of the form $(*)$ such that $v_1, ..., v_j, w_1, ..., w_m \in TVar$ are all different and either t, t' or both do not occur.

B γ is a Condition

In this section we prove that the collection $\gamma(p, q)$, which has been defined in Paragraph 3.4, is a condition. It is easy to see that for $p \in \mathcal{T}$ the set

$$\{\sigma \in \Sigma \mid \sigma(p) \text{ is a basic term}\}$$

is a condition. We prove that $\{\sigma \in \Sigma \mid \sigma(p) \underline{\leftrightarrow} \sigma(q)\}$ is a condition for all $p, q \in \mathcal{T}$, using induction to the depth of p and q.

Suppose that we have proven $\gamma(p,q)$ to be a condition for pairs p, q of depth $\leq n$. Note that then we can construct a normal form for process terms of depth $\leq n+1$, because in the construction of Paragraph 3.6 only expressions $\gamma(p,q)$ with $depth(p), depth(q) \leq n$ occur. Furthermore, Theorem 4.3 can be proven for time-closed process terms of depth $\leq n+1$.

We show how a process term of depth $\leq n+1$ can be reduced to a basic conditional term $\Sigma_i\{\alpha_i :\rightarrow p_i\}$ with $depth(p_i) \leq n+1$ such that

$$\sigma \in [\alpha_i] \quad \Longrightarrow \quad \sigma(p_i)\downarrow \cong \sigma^*(p_i)$$

Then we are done. For if we reduce $p, q \in \mathcal{T}$ of depth $\leq n+1$ to such forms $\Sigma_i\{\alpha_i :\rightarrow p_i\}$ resp. $\Sigma_j\{\beta_j :\rightarrow q_j\}$, then according to Theorem 4.3 the collection $\{\sigma \in \Sigma \mid \sigma(p) \underline{\leftrightarrow} \sigma(q)\}$ is equal to

$$\bigcup_{(i,j)} \{\sigma \in [\alpha_i] \cap [\beta_j] \mid \sigma^*(p_i) \cong \sigma^*(q_j)\}$$

and it is easy to see that this is a condition.

Let $p \in \mathcal{T}$ have depth $\leq n+1$. Using the construction from Paragraph 3.6, p can be reduced to a normal form $\Sigma_i\{\alpha_i :\rightarrow p_i\}$. Fix an i. Ensure by applying α-conversion that if $w \in FV(p_i)$, then w does not occur as a binding variable in p_i, i.e. p_i does not contain an integral of the form $\int_{w \in W}$. Let in the sequel $\tilde{\sigma}$ denote σ restricted to $FV(p_i)$. Then if p_i contains a subterm $\int_{v \in V} a(v) \cdot q$, it follows that $\int_{v \in \tilde{\sigma}(V)} a(v) \cdot \tilde{\sigma}(q)$ is a subterm of $\sigma(p_i)$.

Now suppose that for some $\sigma \in [\alpha_i]$ we have $\sigma(p_i)\downarrow \not\cong \sigma^*(p_i)$. Then one of the following two cases must hold.

1. p_i has a subterm of the form $\int_{v \in [b,c]} a(v) \cdot q$ with $v \in FV(q)$ and $\tilde{\sigma}^*(b) \cong \tilde{\sigma}^*(c)$.

2. p_i has a subterm of the form $\int_{v \in V_0} a(v) \cdot q_0 + \int_{v \in V_1} a(v) \cdot q_1$ with $q_0 \not\cong q_1$ and $\tilde{\sigma}^*(q_0) \cong \tilde{\sigma}^*(q_1)$.

We rewrite $\{\alpha_i :\rightarrow p_i\}$ so that such subterms are reduced, using rewrite rules 2-5 together with adapted versions of rules 6 and 7. For $b, c \in Bound$ and $p, q \in \mathcal{T}$ let

$$\alpha_i(b,c) := \{\sigma \in [\alpha_i] \mid \tilde{\sigma}^*(b) \cong \tilde{\sigma}^*(c)\}$$
$$\alpha_i(p,q) := \{\sigma \in [\alpha_i] \mid \tilde{\sigma}^*(p) \cong \tilde{\sigma}^*(q)\}$$

We define rewrite rules 6′ and 7′ as follows. In rule 6′ it is required that $v \notin tvar(b)$ and that none of the variables occurring in b are bound by integrals in q.

6′. $\int_{v \in [b,c]} a(v) \cdot q \longrightarrow \{\alpha_i(b,c) :\rightarrow \int_{v \in [b,b]} a(v) \cdot q[b/v]\} + \{\neg\alpha_i(b,c) :\rightarrow \int_{v \in [b,c]} a(v) \cdot q\}$
7′. $\int_{v \in V_0} a(v) \cdot q_0 + \int_{v \in V_1} a(v) \cdot q_1 \longrightarrow \{\alpha_i(q_0,q_1) \wedge V_0 \sim V_1 :\rightarrow \int_{v \in V_0 \cup V_1} a(v) \cdot q_0\}$ $+ \{\neg\alpha_i(q_0,q_1) \vee V_0 \not\sim V_1 :\rightarrow \int_{v \in V_0} a(v) \cdot q_0 + \int_{v \in V_1} a(v) \cdot q_1\}$

We repeat the construction from Paragraph 3.6 for these new rewrite rules in order to reduce p_i. Suppose that we have already reduced the subterms of p_i of depth $\leq m$ and let $\Sigma_j \int_{v\in V_j} a_j(v) \cdot q_j$ be a subterm of p_i of depth $m+1$. This subterm is reduced as follows.

- Replace the q_j by their reducts, which have already been constructed according to the induction hypothesis. Lift the conditions that occur in the reduct of q_j over $\int_{v\in V_j}$ by applying rewrite rule 5. Thus we have constructed a basic conditional term of the form

$$\Sigma_k\{\beta_k :\rightarrow \Sigma_l \int_{v\in V_{kl}} a_k l(v) \cdot q_k l\}$$

- Apply rewrite rule 6′ to each term $\int_{v\in V_{kl}} a_{kl}(v) \cdot q_{kl}$ for which $v \in FV(q_{kl})$ and V_{kl} is of the form $[b, c]$. Replace $q_{kl}[b/v]$ by its reduct (which can be constructed according to the induction hypothesis) and lift the conditions that occur in this reduct over $\int_{v\in V_{kl}}$ using rule 5. Apply rules 2-4 to reduce the conditions.

- Apply rule 7′ to each pair

$$\int_{v\in V_{kl}} a_{kl}(v) \cdot q_{kl} \;+\; \int_{v\in V_{kl'}} a_{kl'}(v) \cdot q_{kl'}$$

  for which $a_{kl} \equiv a_{kl'}$ and $q_{kl} \not\cong q_{kl'}$. Reduce the conditions using rules 2-4.

Performing this construction up to depth $n+1$, we end up with a reduction $\{\beta_j :\rightarrow q_j\}$ of $\{\alpha_i :\rightarrow p_i\}$ with $\sigma \in [\beta_j] \Rightarrow \sigma(q_j)\downarrow \cong \sigma^*(q_j)$.

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