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Discretization of Morphological Operators

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This paper presents a comprehensive theory of discretization of images, image operators and image functionals in mathematical morphology. The procedure of image discretization discussed here consists of two steps, (i) the definition of a sequence of discrete images obtained by sampling the original continuous image on grids with finer and finer mesh width, (ii) the representation of the discrete images as continuous images. The hit-or-miss topology on the space of closed Euclidean sets is used to show that the thus obtained discretized images approach the original continuous image if the mesh width tends to zero. The image discretization procedure is then used to obtain discretizations of image operators and functionals. Particular attention is given to discretizations with nice monotonicity properties, the so-called constricting discretizations.

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1. Statement of the problem

Images processed by a computer are discrete (or digital) images: they consist of a finite number of elements. Usually these picture elements (pixels) are regularly arranged in a grid, for example a square or hexagonal grid, but they may also constitute an image in a different way, e.g. according to some graph structure. In this paper we restrict ourselves to binary images and in this case a discrete image can be considered as a subset of $\mathbb{Z}^d$ where d is the dimension of the underlying space; usually $d = 2$. The ultimate goal of a computerized image analysis system is the extraction of useful information from an image. Cell biologists for example, are often interested in the statistics of certain cell characteristics such as the mean curvature of the cell boundary or the area of the cell nucleus. Mathematical morphology is a branch of image processing which is particularly suited for the extraction of geometric information from an image. It is interweaved with other mathematical areas such as integral geometry, stereology and stochastic geometry. As a result, mathematical morphology is basically a continuous theory in the sense that it deals with continuous images. In the binary case considered here, continuous images can be modelled as subsets of $\mathbb{R}^d$. On the other hand, many morphological operators such as dilation, erosion,

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hit-or-miss transform, closing, opening, etc "only" use the group structure of the underlying space and hence can also be defined for discrete images. In addition, algorithms deriving from morphology are developped for discrete images because in practice one has a finite amount of data at one's disposal and because the computers on which these data are to be processed are digital. So there is a gap between the mathematical tools which are of a continuous nature, and the practical solution which requires discrete computations.

Let us illustrate this point by the following concrete example. Suppose we are interested in the perimeter p of a closed set $X \subseteq \mathbb{R}^2$. We can try to approximate this quantity by choosing a square grid in $\mathbb{R}^2$ with mesh width λ and replacing X by Y_λ, the set of all pixels which intersect X; see Fig. 1.1(a). Does the perimeter of Y_λ, which is a multiple of λ, converge to p as λ tends to zero?

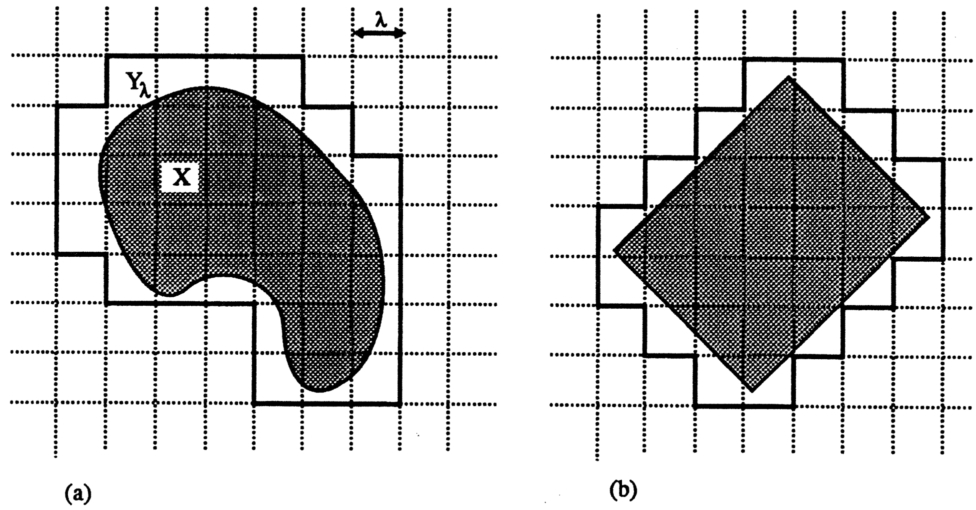

FIGURE 1.1. *"Discretization" of the perimeter*

The example in Fig. 1.1(b) shows that in general, the answer is *no*: in this example it is clear that the perimeter of Y_λ converges to $\sqrt{2}\cdot$(perimeter of X).

The gap between continuous and discrete morphology needs to be bridged by a solid theory of discretization, both of images and of image transformations (or operators as we shall call them here). Such a theory must contain the following three ingredients. First it must explain how to "approximate" a continuous image by a discrete one consisting of at most countably many picture elements (in practice, if the original image is bounded then the discretized image will be finite). In our approach, the approximation will consist of the following two steps. First we introduce a sampling procedure which transforms a continuous image (subset of $\mathbb{R}^d$) into a discrete one (subset of $\mathbb{Z}^d$). Then, in order to be able to compare the discretized image with the original one, we make a representation of the discrete image as a continuous one. Or in formal terms, we embed the discrete images into the space of continuous ones. Secondly, in order to formalize the concept of approximation we need a topology on the space of images. For this purpose we use the hit-or-miss topology which is well-known in the context of mathematical morphology. And finally, the theory must deal with the question under which conditions a continuous morphological operator can be approximated by a sequence of discrete ones. At this point one has to face the following dilemma: should one require that the discretized operator has essentially the same structure as the original one? More specifically, must the discretization of a dilation be again a dilation? It will be clear that such requirements would make our task a considerably more difficult one. In the remainder of this paper we shall speak of Euclidean images and Euclidean operators rather than of continuous images resp. operators, and we will reserve the adjective "continuous" for operators which are continuous in a topological sense; see Sect. 4.

The theory developped in this paper is to a large extent motivated by Chapter VII of Serra's first book [11]. This chapter can be seen as a first attempt to formalize the concept of discretization in the context of mathematical morphology. The principal idea underlying our approach can already be found in this chapter. Our contribution consists hereof that we put our ideas in a rigorous algebraic and topological framework which enables us to develop a concise and rather general theory of discretization, and to avoid some of the ambiguities occurring in [11].

In Sect.2 we shall present a brief overview of the notions from mathematical morphology which we need in the sequel. Particular attention is given to the notion of adjunctions (pairs of operators formed by erosions and dilations) which play an essential role in the rest of this paper. In Sect.3 we discuss the sampling strategy which will form the basis for the discretization procedure discussed here. It turns out that the sampling and representation operator together form an adjunction. A major consequence of this fact is that the reconstruction operator obtained by the composition of these two operators forms a closing. The hit-or-miss topology will be discussed in Sect. 4. At that place we shall also introduce the notion of semi-continuity of a morphological operator which turns out to be important in relation to the question which operators can be discretized. In Sect. 5 and 6 we explain how the sampling and representation procedure of Sect.3 can be used to formalize the discretization of images (Sect.5) and operators (Sect. 6). Sect. 7 is devoted to a discretization procedure, called the *covering discretization*, which has some particularly nice properties. The main feature of this procedure is that the approximations decrease monotonically towards the original image; such a discretization will be called *constricting*. In Sect. 8 we will extend the previous results for image functionals, i.e., mappings from the space of images to the (extended) real line. Finally, in Sect. 9 we summarize our results, and make some final remarks.

2. Basic notions from mathematical morphology

We assume that the reader is acquainted with the basic morphological concepts such as dilations, erosions, closings and openings. For the convenience of the reader we shall recall some of these concepts without any further explanation; a comprehensive account of the theory of mathematical morphology may be found in [11]. Let E be the continuous Euclidean space $\mathbb{R}^d$ or the discrete space $\mathbb{Z}^d$, and denote by $\mathcal{P}(E)$ the family consisting of all subsets of E. Let $X \in \mathcal{P}(E)$ and let $A \subseteq E$ be a structuring element. The dilation and erosion of X by A are respectively defined as

$$\delta_A(X) = X \oplus A = \bigcup_{a \in A} X_a = \{x + a \mid x \in X, a \in A\} \tag{2.1}$$

$$\varepsilon_A(X) = X \ominus A = \bigcap_{a \in A} X_{-a}. \tag{2.2}$$

Here X_a denotes the translate of X along the vector a, i.e., $X_a = \{x+a \mid x \in X\}$. The operators δ_A, ε_A are known to form and adjunction on $\mathcal{P}(E)$: see Def. 2.1 below. Furthermore these are examples of operators which are translation-invariant: an operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is called translation-invariant if $\psi(X_h) = [\psi(X)]_h$ for $X \subseteq E$ and $h \in E$.

In [6,10,12] the complete lattice framework for morphology has been introduced. The main motivation for adopting such an abstract approach is the observation that the complete lattice structure of the image space is a minimal requirement to define a theory of morphology which includes notions as increasingness, dilations, erosions and (anti-)extensivity. A set $\mathcal{L}$ with a partial order $\leq$ is called a complete lattice if every collection $X_i \in \mathcal{L}$, $i \in I$ has a supremum (least upper bound) $\bigvee_{i\in I} X_i$ and an infimum (greatest lower bound) $\bigwedge_{i\in I} X_i$. Examples of complete lattices are $\mathcal{P}(E)$, the closed subsets of $\mathbb{R}^d$, and the functions mapping a set E into the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For other examples we refer to [6,10].

2.1. Definition. Let $\mathcal{L}, \mathcal{M}$ be complete lattices.

(a) The operator $\psi : \mathcal{L} \to \mathcal{M}$ is said to be *increasing* if $X \leq Y$ implies that $\psi(X) \leq \psi(Y)$.

(b) The operator $\varepsilon : \mathcal{L} \to \mathcal{M}$ is called an *erosion* if for every family $X_i \in \mathcal{L}$, $i \in I$ we have $\varepsilon(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \varepsilon(X_i)$.

(c) The operator $\delta : \mathcal{M} \to \mathcal{L}$ is called a *dilation* if for every family $Y_i \in \mathcal{M}$, $i \in I$ we have $\delta(\bigvee_{i\in I} Y_i) = \bigvee_{i\in I} \delta(Y_i)$.

(d) Let $\varepsilon : \mathcal{L} \to \mathcal{M}$ and $\delta : \mathcal{M} \to \mathcal{L}$. Then (ε, δ) is called an *adjunction* between $\mathcal{L}$ and $\mathcal{M}$ if for every $X \in \mathcal{L}$ and $Y \in \mathcal{M}$

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y).$$

(e) The operator $\psi : \mathcal{L} \to \mathcal{L}$ is called an *opening* if ψ is increasing, $\psi(X) \leq X$ for every $X \in \mathcal{L}$, and $\psi^2 = \psi$.

(f) The operator $\psi : \mathcal{L} \to \mathcal{L}$ is called a *closing* if ψ is increasing, $\psi(X) \geq X$ for every $X \in \mathcal{L}$, and $\psi^2 = \psi$.

Here $\psi^2 = \psi \circ \psi$, the operator ψ applied twice. We recall the following basic results about adjunctions. For more details we refer to [6]. If (ε, δ) is an adjunction between the complete lattices $\mathcal{L}$ and $\mathcal{M}$, then ε is an erosion and δ a dilation. For every erosion $\varepsilon : \mathcal{L} \to \mathcal{M}$ there exists a unique dilation $\delta : \mathcal{M} \to \mathcal{L}$ such that (ε, δ) forms an adjunction. And dually, for every dilation $\delta : \mathcal{M} \to \mathcal{L}$ there exists a unique erosion $\varepsilon : \mathcal{L} \to \mathcal{M}$ such that (ε, δ) is an adjunction. If (ε, δ) is an adjunction then ε is called the *left adjoint* of δ, whereas δ is called the *right adjoint* of ε. Furthermore, for every adjunction (ε, δ) between $\mathcal{L}$ and $\mathcal{M}$ the following holds

- $\varepsilon\delta\varepsilon = \varepsilon$, $\quad \delta\varepsilon\delta = \delta$,
- $\delta\varepsilon$ is an opening on $\mathcal{L}$,
- $\varepsilon\delta$ is a closing on $\mathcal{M}$.

3. Sampling and reconstruction

We denote by $\mathcal{F}(\mathbb{R}^d)$ and $\mathcal{G}(\mathbb{R}^d)$ the space of closed resp. open subsets of $\mathbb{R}^d$. Let S be an arbitrary set and $C : S \to \mathcal{G}(\mathbb{R}^d)$. We define the sampling operator $\xi : \mathcal{F}(\mathbb{R}^d) \to \mathcal{P}(S)$ by

$$\xi(X) = \{s \in S \mid C(s) \cap X \neq \emptyset\}. \tag{3.1}$$

Our motivation to restrict to the space of closed sets $\mathcal{F}(\mathbb{R}^d)$ rather than dealing with the entire space $\mathcal{P}(\mathbb{R}^d)$ becomes clear in Sect. 4 where we discuss the hit-or-miss topology. We call S the *sampling set* and C the *sampling function.* Let the reciprocal function $C^* : \mathbb{R}^d \to \mathcal{P}(S)$ be defined as

$$C^*(x) = \{s \in S \mid x \in C(s)\}. \tag{3.2}$$

Then $s \in C^*(x)$ if and only if $x \in C(s)$.

3.1. Lemma. *For every $V \subseteq S$ the set $\{x \in \mathbb{R}^d \mid C^*(x) \subseteq V\}$ is closed.*

PROOF. Let $x_n \to x$ and $C^*(x_n) \subseteq V$. We must show that $C^*(x) \subseteq V$. Assume that $s \in C^*(x)$. We show that $s \in V$. Since $s \in C^*(x)$ we have $x \in C(s)$. Since $C(s)$ is open, $x_n \in C(s)$ eventually. But this implies that $s \in C^*(x_n) \subseteq V$. ∎

We define $\eta : \mathcal{P}(S) \to \mathcal{F}(\mathbb{R}^d)$ by

$$\eta(V) = \{x \in \mathbb{R}^d \mid C^*(x) \subseteq V\}. \tag{3.3}$$

3.2. Proposition. *(η, ξ) defines an adjunction between $\mathcal{P}(S)$ and $\mathcal{F}(\mathbb{R}^d)$.*

PROOF. We show that $\xi(X) \subseteq V \iff X \subseteq \eta(V)$.
"$\Rightarrow$": let $\xi(X) \subseteq V$ and $x \in X$. We must show that $x \in \eta(V)$, i.e., $C^*(x) \subseteq V$. Let $s \in C^*(x)$, then $x \in C(s)$ and so $C(s) \cap X \neq \emptyset$. This yields that $s \in \xi(X)$ hence $s \in V$.
"$\Leftarrow$ ": let $X \subseteq \eta(V)$ and $s \in \xi(X)$. We must show that $s \in V$. From $s \in \xi(X)$ we conclude that $C(s) \cap X \neq \emptyset$. Let $x \in C(s) \cap X$. This implies that $x \in \eta(V)$ and hence that $C^*(x) \subseteq V$. Since $x \in C(s)$ we also have $s \in C^*(x)$ and so $s \in V$. ∎

We have the following alternative characterization of η.

3.3. Proposition. $\eta(V) = \left[\bigcup_{s \in S \setminus V} C(s)\right]^c$.

PROOF. "$\subseteq$": assume that $x \in C(s)$ for some $s \in S \setminus V$, and that $x \in \eta(V)$. Then $C^*(x) \subseteq V$ and, since $s \in C^*(x)$, also $s \in V$ which is a contradiction.
"$\supseteq$": assume that $x \notin \eta(V)$, then $C^*(x) \not\subseteq V$. So there is an $s \in C^*(x)$ such that $s \notin V$. In other words, $x \in C(s)$ for some $s \in S \setminus V$. This yields that $x \notin \left[\bigcup_{s \in S \setminus V} C(s)\right]^c$. ∎

We call ξ the *sampling operator* and η the *representation operator.* We define the reconstruction operator ρ as

$$\rho = \eta\xi. \tag{3.4}$$

From the fact that (η, ξ) is an adjunction we derive immediately that ρ is a closing on $\mathcal{F}(\mathbb{R}^d)$. So we have in particular that

$$X \subseteq \rho(X), \qquad X \in \mathcal{F}(\mathbb{R}^d). \tag{3.5}$$

If $U = \bigcup_{s \in S} C(s)$ is a proper subset of $\mathbb{R}^d$, then $\mathbb{R}^d \setminus U$ is contained in $\rho(X)$, an undesirable situation. From now on we assume that

3.4. Covering Assumption. $\bigcup_{s \in S} C(s) = \mathbb{R}^d$.

Note that this assumption is equivalent to

$$C^*(x) \neq \emptyset, \quad \text{for each } x \in \mathbb{R}^d. \tag{3.6}$$

3.5. Proposition. *Let K be a compact set and assume that for every $s \in S$ there exists an $x(s) \in \mathbb{R}^d$ such that $C(s) \subseteq K_{x(s)}$. Define $L = K \oplus \check{K}$. If Assumption 3.4 is satisfied, then*

$$\rho(X) \subseteq X \oplus L,$$

for every $X \in \mathcal{F}(\mathbb{R}^d)$.

PROOF. Let X be closed, then

$$\begin{aligned}
\rho(X) = \eta\xi(X) &= \{x \in \mathbb{R}^d \mid C^*(x) \subseteq \xi(X)\} \\
&= \{x \in \mathbb{R}^d \mid s \in C^*(x) \implies C(s) \cap X \neq \emptyset\} \\
&= \{x \in \mathbb{R}^d \mid x \in C(s) \implies C(s) \cap X \neq \emptyset\} \\
&\quad [\text{if } \textstyle\bigcup C(s) = \mathbb{R}^d] \\
&\subseteq \textstyle\bigcup\{C(s) \mid C(s) \cap X \neq \emptyset\} \\
&\subseteq \textstyle\bigcup\{K_{x(s)} \mid K_{x(s)} \cap X \neq \emptyset\} \\
&\subseteq \textstyle\bigcup\{K_x \mid K_x \cap X \neq \emptyset\} \\
&= (X \oplus \check{K}) \oplus K = X \oplus L,
\end{aligned}$$

which was to be proved. ∎

Although the theory does not require any further restrictions on C and S besides the Covering Assumption, in practice one usually assumes that S is a regular grid in $\mathbb{R}^d$:

$$S = \{k_1 u_1 + \ldots + k_d u_d \mid k_1, \ldots, k_d \in \mathbb{Z}\},$$

where $u_1, \ldots, u_d$ are linearly independent vectors in $\mathbb{R}^d$. In such cases S is called the *sampling grid.* Furthermore one often assumes that for every grid point $s \in S$, $C(s)$ is obtained by translating some fixed set $C \subseteq \mathbb{R}^d$ along s, in other words, $C(s) = C_s$. In such cases the Covering Assumption can be reformulated as

$$C \oplus S = \mathbb{R}^d. \tag{3.7}$$

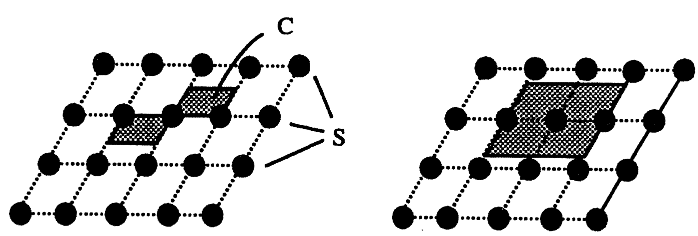

FIGURE 3.1. *Some regular sampling strategies.*

Let $\check{C}$ denote the reflected set $\{-x \mid x \in C\}$. Then $C^*(x) = \check{C}_x \cap S$. In Fig.3.1 we have depicted some regular sampling strategies in the two-dimensional case for which the Covering Assumption is fulfilled. Of particular interest is the case where the sampling element C "inherits" the shape of the grid, that is

$$C = \{x_1 u_1 + \ldots + x_d u_d \mid x_i \in (-a, a)\}$$

where $a > \frac{1}{2}$ in order that (3.7) is satisfied. Fig. 3.2 illustrates the sampling and representation procedure for a regular sampling strategy in the 2-dimensional case.

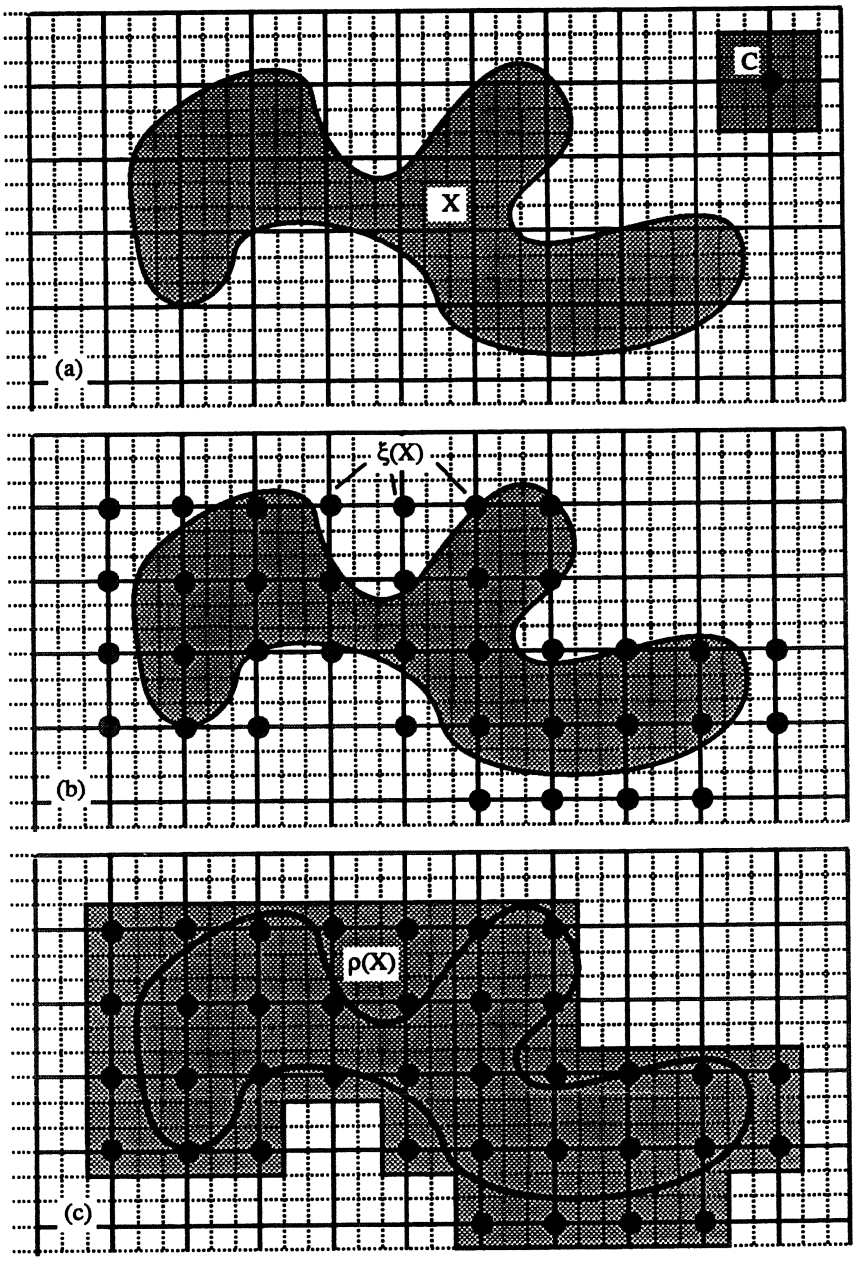

FIGURE 3.2. *Sampling and reconstruction.*
(a) Original set X and sampling element C. (b) The sampled set $\xi(X)$. (c) The reconstructed set $\rho(X)$.

4. Hit-or-miss topology

As we explained in the introduction we need a topology to formalize the concept of discretization. For that purpose we use the hit-or-miss topology which is well-known among researchers working in the field of stochastic geometry or mathematical morphology. In this section all results will be stated without proof. For a comprehensive account on the hit-or-miss topology we refer to the excellent monograph of Matheron [9] which contains a wealth of results concerning this topology and its application to morphological operators and functionals. The hit-or-miss topology is not defined on the entire space $\mathcal{P}(\mathbb{R}^d)$ but only on the subset $\mathcal{F}(\mathbb{R}^d)$, the space of all *closed* subsets of $\mathbb{R}^d$. In the remainder of this paper we shall sometimes use the shorter notation $\mathcal{F}$ instead of $\mathcal{F}(\mathbb{R}^d)$. Furthermore we denote by $\mathcal{G} = \mathcal{G}(\mathbb{R}^d)$ the open subsets of $\mathbb{R}^d$ and by $\mathcal{K} = \mathcal{K}(\mathbb{R}^d)$ the compact subsets. For $A \subseteq \mathbb{R}^d$ we define

$$\mathcal{F}^A = \{X \in \mathcal{F} \mid X \cap A = \emptyset\}$$
$$\mathcal{F}_A = \{X \in \mathcal{F} \mid X \cap A \neq \emptyset\}.$$

The *hit-or-miss topology* is the topology generated by the subbasis $\{\mathcal{F}^K \mid K \in \mathcal{K}\} \cup \{\mathcal{F}_G \mid G \in \mathcal{G}\}$. The basis for this topology can be obtained by taking all finite intersections of subbasis elements and is formed by the sets

$$\mathcal{F}^K_{G_1,G_2,\ldots,G_n} = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \ldots \cap \mathcal{F}_{G_n}, \tag{4.1}$$

where K is compact and $G_1, \ldots, G_n$ are open. It is allowed that $n = 0$ in which case Eq. (4.1) reduces to $\mathcal{F}^K$. Furthermore, if $K = \emptyset$ we get $\mathcal{F}^K_G = \mathcal{F}_G$. It has been shown that the space $\mathcal{F}$ with the hit-or-miss topology is Hausdorff, compact, and has a countable basis: see [9]. The latter also means that sequential convergence is adequate to characterize topological concepts. Let X_n be a sequence in $\mathcal{F}$ and $X \in \mathcal{F}$. Then $X_n \xrightarrow{\mathcal{F}} X$, that is, X_n converges towards X with respect to the hit-or-miss topology, if the following two criteria are satisfied:

(A) if $X \cap G \neq \emptyset$ then $X_n \cap G \neq \emptyset$ eventually, for each open set G,

(B) if $X \cap K = \emptyset$ then $X_n \cap K = \emptyset$ eventually, for each compact set K.

Here "eventually" means "for n sufficiently large". Note that this topology is compatible with the partial order $\subseteq$ in the following sense.

4.1. Proposition. *Let U_n, V_n, X_n be sequences in $\mathcal{F}$. If $U_n \subseteq X_n \subseteq V_n$ and $U_n \xrightarrow{\mathcal{F}} X$, $V_n \xrightarrow{\mathcal{F}} X$, then $X_n \xrightarrow{\mathcal{F}} X$.*

We introduce two different notions of convergence, called *monotone convergence*. These concepts are not a priori related to some topology but make use of the partial ordering "$\subseteq$" on $\mathcal{P}(\mathbb{R}^d)$. Let X_n be subsets in E. We say that

$$X_n \downarrow X \quad \text{if} \quad X_1 \supseteq X_2 \supseteq \ldots \quad \text{and} \quad X = \bigcap_{n\geq 1} X_n,$$

and we pronounce this as "X_n decreases towards X". We say that

$$X_n \uparrow X \quad \text{if} \quad X_1 \subseteq X_2 \subseteq \ldots \quad \text{and} \quad X = \bigcup_{n\geq 1} X_n,$$

and we pronounce this as "X_n increases towards X".

4.2. Proposition. *Let $X_n, X \in \mathcal{F}(E)$ and $Y \in \mathcal{P}(E)$. Then*
(a) $X_n \downarrow X \implies X_n \xrightarrow{\mathcal{F}} X$.
(b) $X_n \uparrow Y \implies X_n \xrightarrow{\mathcal{F}} \overline{Y}$.

Furthermore we introduce the *lower limit* $\underline{\lim}\, X_n$ of a sequence X_n in $\mathcal{F}$ as the largest closed set which satisfies criterion (A). Note that this criterion is easiest satisfied by small sets. On the other hand, criterion (B) has a preference for large sets and we define the *upper limit* $\overline{\lim}\, X_n$ as the smallest closed set which satisfies criterion (B):

$$\underline{\lim}\, X_n = \overline{\bigcup\{F \in \mathcal{F} \mid F \text{ satisfies (A) }\}}$$

$$\overline{\lim}\, X_n = \bigcap\{F \in \mathcal{F} \mid F \text{ satisfies (B) }\}.$$

Let ψ be a mapping from $\mathcal{F}$ into $\mathcal{F}$. We call ψ *upper semi-continuous (u.s.c.)* if $\psi^{-1}(\mathcal{F}^K)$ is open in $\mathcal{F}$ for each $K \in \mathcal{K}$. If, on the other hand $\psi^{-1}(\mathcal{F}_G)$ is open in $\mathcal{F}$ for each $G \in \mathcal{G}$ then we call ψ *lower semi-continuous (l.s.c.)*. The mapping ψ is continuous if it is both u.s.c. and l.s.c. The following results due to Matheron [9, Prop. 1-2-4] provide useful criteria for lower- and upper semi-continuity.

4.3. Proposition. *Let ψ be a mapping from $\mathcal{F}$ into itself.*
(a) ψ is u.s.c. if $X_n \to X$ implies that $\overline{\lim}\, \psi(X_n) \subseteq \psi(X)$.
(b) ψ is l.s.c. if $X_n \to X$ implies that $\psi(X) \subseteq \underline{\lim}\, \psi(X_n)$.

4.4. Proposition. *Let $\psi : \mathcal{F} \to \mathcal{F}$ be an increasing operator. Then ψ is u.s.c. if and only if $X_n \downarrow X$ implies $\psi(X_n) \downarrow \psi(X)$.*

We can also define a topology on $\mathcal{K}$, the compact subsets of $\mathbb{R}^d$. Let $\mathcal{K}^A$ and $\mathcal{K}_A$ be defined as

$$\mathcal{K}^A = \{X \in \mathcal{K} \mid X \cap A = \emptyset\}, \quad \mathcal{K}_A = \{X \in \mathcal{K} \mid X \cap A \neq \emptyset\}.$$

The classes $\mathcal{K}^F$ with F closed, and $\mathcal{K}_G$ with G open, define a topology on $\mathcal{K}$ called the *myope topology*. It can easily be shown that the myope topology is strictly finer than the relative hit-or-miss topology if E is not compact. $\mathcal{K}$ with the myope topology is Hausdorff, locally compact, and has a countable basis. The observation that $\mathcal{K}$ has a countable basis implies among others that the myope topology is completely characterized by sequential convergence. We write $X_n \xrightarrow{\mathcal{K}} X$ to denote that X_n converges towards X with respect to the myope topology. We quote the following results from [9, Sect. 1-4].

4.5. Proposition. *Let X_n be a sequence in $\mathcal{K}$ and $X \in \mathcal{K}$. Then $X_n \xrightarrow{\mathcal{K}} X$ if and only if the following two conditions are satisfied:*
(i) there exists $K \in \mathcal{K}$ such that $X_n \subseteq K$ for each n;
(ii) $X_n \xrightarrow{\mathcal{F}} X$.

4.6. Proposition. *Let X_n be a sequence in $\mathcal{K}$ and $X \in \mathcal{K}$. Then $X_n \downarrow X$ implies that $X_n \xrightarrow{\mathcal{K}} X$.*

Finally we state some continuity results for morphological operators such as dilation, erosion, closing and opening: see [9, Sect. 1-5]. Before doing so we point out that $X \in \mathcal{F}$, $A \in \mathcal{K}$ implies that $X \oplus A, X \ominus A$ are also contained in $\mathcal{F}$. Then by composition it follows that the closing $X \bullet A$ and the opening $X \circ A$ are contained in $\mathcal{F}$ as well. By $\mathcal{F} \times \mathcal{K}$ we denote the product space of $\mathcal{F}$ and $\mathcal{K}$ supplied with the product topology.

4.7. Proposition.

(a) *The mapping* $(X, A) \to X \oplus A$ *from* $\mathcal{F} \times \mathcal{K}$ *into* $\mathcal{F}$ *is continuous.*

(b) *The mapping* $(X, A) \to X \ominus A$ *from* $\mathcal{F} \times \mathcal{K}$ *into* $\mathcal{F}$ *is u.s.c.*

(c) *The mapping* $(X, A) \to X \bullet A$ *from* $\mathcal{F} \times \mathcal{K}$ *into* $\mathcal{F}$ *is u.s.c.*

(d) *The mapping* $(X, A) \to X \circ A$ *from* $\mathcal{F} \times \mathcal{K}$ *into* $\mathcal{F}$ *is u.s.c.*

5. Discretization of sets

By a discretization of a binary image $X \subseteq \mathbb{R}^d$ we mean the approximation of X by a sequence $X_n \subseteq \mathbb{R}^d$, $n \geq 1$, where X_n is a set with a discrete representation.

5.1. Definition. A *discretization* on $\mathcal{F}(\mathbb{R}^d)$ is a collection $\Delta = \{S_n, \xi_n, \eta_n\}_{n \geq 1}$ where S_n is a sequence of countable sets, ξ_n is an operator from $\mathcal{F}(\mathbb{R}^d)$ to $\mathcal{P}(S_n)$ and η_n is an operator from $\mathcal{P}(S_n)$ to $\mathcal{F}(\mathbb{R}^d)$ such that $\eta_n \xi_n(X) \xrightarrow{\mathcal{F}} X$ as $n \to \infty$, for every $X \in \mathcal{F}(\mathbb{R}^d)$.

In practical cases, S_n is taken to be a subset of $\mathbb{R}^d$, e.g. a regular grid. Note that, although $\eta_n \xi_n(X)$ lies in the original space $\mathcal{F}(\mathbb{R}^d)$, the operator $\eta_n \xi_n$ only takes values in a "discrete" subspace of $\mathcal{F}(\mathbb{R}^d)$. In this section we shall describe a discretization procedure based on the sampling strategy discussed in the previous section. Throughout the remainder of this section we make the following assumptions.

5.2. Assumption. Let for $n \geq 1$, S_n be a sampling set and $C_n : S_n \to \mathcal{G}(\mathbb{R}^d)$ be a sampling function such that the following hold:

- for every n the Covering Assumption is satisfied;
- for every n there exists a compact set $K_n \subseteq \mathbb{R}^d$ such that for every $s \in S_n$ there is an $x(s) \in \mathbb{R}^d$ such that $C_n(s) \subseteq (K_n)_{x(s)}$;
- $K_n \xrightarrow{\mathcal{K}} \{0\}$.

These assumptions may seem rather technical at first sight but below we shall see that in the case where S_n is a regular grid and $C_n(s)$ is the translate of some fixed set they are quite natural and easy to check. Before we continue our exposition we state a simple lemma which we shall need in the sequel.

5.3. Lemma. *Let* K_n *be a sequence of sets in* $\mathbb{R}^d$ *which satisfies* $K_n \xrightarrow{\mathcal{K}} \{0\}$. *Let* $L_n = K_n \oplus \check{K}_n$. *Then* L_n *is a compact set, and* $L_n \xrightarrow{\mathcal{K}} \{0\}$.

PROOF. It is obvious that $\check{K}_n \to \{0\}$ both with respect to the hit-or-miss and the myope topology. Thus we conclude from Prop. 4.7(a) that $L_n = K_n \oplus \check{K}_n \to \{0\} \oplus \{0\} = \{0\}$ with respect to the hit-or-miss topology. Since $L_n \subseteq K \oplus \check{K}$, where K is the uniform bound of the sequence K_n, we may apply Prop. 4.5 and conclude that $L_n \to \{0\}$ with respect to the myope topology. ∎

Let ξ_n be the sampling operator given by Eq. (3.1) with $S = S_n$, that is

$$\xi_n(X) = \{s \in S_n \mid C_n(s) \cap X \neq \emptyset\},$$

and let η_n be its left adjoint as given by Eq. (3.3),

$$\eta_n(V) = \{x \in \mathbb{R}^d \mid C_n^*(x) \subseteq V\}.$$

Finally, let ρ_n be the reconstruction operator

$$\rho_n = \eta_n \xi_n.$$

5.4. Proposition. *$\rho_n(X) \xrightarrow{\mathcal{F}} X$ as $n \to \infty$ for every $X \in \mathcal{F}(\mathbb{R}^d)$.*

PROOF. Let $X \in \mathcal{F}(\mathbb{R}^d)$. From Prop. 3.5 we know that

$$X \subseteq \rho_n(X) \subseteq X \oplus L_n.$$

From Lemma 5.3 and Prop. 4.7(a) we conclude that $X \oplus L_n \xrightarrow{\mathcal{F}} X$. Because of Prop. 4.1 this also proves that $\rho_n(X) \xrightarrow{\mathcal{F}} X$. ∎

5.5. Corollary. *The sequence $\Delta = \{S_n, \xi_n, \eta_n\}_{n\geq 1}$ defines a discretization on $\mathcal{F}(\mathbb{R}^d)$.*

Let us, for example, consider the case where S_n is a regular grid. Let $u_1, \ldots, u_d$ be linearly independent vectors in $\mathbb{R}^d$ and let $u_i^n = \lambda_n u_i$, $i = 1, \ldots, d$, where $\{\lambda_n\}$ is a sequence of positive reals converging to zero. Let S_n be the sampling grid

$$S_n = \{k_1 u_1^n + \cdots + k_d u_d^n \mid k_1, \ldots, k_d \in \mathbb{Z}\}.$$

Furthermore let $C_n(s)$ be the translate of the open set $C_n = \{x_1 u_1^n + \cdots + x_d u_d^n \mid x_1, \ldots, x_d \in (-a_n, a_n)\}$ along s. Here $a_n > \frac{1}{2}$ in order that the Covering Assumption is fulfilled. Note that we may also write

$$C_n = (-a_n u_1^n, a_n u_1^n) \oplus \cdots \oplus (-a_n u_d^n, a_n u_d^n).$$

Here $(-a_n u_i^n, a_n u_i^n)$ is the collection of all points $x u_i^n$ with $-a_n < x < a_n$.

In general the convergence of the sequence $\rho_n(X)$ of reconstructions towards X is *not* monotone. One may wonder (and below we shall see that indeed this is a useful question) under what extra conditions one has

$$\cdots \subseteq \rho_{n+1}(X) \subseteq \rho_n(X) \subseteq \rho_{n-1}(X) \subseteq \cdots,$$

and hence that $\rho_n(X) \downarrow X$. In that case the reconstructions $\rho_n(X)$ approach X from above in a monotone fashion and we call Δ a *constricting discretization.* Note that $X \subseteq \rho_n(X)$ is satisfied automatically since every ρ_n is a closing. It turns out that we can give a complete solution to this problem in terms of the following condition.

$$\forall x \in \mathbb{R}^d \; \forall s \in C_n^*(x) \; \exists s' \in C_{n+1}^*(x): \quad C_{n+1}(s') \subseteq C_n(s). \tag{5.1}$$

5.6. Proposition. *The discretization $\Delta = \{S_n, \xi_n, \eta_n\}_{n\geq 1}$ is constricting if and only if condition (5.1) is satisfied.*

PROOF. Recall that $\xi_n(X) = \{s \in S_n \mid C_n(s) \cap X \neq \emptyset\}$ and that $\rho_n(X) = \{x \in \mathbb{R}^d \mid C_n^*(x) \subseteq \xi_n(X)\}$.
"if": let (5.1) hold and suppose that $x \in \rho_{n+1}(X)$. We show that $x \in \rho_n(X)$. This means that we have to show that $s \in C_n^*(x) \implies C_n(s) \cap X \neq \emptyset$. Let $s \in C_n^*(x)$. Then, by (5.1), there is an $s' \in C_{n+1}^*(x)$ such that $C_{n+1}(s') \subseteq C_n(s)$. Now $C_{n+1}(s') \cap X \neq \emptyset$ and therefore $C_n(s) \cap X \neq \emptyset$.
"only if": assume that $\rho_{n+1}(X) \subseteq \rho_n(X)$, let $x \in \mathbb{R}^d$ and $s \in C_n^*(x)$. Define $X := [C_n(s)]^c$. Then $s \notin \xi_n(X)$, so $C_n^*(x) \not\subseteq \xi_n(X)$ and thus $x \notin \rho_n(X)$. This implies that $x \notin \rho_{n+1}(X)$ and so $C_{n+1}^*(x) \not\subseteq \xi_{n+1}(X)$. Thus there must exist an $s' \in C_{n+1}^*(x)$ so that $s' \notin \xi_{n+1}(X)$. The latter means that $C_{n+1}(s') \cap X = \emptyset$, that is $C_{n+1}(s') \subseteq C_n(s)$ which was to be proved. ∎

Section 7 will be entirely devoted to a regular sampling strategy for which condition (5.1) will be satisfied. There the reader can also find illustrations which may help him to capture the underlying idea.

6. Discretization of operators

Now that we know how to approximate Euclidean closed sets by discrete ones we may consider the problem of approximating Euclidean set operators by discrete set operators. The following definition formalizes this problem.

6.1. Definition. Let $\Delta = \{S_n, \xi_n, \eta_n\}_{n\geq 1}$ be a discretization of $\mathcal{F}(\mathbb{R}^d)$, and let ψ be an operator mapping $\mathcal{F}(\mathbb{R}^d)$ into itself. Let ψ_n be a sequence of operators on $\mathcal{P}(S_n)$ such that for every $X \in \mathcal{F}(\mathbb{R}^d)$,

$$\eta_n \psi_n \xi_n(X) \xrightarrow{\mathcal{F}} \psi(X) \text{ as } n \to \infty.$$

Then $\{\psi_n\}_{n\geq 1}$ is called a *discretization of ψ with respect to Δ*, or also a *Δ-discretization of ψ*.

The question if a certain operator is discretizable depends on the discretization Δ of the space $\mathcal{F}(\mathbb{R}^d)$. Our definition allows that ψ is discretizable with respect to one discretization Δ, but not with respect to another Δ'. In Thm. 6.2 below for example it is shown that constricting discretizations allow a larger class of discretizable operators than discretizations which lack this property. Now suppose that $\Delta = \{S_n, \xi_n, \eta_n\}_{n\geq 1}$ is a discretization of $\mathcal{F}(\mathbb{R}^d)$, and let ψ be an operator on $\mathcal{F}(\mathbb{R}^d)$. The diagram below suggests how one may obtain a sequence of discrete operators ψ_n.

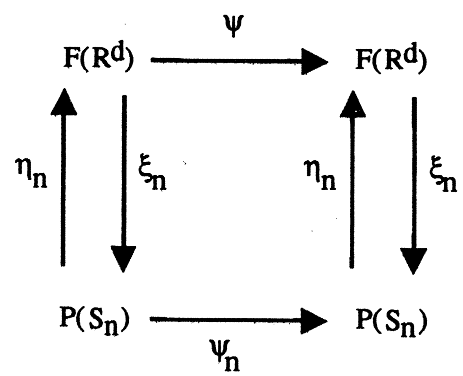

FIGURE 6.1. *Diagram for the discretization of an operator.*

Putting

$$\psi_n = \xi_n \psi \eta_n$$

one finds that

$$\eta_n \psi_n \xi_n = \rho_n \psi \rho_n,$$

where ρ_n is the reconstruction operator $\eta_n \xi_n$.

6.2. Theorem. *Let $\Delta = \{S_n, \xi_n, \eta_n\}_{n\geq 1}$ be a discretization of $\mathcal{F}(\mathbb{R}^d)$, let ψ be an operator on $\mathcal{F}(\mathbb{R}^d)$, and let $\psi_n : \mathcal{P}(S_n) \to \mathcal{P}(S_n)$ be given by*

$$\psi_n = \xi_n \psi \eta_n.$$

Then $\{\psi_n\}_{n\geq 1}$ is a Δ-discretization of ψ in either of the following situations:

(a) ψ is continuous;

(b) Δ is a constricting discretization and ψ is increasing and u.s.c.

PROOF. The proof of (a) is trivial. To prove (b), let $X \in \mathcal{F}(\mathbb{R}^d)$. Then $\rho_n(X) \downarrow X$ and since ψ is u.s.c., $\psi\rho_n(X) \downarrow \psi(X)$. We show that

$$\bigcap_{n\geq 1} \rho_n \psi \rho_n(X) = \psi(X).$$

It is clear that "$\supseteq$" always holds. On the other hand, for $m \geq 1$,

$$\bigcap_{n\geq 1} \rho_n \psi \rho_n(X) \subseteq \bigcap_{n\geq 1} \rho_n \psi \rho_m(X) = \psi\rho_m(X).$$

Now we take the intersection over $m \geq 1$ at the right-hand-side and use the upper semi-continuity of ψ to get that

$$\bigcap_{n\geq 1} \rho_n \psi \rho_n(X) \subseteq \psi(X).$$

This proves the result ∎

Although this theorem provides only a partial answer to the question which morphological operators are discretizable it applies to some of the most important morphological transformations such as the dilation (which is a continuous operator), erosion, closing and opening (which are u.s.c. operators): see Prop. 4.7.

7. A special case: the covering discretization

In this section we consider a regular discretization procedure which has some particularly nice properties including constrictingness: see Sect. 5. Let $u_1, \ldots, u_d$ be independent vectors in $\mathbb{R}^d$ and let S be the grid spanned by these vectors. We take

$$C = (-u_1, u_1) \oplus \cdots \oplus (-u_d, u_d), \tag{7.1}$$

or alternatively $C = \{x_1u_1 + \cdots + x_du_d \mid x_i \in (-1, 1)\}$. It is obvious that $C \oplus S = \mathbb{R}^d$. Furthermore

$$C \cap S = \{0\}. \tag{7.2}$$

Let ξ,η be as before, that is

$$\xi(X) = \{s \in S \mid C(s) \cap X \neq \emptyset\}, \quad X \in \mathcal{F}(\mathbb{R}^d),$$
$$\eta(V) = \{x \in \mathbb{R}^d \mid C^*(x) \subseteq V\}, \quad V \in \mathcal{P}(S).$$

Let $\rho = \eta\xi$ and denote the range of ρ by $\mathcal{F}_\rho$,

$$\mathcal{F}_\rho = \{\rho(X) \mid X \in \mathcal{F}(\mathbb{R}^d)\},$$

which is a subspace of $\mathcal{F}(\mathbb{R}^d)$.

7.1. Lemma. *For $X \in \mathcal{F}_\rho$ the following hold*

(a) $\xi(X) = X \cap S$;

(b) $X = \eta(X \cap S)$.

PROOF. First we observe that for $s \in S$, $C^*(s) = \{s\}$. It is clear that indeed $s \in C^*(s)$. Assume on the other hand that $s' \in C^*(s)$. Then $s \in C_{s'}$, hence $s - s' \in C$. But $s - s' \in S$ and $C \cap S = \{0\}$ which implies that $s = s'$.
(a): let $X \in \mathcal{F}_\rho$, then $X = \rho(X) = \eta\xi(X)$. Thus

$$X \cap S = \eta\xi(X) \cap S = \{s \in S \mid C^*(s) \subseteq \xi(X)\} = \xi(X).$$

(b): let $X \in \mathcal{F}_\rho$, then, by (a), $\xi(X) = X \cap S$. So $\eta\xi(X) = \rho(X) = X = \eta(X \cap S)$. ∎

Note that in this lemma we only use the fact that $C \cap S = \{0\}$. For our next lemma we explicitly need that C is of the form described in Eq. (7.1).

7.2. Lemma. *If $X, Y \in \mathcal{F}_\rho$ then $X \oplus Y, X \ominus Y \in \mathcal{F}_\rho$ as well and*

$$(X \cap S) \oplus (Y \cap S) = (X \oplus Y) \cap S.$$

PROOF. To prove this lemma we first note that for $x, y \in \mathbb{R}^d$,

$$C^*(x + y) \subseteq C^*(x) \oplus C^*(y).$$

We give a sketch of the proof of this fact for the case $d = 2$, but our argument easily carries over to the general case. We only consider the situation where neither x nor y are positioned on one of the grid lines. Then both $C^*(x)$ and $C^*(y)$ consist of four grid points, namely those grid points which are closest to x resp. y. Now $C^*(x) \oplus C^*(y)$ consists of nine grid points, a three by three square, and $x + y$ lies in the convex hull formed by these nine points. Thus it is obvious that $C^*(x + y) \subseteq C^*(x) \oplus C^*(y)$.
We are now ready to prove the lemma. First we observe that the inclusion "$\subseteq$" is trivial and we only prove "$\supseteq$". Let $s \in (X \oplus Y) \cap S$. Hence $s = x + y$ for some $x \in X$ and $y \in Y$. From the fact that $X \in \mathcal{F}_\rho$ and Lemma 7.1(b) we conclude that $x \in \eta(X \cap S)$, that is, $C^*(x) \subseteq X \cap S$. For the same reasons, $C^*(x) \subseteq Y \cap S$. Then

$$\{s\} = C^*(s) = C^*(x + y) \subseteq C^*(x) \oplus C^*(y) \subseteq (X \cap S) \oplus (Y \cap S).$$

This concludes the proof. ∎

We now define a discretization in the following way. Let $S_1 := S$ and $C_1 := C$ and define

$$S_{n+1} := \frac{1}{2}S_n, \quad C_{n+1} := \frac{1}{2}C_n. \tag{7.3}$$

See Fig. 7.1(a) for an example.

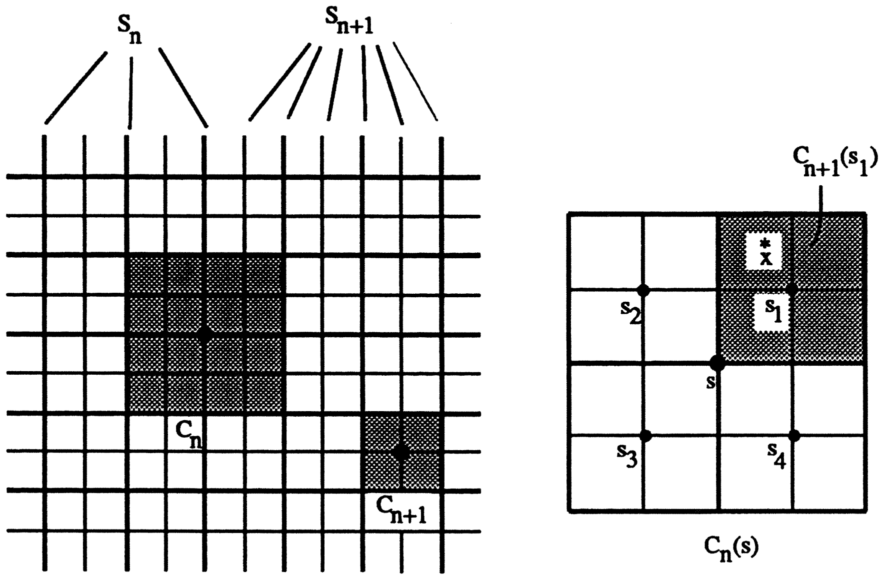

FIGURE 7.1. *(a) A discretization scheme for which the constriction property is satisfied. (b) If $x \in C_n(s)$ then there exists an $s' \in C^*_{n+1}(x)$ such that $C_{n+1}(s') \subseteq C_n(s)$. In the case above $s' = s_1$. See also the text.*

Then Assumption 5.2 is satisfied. Furthermore we can show that also condition (5.1) is satisfied. Namely, let $x \in \mathbb{R}^d$ and take $s \in C^*_n(x)$. We must show that there exists an $s' \in C^*_{n+1}(x)$ such that such that $C_{n+1}(s') \subseteq C_n(s)$. We restrict to $d = 2$ for the sake of exposition. From Fig.7.1(b) it is immediately clear how to choose s' in this case. If x does not lie on one of the grid lines of S_n through s, then x is contained in one of the cells $C_{n+1}(s_1), C_{n+1}(s_2), C_{n+1}(s_3), C_{n+1}(s_4)$. If x does lie on of these grid lines then one has to choose one of the other four $(n+1)$-cells contained within $C_n(s)$.

We point out that that one may also choose a multiplication factor $\frac{1}{p}$ (p a positive integer) in Eq. (7.3) instead of $\frac{1}{2}$ without destroying the validity of condition (5.1). However, for all other multiplication factors condition (5.1) is not satisfied. We call the resulting discretization procedure the *covering discretization.* This discretization has also been discussed by Serra in [11, Chap. VII]. We point out that our definition is different from the one given by Serra, although this distinction is rather subtle. One of the differences consists hereof that in our case the reconstruction operator ρ may yield isolated grid points and grid segments whereas in Serra's approach the discretization consists of closed cells only. In Fig. 7.2 we have applied the covering discretization to a typical set.

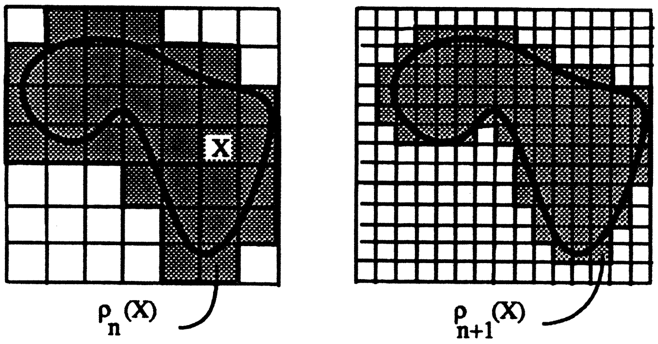

FIGURE 7.2. *Covering discretization.*

The monotonicity property $\rho_1(X) \subseteq \rho_2(X) \subseteq \cdots \subseteq X$ in combination with the properties of the range space $\mathcal{F}_\rho$ mentioned in Lemmas 7.1 and 7.2 enables us to introduce another discretization procedure for increasing translation-invariant operators. As a first step we consider the dilation operator. In contrast to the discretization discussed in the previous section the covering discretization of a dilation is again a dilation.

7.3. Proposition. *Consider the dilation $\delta(X) = X \oplus A$ on $\mathcal{F}(\mathbb{R}^d)$, where $A \subseteq \mathbb{R}^d$ is a compact structuring element. Define the discrete dilation δ_n on $\mathcal{P}(S_n)$ by $\delta_n(V) = V \oplus \xi_n(A)$. Then δ_n is a discretization of δ with respect to the covering discretization of $\mathcal{F}(\mathbb{R}^d)$.*

PROOF. Let δ_n be as above. Then

$$\eta_n \delta_n \xi_n(X) = \eta_n \left[\xi_n(X) \oplus \xi_n(A)\right].$$

Using that $\xi_n \rho_n = \xi_n$ we obtain that

$$\begin{aligned} \eta_n \delta_n \xi_n(X) &= \eta_n \left[\xi_n \rho_n(X) \oplus \xi_n \rho_n(A)\right] \\ &\quad \text{[by Lemma 7.1(a)]} \\ &= \eta_n \left[(\rho_n(X) \cap S_n) \oplus (\rho_n(A) \cap S_n)\right] \\ &\quad \text{[by Lemma 7.2]} \\ &= \eta_n \left[(\rho_n(X) \oplus \rho_n(A)) \cap S_n\right] \\ &\quad \text{[by Lemma 7.1(b)]} \\ &= \rho_n(X) \oplus \rho_n(A). \end{aligned}$$

From the fact that the covering discretization is constricting we get that $\rho_n(X) \downarrow X$ and $\rho_n(A) \downarrow A$, hence $\rho_n(X) \xrightarrow{\mathcal{F}} X$ and $\rho_n(A) \xrightarrow{\mathcal{K}} A$ (see Prop. 4.2(a) and Prop. 4.6), and from Prop. 4.7(a) we conclude that

$$\rho_n(X) \oplus \rho_n(A) \xrightarrow{\mathcal{F}} X \oplus A.$$

Therefore δ_n defines a discretization of δ. ∎

Now consider the operator ψ given by

$$\psi(X) = \bigcap_{A \in \mathcal{A}} X \oplus A, \tag{7.4}$$

where $\mathcal{A}$ is a collection of compact structuring elements. We show that ψ is discretizable with respect to the covering discretization discussed above. Define, for $V \subseteq S_n$,

$$\psi_n(V) = \bigcap_{A \in \mathcal{A}} V \oplus \xi_n(A).$$

Then, using that η_n is an erosion,

$$\begin{aligned}
\eta_n \psi_n \xi_n(X) &= \eta_n\Big(\bigcap_{A \in \mathcal{A}} \xi_n(X) \oplus \xi_n(A)\Big) \\
&= \bigcap_{A \in \mathcal{A}} \eta_n(\xi_n(X) \oplus \xi_n(A)) \\
&= \bigcap_{A \in \mathcal{A}} \rho_n(X) \oplus \rho_n(A).
\end{aligned}$$

Here we have used the same argument as in the proof of the previous proposition. It is obvious that this defines a decreasing sequence and, using Prop. 7.3,

$$\begin{aligned}
\bigcap_{n \geq 1} \eta_n \psi_n \xi_n(X) &= \bigcap_{A \in \mathcal{A}} \bigcap_{n \geq 1} \rho_n(X) \oplus \rho_n(A) \\
&= \bigcap_{A \in \mathcal{A}} X \oplus A \\
&= \psi(X).
\end{aligned}$$

Thus ψ_n defines a discretization of ψ. The operator ψ in Eq. (7.4) is increasing, translation-invariant and u.s.c. (in fact, any intersection of u.s.c. operators is u.s.c.). On the other hand, a theorem due to Matheron [9, Corollary to Prop. 8-2-2] states that an increasing translation-invariant operator $\psi : \mathcal{F} \to \mathcal{F}$ is u.s.c. if and only if it admits the representation (7.4) for a family $\mathcal{A} \subseteq \mathcal{K}$ which is closed with respect to the myope topology. We summarize our conclusions in the following proposition.

7.4. Proposition. *Let $\psi : \mathcal{F} \to \mathcal{F}$ be an increasing translation-invariant u.s.c. operator. Then ψ can be represented as*

$$\psi(X) = \bigcap_{A \in \mathcal{A}} X \oplus \mathcal{A},$$

where $\mathcal{A} \subseteq \mathcal{K}$ is myope-closed. Let $\Delta = \{S_n, \xi_n \eta_n\}$ be the covering discretization of $\mathcal{F}$. Then the operators

$$\psi_n(V) = \bigcap_{A \in \mathcal{A}} V \oplus \xi_n(A)$$

define a Δ-discretization of ψ.

8. Discretization of functionals

Let $\mathcal{L}$ be an arbitrary complete lattice. By a functional on $\mathcal{L}$ we mean a mapping $f : \mathcal{L} \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ is the extended real line $\mathbb{R} \cup \{-\infty, \infty\}$. The functional f is said to be increasing if $X \le Y$ implies that $f(X) \le f(Y)$. For example, if $\mathcal{L} = \mathcal{P}(\mathbb{Z}^2)$, the space of discrete binary 2-dimensional images, then the function f which returns the number of pixels in a set is an increasing functional. It can be regarded as the discrete analogon of the functional on $\mathcal{F}(\mathbb{R}^d)$ which gives the area (Lebesgue measure) of a set. The latter functional is an example of a Minkowski functional: see e.g. [9, Sect. 4-1].

In this section we will show that the discretization procedure for image operators can easily be carried over to image functionals. To do this we need the notion of the lower and upper limit in $\overline{\mathbb{R}}$. But such notions already exist and are known as the "lim inf" and "lim sup". We shall stick to our notation and use "$\underline{\lim}$" and "$\overline{\lim}$" respectively. Let t_n be a sequence in $\overline{\mathbb{R}}$. Then $\underline{\lim}\ t_n$ is the smallest limit point of the sequence t_n, whereas $\overline{\lim}\ t_n$ is the largest limit point of this sequence. A functional f on $\mathcal{F}(\mathbb{R}^d)$ is said to be *l.s.c.* if $X_n \xrightarrow{\mathcal{F}} X$ implies that $f(X) \le \underline{\lim}\ f(X_n)$, and *u.s.c.* if $X_n \xrightarrow{\mathcal{F}} X$ implies that $\overline{\lim}\ f(X_n) \le f(X)$. If f is both l.s.c. and u.s.c. then f is called *continuous*. Consider the following analogue of Prop. 4.4.

8.1. Proposition. *Let f be an increasing functional on $\mathcal{F}(\mathbb{R}^d)$. Then f is u.s.c. if and only if $X_n \downarrow X$ in $\mathcal{F}(\mathbb{R}^d)$ implies that $f(X_n) \downarrow f(X)$.*

For a proof we refer to [9, Prop. 1-2-5]. Following Def. 6.1 we can define the discretization of a functional.

8.2. Definition. Let $\Delta = \{S_n, \xi_n, \eta_n\}_{n \ge 1}$ be a discretization of $\mathcal{F}(\mathbb{R}^d)$, and let f be a functional on $\mathcal{F}(\mathbb{R}^d)$. Let f_n be a sequence of functionals on $\mathcal{P}(S_n)$ such that for every $X \in \mathcal{F}(\mathbb{R}^d)$,

$$f_n \xi_n(X) \to f(X) \text{ as } n \to \infty.$$

Then $\{f_n\}_{n \ge 1}$ is called a *Δ-discretization of f.*

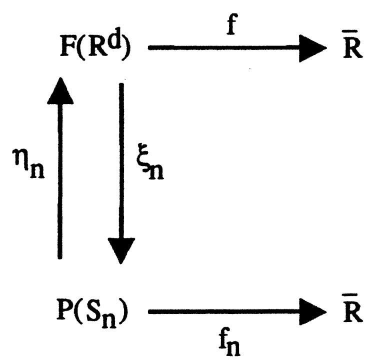

FIGURE 8.1. *Diagram for the discretization of a functional.*

A candidate for f_n is the functional $f\eta_n$. Then

$$f_n \xi_n = f \eta_n \xi_n = f \rho_n.$$

We immediately obtain the following analogon of Thm. 6.2.

8.3. Theorem. *Let $\Delta = \{S_n, \xi_n, \eta_n\}_{n\geq 1}$ be a discretization of $\mathcal{F}(\mathbb{R}^d)$, let f be a functional on $\mathcal{F}(\mathbb{R}^d)$, and let the functional f_n on $\mathcal{P}(S_n)$ be given by*

$$f_n = f\eta_n.$$

Then $\{f_n\}_{n\geq 1}$ is a Δ-discretization of f in either of the following situations:
(a) f is continuous;
(b) Δ is a constricting discretization and f is increasing and u.s.c.

In the introduction we have seen how an attempt to discretize the perimeter gives rise to serious difficulties. Indeed, Thm. 8.3 does not apply to the perimeter since this functional is neither increasing nor continuous. In the literature several approximations of the perimeter can be found. The validity and quality of these estimates depends among others on the regularity of the sets under consideration. This subject, however, falls outside the scope of this paper and we refer the interested reader to [11, Sect. VII.F.2] or [2].

9. Discussion

In this paper we have described a discretization procedure for binary images (sets), image operators and image functionals. This procedure consists of two elements. First we consider a sampling scheme which transforms a Euclidean image into a discrete image. The sampling operator ξ_n chosen in this paper is a dilation, i.e., it acts distributively over unions. The left adjoint η_n of this sampling operator, which is an erosion, can be interpreted as as a representation operator. It defines an embedding of the discrete images into the space of Euclidean images. The composition of both operators, $\rho_n = \eta_n\xi_n$, is a closing and is called the reconstruction operator. The second element of our discretization procedure is the introduction of a topology. In accordance with the common practice of mathematical morphology we use the hit-or-miss topology. A drawback of this choice is that we can no longer work on the space of all Euclidean sets but have to limit ourselves to the space of closed sets. It is possible to consider the discretization problem from the dual point of view. In that case one has to define the sampling operator ξ as

$$\xi(X) = \{s \in S \mid C(s) \subseteq X\}$$

instead of Eq. (3.1). Obviously, ξ has now become an erosion, and the representation operator given by

$$\eta(V) = \{x \in \mathbb{R}^d \mid C^*(x) \cap V \neq \emptyset\},$$

is a dilation. In this case the reconstruction operator $\eta\xi$ is an opening. In this dual approach one should rather work with open than with closed sets, meaning in particular that one has to introduce the analogue of the hit-or-miss topology on $\mathcal{G}(\mathbb{R}^d)$, the open subsets of $\mathbb{R}^d$: see [9, Sect. 1-3].

Our results can readily be extended to grey-level images (functions). In that case one has to work on the space of u.s.c. functions. It is well-known that this space can be endowed with a hit-or-miss like topology; see e.g. [11, Sect. XII.H] or [13]. We do not give any further details here.

In [3], Dougherty and Giardina make estimates of the type

$$\|f(\cdot, X) - f_n(\cdot, X_n)\| \le \tau(X, n),$$

where $f(\lambda, \cdot)$ is a parametrized family of Euclidean functionals (size distributions) with discretizations $f_n(\lambda, \cdot)$, and $\|\cdot\|$ is a norm on the set of all distribution functions $f(\lambda)$, for instance a sup-norm or L^1-norm. Dougherty and Giardina restrict their analysis to size distributions obtained from dilation, erosion or opening by a parametrized family of (convex) structuring elements. Furthermore, they use a discretization very similar to the covering discretization discussed in Sect. 7.

Computer graphics is another field where discretization (or digitization) is of great importance. We refer in particular to the work of van Lierop [8]. The goal there, however, is different from ours. Our primary interest is in Euclidean image transformations and functionals: we regard their discrete versions only as a tool to make computer approximations. Of course, discrete image transformations may be of independent interest themselves; they can for instance be used to clean noisy images or to perform some other image pre-processing task, but this is not the view point taken here. In computer graphics, however, discrete objects form the basic entity, and Euclidean objects may only be of interest as an intermediate step. We illustrate this point by an example. Suppose one wants to define discrete rotations. Let V be some discrete object represented by a set of square pixels. Rotating this set over an angle α which is not a multiple of $\frac{\pi}{2}$, one obtains a set V' which is no longer a set of pixels (unless one rotates the whole computer screen ...). To represent V' as a pixel set one may regard V' as a continuous object, perform some discretization of V', and define the outcome to be the discrete rotation over the angle α. In our notation a discrete rotation θ_α^d could be defined as

$$\theta_\alpha^d(V) = \xi\theta_\alpha\eta(V), \tag{9.1}$$

where θ_α denotes the continuous rotation.

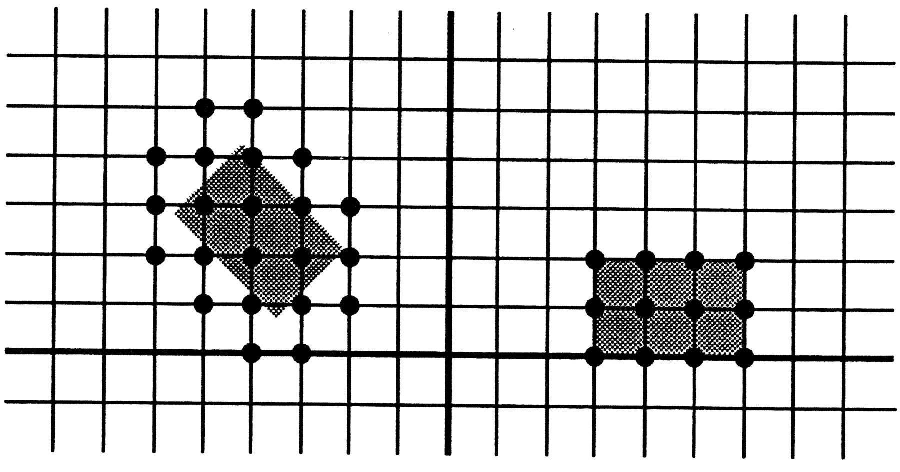

FIGURE 9.1. *Discrete rotation* $\theta_\alpha^d = \xi\theta_\alpha\eta$. *The discretization scheme is the one depicted in Fig. 7.1.*

We refer to Fig. 9.1 for an illustration. The pixel set at the right (black dots) is our original image V. The continuous set in shaded grey represents its continuous representation $\eta(V)$. This set $\eta(V)$ is rotated over an angle α (135° in our example), and finally the outcome is discretized again (black dots at the left).

Unfortunately, discrete rotations lack most of the properties of their continuous counterparts. They may destroy connectedness as well as disconnectedness and they may change the homotopy of a set. Furthermore, discrete rotations as defined in Eq. (9.1) do not have the semigroup property

$$\theta^d_\beta \circ \theta^d_\alpha = \theta^d_{\alpha+\beta}.$$

Note that the definition in Eq. (9.1) would yield $\theta^d_\beta \circ \theta^d_\alpha = \xi\theta_\beta\rho\theta_\alpha\eta$, and in fact, the term ρ in the middle of this expression has the effect of an enlargement of the original set. Starting with one pixel and performing 360 rotations over 1 degree may result in a set containing many elements instead of returning the one pixel.

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