1992

J.R. Kennaway

On transfinite abstract reduction systems

Computer Science/Department of Software Technology Report CS-R9205 January

CWI is the research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a non-profit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands organization for scientific research (NWO).

On Transfinite Abstract Reduction Systems

J.R. KENNAWAY

School of Information Systems, University of East Anglia,
Norwich NR4 7TJ, United Kingdom

Email: jrk@sys.uea.ac.uk

ABSTRACT We generalise the notion of an abstract reduction system to allow transfinite reductions: reduction sequences of any ordinal length. We find that while typical properties of abstract reduction systems can be extended in a natural way to the transfinite case, most of the standard theorems relating them no longer hold.

1991 Mathematics Subject Classification: 68Q50, 68Q55.
1991 CR Categories: F.1.1, F.4.1, F.4.2.
Keywords and Phrases: Abstract reduction system, transfinite reduction.

Note: This work was partially supported by ESPRIT Basic Research Action no. 3074 (Semagraph), and by a SERC Advanced Fellowship and SERC Research Grant GR/F91582. This report was completed during a visit by the author to CWI (Centre for Mathematics and Computer Science), Amsterdam, whose hospitality is gratefully acknowledged.

1. Introduction

Several properties and theorems about term rewrite systems and lambda calculus can be expressed in terms of abstract reduction systems, which consist of just a set of uninterpreted objects and a binary relation on them. A term rewrite system gives an abstract reduction system in which the objects are the terms, and t is related to t' iff t reduces to t' in one step. Certain theorems, such as Newman's lemma (stating that if a system is strongly normalising and weakly Church-Rosser, then it is Church-Rosser) can be proved for abstract reduction systems, as they are independent of the mechanism of rewriting. Several such results are proved in [Klo80].

This note considers the possibility of defining a notion of transfinite abstract reduction system, to stand in the same relation to transfinite term rewriting [Ken91] as abstract reduction systems do to finitary term rewriting. We introduce an ultrametric on the set of objects and a measure on the reduction steps, by analogy with the ultrametric on the space of finite and infinite terms and a measure of depth of a redex in a term. The notion of strongly convergent reduction sequence from [Ken91] then carries over to the abstract setting. We find that while the concepts of strong normalisation, Church-Rosser property, etc. have natural generalisations to the transfinite case, most of the known theorems relating these properties no longer hold. We then consider some further axioms that might be imposed on transfinite abstract reduction systems.

Report CS-R9205
ISSN 0169-118X
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

DEFINITION. An *abstract reduction system* (or *ARS*) is a set A, and a collection { $\rightarrow_s$ | s in S } of binary relations on A, indexed by some set S. We write $\rightarrow$ for the union of all the relations.

DEFINITION. A *metric abstract reduction system* (or *MARS*) is an ARS (A,{ $\rightarrow_s$ | s in S }), such that

(i) A is a complete ultrametric space, that is, a complete metric space whose distance measure **d** satisfies the following stronger form of the triangle inequality: $\mathbf{d}(a,c) \le \max(\mathbf{d}(a,b), \mathbf{d}(b,c))$.

(ii) S is a set of positive reals.

(iii) if a $\rightarrow_r$ b then $\mathbf{d}(a,b) \le r$.

A *reduction step* of the system is an instance of one of its relations a $\rightarrow_r$ b.

Imagine A as being a plane, over which jumps an insect. If the insect can jump from a to b, this means that a $\rightarrow_r$ b for some r. r is the length of the insect's trajectory. This must be at least as large as the distance from a to b. See Figure 1.

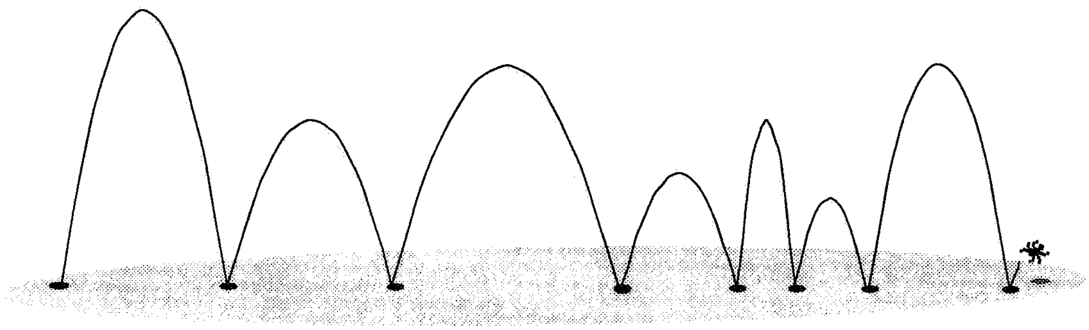

Figure 1. Reduction sequence.

The next definition uses the analogy of the jumping insect.

DEFINITION. The *height* of a reduction step a $\rightarrow_s$ b of a MARS is s; the *length* of the step is $\mathbf{d}(a,b)$.

DEFINITION. A *transfinite sequence* of elements of a set A is a function from some ordinal to A. The ordinal is the *length* of the sequence.

DEFINITION. A *continuous* transfinite sequence of elements of a metric space A is a continuous function from some ordinal to A (with the usual topology on the ordinal, i.e. limit ordinals are exactly the topological condensation points).

DEFINITION. An *open transfinite reduction sequence* in an ARS A is a transfinite sequence of reduction steps $a_i \rightarrow_{s_i} b_i$ (i in some ordinal α) of A, such that for all applicable i, $b_i = a_{i+1}$. Its *length* is α. A *closed transfinite reduction sequence* is either an open sequence of length α (where α is a successor ordinal), or an open sequence of length λ (a limit ordinal) together with an object a_λ. In either case its *length* is the length of the underlying open sequence.

Note that the number of objects in a closed sequence of length α is $\alpha+1$, and the number of steps is α, whether α is a successor or a limit ordinal. For an open sequence, the number of steps is always α, but the number of objects is either $\alpha+1$ (if α is a successor) of α (if α is a limit).

DEFINITION. A (closed or open) transfinite reduction sequence as above is *weakly continuous* if the sequence { $a_i \mid i \in \alpha$ } of elements of A is continuous, considered as a function from the ordinal α (with the usual ordinal topology) to A.

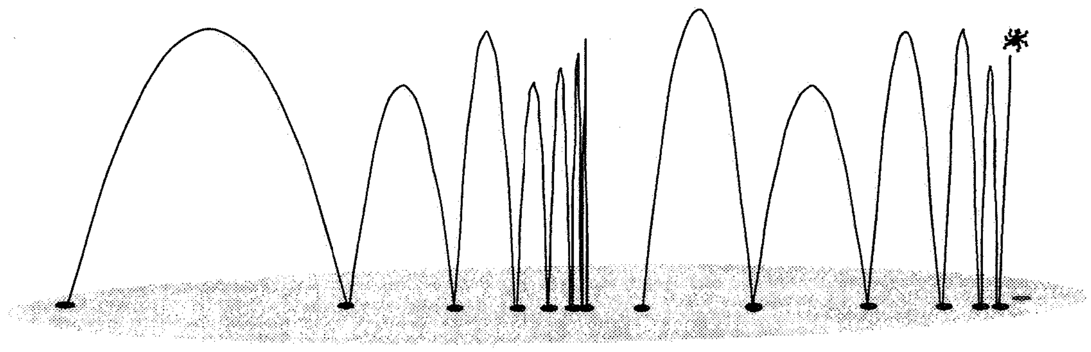

Figure 2: Discontinuous sequence.

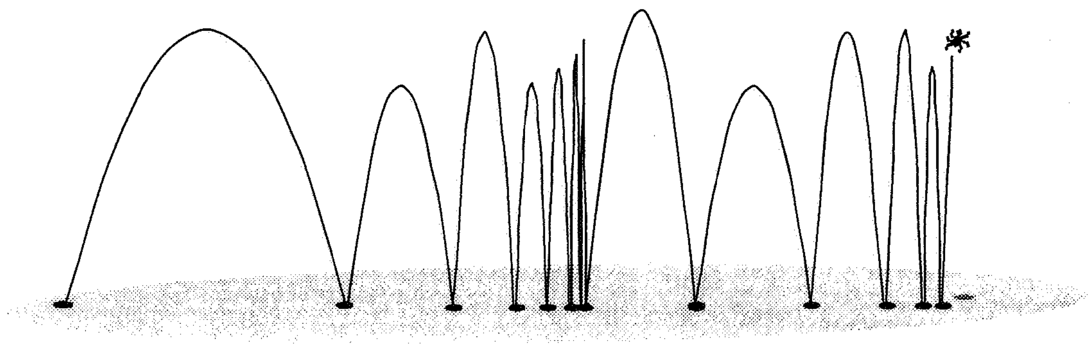

Figure 3: Weakly continuous sequence.

DEFINITION. A transfinite reduction sequence is *strongly continuous* if it is weakly continuous, and the sequence { s_i | $i \in \alpha$ } of positive reals converges to zero at each limit ordinal $\leq \alpha$. We write $a \rightarrow^{\infty} b$ to denote a strongly continuous reduction sequence from a to b, or to assert that some such sequence exists.

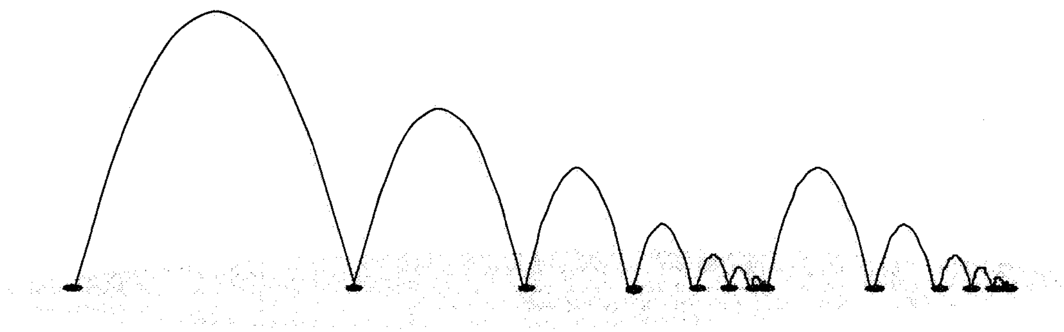

Figure 4: Strongly continuous sequence.

The next proposition states an important but subtle technicality implicit in these definitions.

PROPOSITION. A strongly continuous sequence converges to a limit; a weakly convergent reduction sequence need not.

PROOF. This is a consequence of the property of ultrametric spaces, that if the distances between successive members of a sequence tend to zero, the sequence converges. (We thank Jan Vis of the University of Nijmegen for pointing out that this proposition depends on the space being ultrametric, and not merely metric.) □

THEOREM. In any strongly continuous reduction sequence, for any real number $d>0$, there are at most finitely many steps of the sequence whose height is greater than d.

PROOF. Suppose that for some sequence and some d there are infinitely many such steps. Consider the least upper bound of the set of indexes of these steps. This must be a limit ordinal λ, hence by strong continuity there must be some $\alpha<\lambda$ such that every step between α and λ has height less than or equal to d. Hence α is an upper bound on the indexes of steps of height greater than d. But λ was assumed to be the least upper bound, contradiction. □

THEOREM. A strongly continuous reduction sequence has countable length.

PROOF. Consider a countable sequence of positive reals tending to zero. For each real d in the sequence, there is only a finite set of steps of a given strongly continuous reduction sequence of height greater than d. The union of these finite sets, for all members of the sequence of reals, must be the set of all steps of the reduction sequence. But a countable union of finite sets is countable. □

COUNTEREXAMPLE. A weakly continuous reduction sequence may have any ordinal length.

PROOF. Consider the MARS with a single object •, and a single reduction relation given by $\bullet \rightarrow_1 \bullet$. For any ordinal α, the function from α to the single possible reduction step of this system defines a weakly continuous sequence. □

PROPOSITION. A term rewrite system can be viewed as a MARS by defining the height of a reduction step to be $2^{-\text{depth of redex}}$. Then the weakly, resp. strongly continuous sequences are the same as the weakly convergent, resp. strongly convergent term rewriting sequences defined in [Ken91].

2. Transfinite counterparts of existing concepts

In this section we shall consider a collection of concepts which have been defined for abstract reduction systems with finite reduction, and generalise them to the transfinite case.

DEFINITION. $A \models a=b \Leftrightarrow$ a is related to b by the equivalence closure of $\rightarrow^\infty$. Equivalently, $A \models a=b \Leftrightarrow$ there is a finite sequence a_0 R a_1 R … R a_n, such that $a_0 = a$, $a_n = b$, and R is the relation $\rightarrow^\infty \cup (\rightarrow^\infty)^\dagger$, where $\dagger$ is transposition: $a\ S^\dagger\ b \Leftrightarrow b\ S\ a$.

Note that we require that we can get from a to b by a *finite* sequence of steps of the form $a_i \rightarrow^\infty a_{i+1}$ or $a_{i+1} \rightarrow^\infty a_i$. We do not allow the taking of limits here. In particular, note that R is a proper subrelation of $(\rightarrow \cup \leftarrow)^\infty$. The reason for this restriction is that without it, the equality relation has undesirable properties for the case of orthogonal term rewrite systems. For example, consider the following set of rewrite rules:

$$F(0) = 0$$
$$F(1) = 1$$

We have the following "backwards infinite" sequences:

$$0 \leftarrow F(0) \leftarrow F(F(0)) \leftarrow F(F(F(0))) \leftarrow \ldots F(F(F(F(\ldots))))$$
$$1 \leftarrow F(1) \leftarrow F(F(1)) \leftarrow F(F(F(1))) \leftarrow \ldots F(F(F(F(\ldots))))$$

Thus 0 and 1 can both be reduced to the same infinite term F^ω, if backwards infinite rewriting is allowed. Note also that 0, 1, and F^ω are normal forms — none contains a redex. A concept of equality that made all three "equal" does not appear useful. Notice that this implies that the equivalence classes of the '=' relation are in general not topologically closed.

DEFINITION.

(0) a has a normal form if there is a b such that $a \to^\infty b$.

(1) SN is the property: every reduction sequence is finite.

SN^∞ is the property: every weakly continuous reduction sequence is strongly continuous.

(2) WCR is the property: $\forall a,b,c.\ a \to b \wedge a \to c \Rightarrow \exists d.\ a \to^* d \wedge b \to^* d$.

WCR^∞ is the property: $\forall a,b,c.\ a \to b \wedge a \to c \Rightarrow \exists d.\ a \to^\infty d \wedge b \to^\infty d$.

(3) CR is the property: $\forall a,b,c.\ a \to^* b \wedge a \to^* c \Rightarrow \exists d.\ a \to^* d \wedge b \to^* d$.

CR^∞ is the property: $\forall a,b,c.\ a \to^\infty b \wedge a \to^\infty c \Rightarrow \exists d.\ a \to^\infty d \wedge b \to^\infty d$.

(4) WN is the property: $\forall a\, \exists b.\ a \to^* b \wedge b$ is a normal form.

WN^∞ is the property: $\forall a\, \exists b.\ a \to^\infty b \wedge b$ is a normal form.

(5) NF is the property: $\forall a,b.$ (a is a normal form $\wedge$ a=b) $\Rightarrow b \to^* a$.

NF^∞ is the property: $\forall a,b.$ (a is a normal form $\wedge$ a=b) $\Rightarrow b \to^\infty a$.

(6) Given two reduction relations $\to_\alpha$ and $\to_\beta$, $\to_{\alpha\beta}$ is their union.

$PP_{\alpha,\beta}$ is the property: $a \to^*_{\alpha\beta} b \Rightarrow \exists c.\ a \to^*_\alpha c \to^*_\beta b$.

$PP^\infty_{\alpha,\beta}$ is the property: $a \to^\infty_{\alpha\beta} b \Rightarrow \exists c.\ a \to^\infty_\alpha c \to^\infty_\beta b$.

(7) PE is the property $PP_{\to,\leftarrow}$.(Postponement of expansions after contractions.)

PE^∞ is the following property: Let R be the relation $(\to^\infty \cup \leftarrow^*)^*$ Then a R b $\Rightarrow \exists c.\ a \to^\infty c \leftarrow^* b$. (Postponement of finite expansions after transfinite contractions.)

(8) Given two reduction relations α and β, $\alpha \otimes^\infty \beta$ is the property: $a \to_\alpha b \wedge a \to_\beta c \Rightarrow \exists d.\ b \to^\infty_\beta d \wedge c \to^\infty_\alpha d$. ($\alpha$ *weakly commutes with* β.)

(9) α *commutes with* β if $\to^\infty_\alpha \otimes \to^\infty_\beta$.

(10) UN^∞ is the property: (a and b are normal forms $\wedge$ a=b) $\Rightarrow$ a≡b.

$PP^\infty_{\alpha,\beta}$ is a straightforward adaptation of the $PP_{\alpha,\beta}$ property of [Klo80]. However, the particular case of $PP_{\alpha,\beta}$ used in [Klo80] takes β to be $\leftarrow$. In view of the discussion above of the pathological nature of backwards infinite sequences, PE^∞ appears to be a more appropriate transfinite generalisation of that instance of $PP_{\alpha,\beta}$.

One might expect the transfinite generalisation of SN to merely require that every maximal reduction sequence end with a normal form. However, SN^∞ as defined above seems more appropriate as the transfinite version of SN. It embodies the idea that if you start from an object and construct a reduction sequence in any way you like, you cannot avoid converging to a normal form. The next proposition shows that this implies that every object does have a normal form.

It may seem surprising that SN^∞ as defined above makes no explicit reference to normal forms. The following proposition makes the connection clearer.

PROPOSITION. SN^∞ implies that every object has a normal form.

PROOF. Suppose some object a has no normal form. Construct a strongly convergent transfinite reduction sequence from a by induction:

- Given a closed strongly convergent reduction sequence of length α starting from a and ending with b, extend it by one step $b \to c$ (which must be possible, since a has no normal form).

- Given an open strongly convergent reduction sequence of length λ (a limit ordinal), close it by appending its limit (which exists by strong convergence).

This defines a weakly continuous (hence by SN^∞, strongly continuous) reduction sequence starting from a, and of length not bounded by any ordinal. But such a sequence has at most countable length, contradiction. Therefore a has a normal form. □

The converse is not true. The following picture of a MARS provides a counterexample:

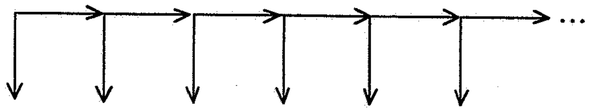

Figure 5. Every element has a normal form, but $\neg SN^\infty$.

In pictures such as these, we adopt the following conventions:

- the length of an arrow is the height of the reduction step it represents
- the distance between two objects in the plane is their distance according to the metric
- there is an object at each end of every arrow, and wherever else indicated by a heavy dot or a letter
- when a set of objects is drawn so as to suggest an accumulation point, there is an object at that accumulation point only if it is at either end of some arrow, or is marked by a dot or letter.

THEOREM (cf. prop. I.5.3 of [Klop80]). Consider the following properties which a MARS may have:

(i) $\to$ is CR^∞

(ii) $\twoheadrightarrow^\infty$ is WCR

(iii) $\twoheadrightarrow^\infty$ is self-commuting

(iv) $\twoheadrightarrow^\infty$ is $WCR^{\leq 1}$

(va) $PP^\infty_{\twoheadrightarrow,\leftarrow}$

(vb) PE^∞

(vi) PM^∞

(vii) $\forall a,b.\ \exists c.\ a=b \Rightarrow a \twoheadrightarrow^\infty c \wedge b \twoheadrightarrow^\infty c$

Then the following implications, and only these, hold among them:

(i), (ii), (iii), (iv), (vb), and (vii) are equivalent.

(va) $\Rightarrow$ { (i), (ii), (iii), (iv), (vb), (vii) } $\Rightarrow$ (vi)

PROOF. The implications are all routine.

Figure 6 provides a counterexample to (vi) $\Rightarrow$ (i), and Figure 7 to (i) $\Rightarrow$ (va). □

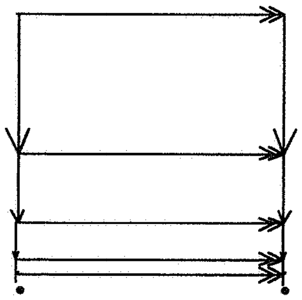

Figure 6. $PM^{\infty} \not\Rightarrow CR^{\infty}$.

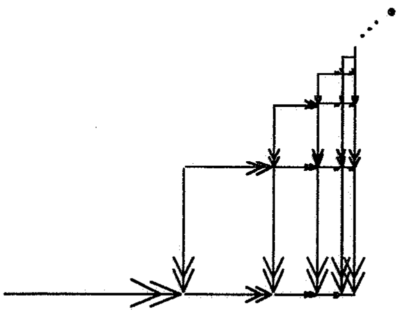

Figure 7. $CR \not\Rightarrow PP^{\infty}_{\twoheadrightarrow,\leftarrow}$.

THEOREM. (cf. Lemma I.5.7 of [Klop80]) (1) (failure of Newman's lemma) $SN^{\infty} \wedge WCR^{\infty} \not\Rightarrow CR^{\infty}$.

(2) $WN^{\infty} \wedge UN^{\infty} \Rightarrow CR^{\infty}$

(3) $WCR^{\leq 1} \not\Rightarrow CR^{\infty}$.

(4) (failure of Hindley-Rosen lemma) Let $\rightarrow\ = \ \rightarrow_{\alpha_1} \cup \rightarrow_{\alpha_2}$. Let $\alpha_i \otimes^{\infty} \alpha_j$ for $i,j \in \{1,2\}$. Then $\rightarrow$ is CR^{∞}.

PROOF.

(1) Figure 8 shows a counterexample. Note that the right hand side of this diagram is not a reduction sequence; the objects at the top right and bottom right corners are both normal forms.

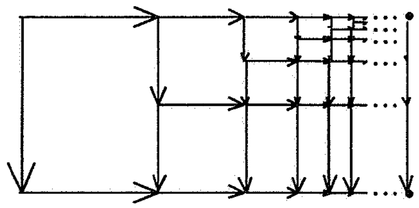

Figure 8. $SN^{\infty} \wedge WCR^{\infty} \not\Rightarrow CR^{\infty}$.

(2) Let $a \rightarrow^{\infty} b$ and $a \rightarrow^{\infty} c$. By WN^{∞}, b and c have normal forms b' and c' respectively. These are normal forms of a. By UN^{∞}, b'=c'. Hence CR^{∞}. (This is the same proof as for the finite case.)

(3) Figure 9 provides a counterexample. The system is $WCR^{\leq 1}$, but b and c are distinct normal forms of a.

(4) In Figure 9, take the first, third, fifth, etc. arrows in each horizontal and vertical row to be $\rightarrow_{\alpha}$, and the remainder to be $\rightarrow_{\beta}$. □

We see that many useful relations among properties of abstract reduction relations fail in the infinite case. These properties, however, hold for orthogonal TRSs and TGRSs. We may wonder whether they hold for non-orthogonal TRSs. The answer is no.

A counterexample to Newman's lemma is provided by the following system:

Rules: $F(x) \rightarrow A(F(x))$ $\quad F(B) \rightarrow B$ $\quad A(B) \rightarrow B$

Starting term: $F(B)$

The system is SN^{∞} and WCR^{∞} (indeed, it is $WCR^{\leq 1}$), but not CR^{∞}. The reduction graph of this term is similar to the abstract system pictured in the proof of part 1. F(B) has the two normal forms A^ω and B.

The same system is also an example of part 3 of the theorem.

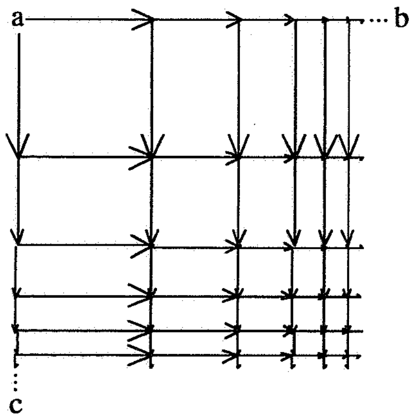

Figure 9. $WCR^{\leq 1} \nRightarrow CR^\infty$.

3. Further axioms.

We have seen that many useful relations among properties of reduction relations fail in the infinite case. This is not due to the freedom to construct pathologies that one has in abstract systems, as they are manifested even in quite simple, though non-orthogonal, term rewrite systems. Nonetheless, it does suggest that we are yet missing some important well-behavedness property, imposing which would avoid these pathologies. Such a property should guarantee that when constructing infinite tiling diagrams such as the figures above, we can in some way take a limit of a sequence of sequences — the successive vertical sequences of the last diagram, for example. But it must also be weak enough that it is satisfied by more than just the orthogonal TRSs and TGRSs.

We begin by considering a norm and a metric on the space of reduction sequences. Let $\mathbf{a} = (\{a_i \mid i \leq \alpha\}, \{ a_i \rightarrow_{s_i} a_{i+1} \mid i < \alpha \})$, $\mathbf{b} = (\{b_i \mid i \leq \alpha\}, \{ b_i \rightarrow_{t_i} b_{i+1} \mid i < \beta \})$ be two reduction sequences.

Define a norm on reduction sequences: $|\mathbf{a}|$, the norm of **a**, is the diameter of the set of terms in **a**, or the height of the largest step of **a**, whichever is larger.

PROPOSITION. Let $\{s_i \mid i < \omega\}$ be a sequence of strongly continuous closed reduction sequences, such that the final term of s_i is the initial term of s_{i+1} for all i. If the sum $\sum |s_i|$ is convergent, then the concatenation of the sequences is strongly continuous.

PROOF. If the sum is convergent then the sequence of endpoints of the component sequences is a Cauchy sequence, hence has a limit. In addition, the norms $|s_i|$ must tend to zero, hence the heights of the reduction steps in the concatenated sequence must tend to zero. □

The metric we define on reduction sequences is a special case of a metric on the set of closed subsets of a metric space. Given a metric space X with metric **d**, and two subsets A and B, define

$$\mathbf{d}(A,B) = \max(\sup \{ \inf(\{ \mathbf{d}(a,b) \mid a \in A \}) \mid b \in B \}, \sup \{ \inf(\{ \mathbf{d}(a,b) \mid b \in B \}) \mid a \in A \})$$

Note that this is not necessarily equal to the minimum distance between points of the two sets. A geometric example may assist the intuition. The distance between the two arcs in Figure 10, according to the above metric, is the maximum vertical distance between them.

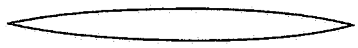

Figure 10. Example of the metric on sequences.

The distance between two reduction sequences is then defined as the distance between their sets of terms.

DEFINITION. CS (completeness of the space of sequences) is the following property: the set of closed reduction sequences with the metric defined above is complete. □

Note that CS is violated by the non-orthogonal TRS we used as a counterexample to Newman's lemma and other properties. The vertical reduction sequences of the diagram form a Cauchy sequence, but it has no limit. Unfortunately for the usefulness of the CS property, it is also violated by orthogonal systems.

COUNTEREXAMPLE. The following orthogonal TRS does not satisfy CS.

Rule: $F(x) \to x$

Sequences: $F^i(A) \to F^{i-1}(A) \to \dots \to F(A) \to A$ in i steps ($i \geq 0$), each of which reduces the innermost F.

The distance between the i'th and j'th sequence is $2^{-\min(i,j)}$. The sequence of sequences is therefore a Cauchy sequence. But it has no limit in the space of sequences.

The problem here is not that the rule is a "collapsing" rule, since we can easily find a counterexample using only non-collapsing rules.

Rule: $F(A(x)) \to A(x)$

Sequences: $F(A^i(B)) \to F(A^{i-1}(B)) \to \dots \to F(A(B)) \to F(B)$ in i steps ($i \geq 0$).

The distance between the i'th and j'th sequence is $2^{-\min(i,j)-1}$. Again we have a Cauchy sequence of sequences with no limit.

The above counterexamples work as TGRSs as well.

However, suppose we consider *non-decreasing* TRSs — that is, TRSs where in each rule, each variable on the left hand side can appear on the right hand side only at depths greater than or equal to its depth on the left hand side.

CONJECTURE. Every non-decreasing OTRS and OTGRS has the property CS.

So perhaps there are interesting CS systems. Does the CS property help us to extend finitary theorems to the transfinite case? We shall consider Newman's lemma, $WCR^{\leq 1} \Rightarrow CR$, and the Hindley-Rosen lemma. The first two fail; the status of the third is not known.

THEOREM. (1) (Infinitary Newman's lemma.) $SN^\infty \wedge WCR^\infty \wedge CS \not\Rightarrow CR^\infty$.

(2) $WCR^{\leq 1} \wedge CS \not\Rightarrow CR^\infty$.

PROOF.

(1) Figure 11 provides a counterexample. This diagram is constructed recursively from the hyperbola-like diagram of Figure 12 (which does not even need to have infinite arms) by repeatedly adding a copy of itself to each of its concavities, and adding all accumulation points. The resulting TARS is clearly SN^∞

and WCR^∞. It is also CS, since in fact there are no “interesting” limits of sequences of reduction sequences. However, the top left object has uncountably many normal forms. (Uncountability is not the problem — if we generate a diagram from a “one-armed hyperbola” instead, we get another counterexample, with only countably many objects.)

In fact, we can improve this, by strengthening the hypothesis WCR^∞ to $WCR^{(\le 1, \le 2)}$, where by $WCR^{(\le p, \le q)}$ is meant the property that when a→b and a→c, there is a d such that either b $\to^{\le p}$ d and c $\to^{\le q}$ d or b $\to^{\le q}$ d and c $\to^{\le p}$ d. Figure 13 is a counterexample to $SN^\infty \wedge WCR^{(\le 1, \le 2)} \wedge CS \Rightarrow CR^\infty$. It is constructed from an infinite number of copies of the building block of Figure 14.

(2) Figure 15 provides a counterexample. □

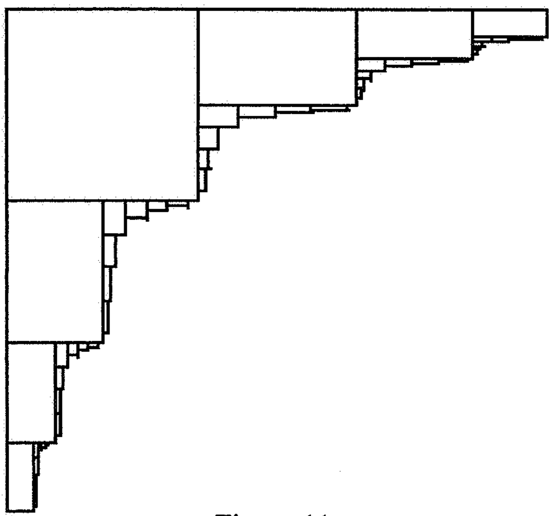

Figure 11.
$SN^\infty \wedge WCR^\infty \wedge CS \not\Rightarrow CR^\infty$.

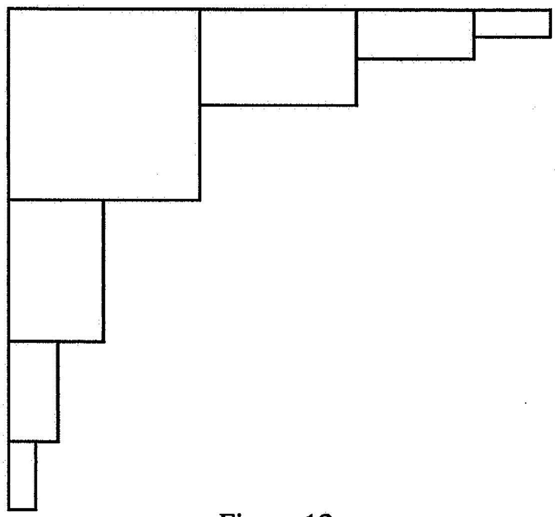

Figure 12.
Building block for Figure 11.

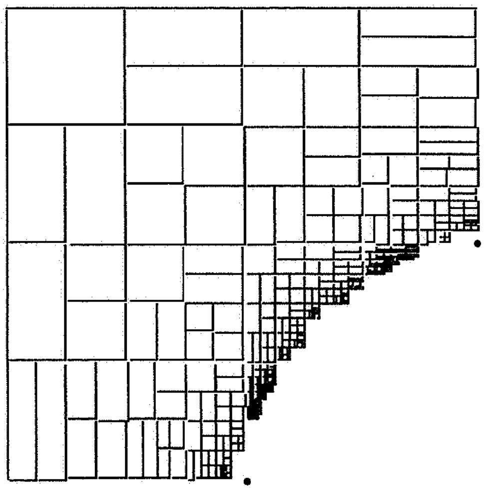

Figure 13.
$SN^\infty \wedge WCR^{(\le 1, \le 2)} \wedge CS \not\Rightarrow CR^\infty$

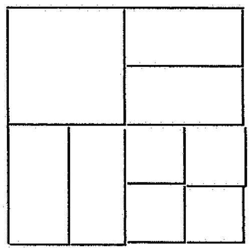

Figure 14.
Building block for Figure 13

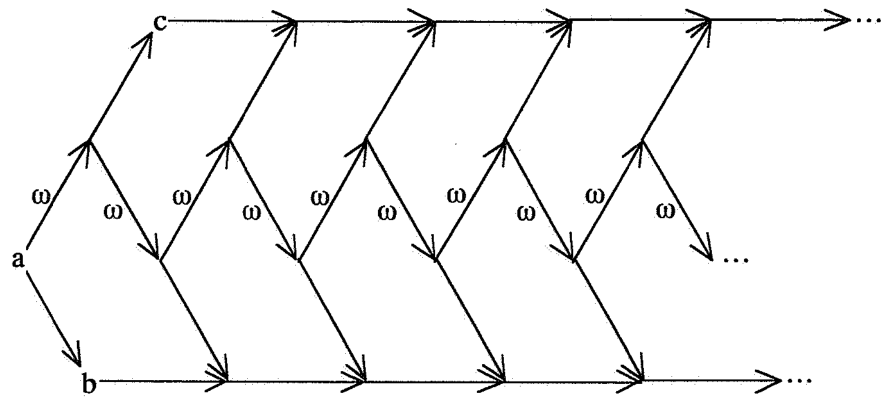

Figure 15. $WCR^{\leq 1} \wedge CS \not\Rightarrow CR^{\infty}$.

ANOTHER QUESTION. Would the answer to (1) be yes if the TARS is a (perhaps non-orthogonal) TRS? I suspect not, but have no counterexample.

We can also consider metric versions of the various tiling properties (WCR, PM, CR, etc.). The idea is that if the top side of a tile is small, then the distance between left and right hand sides should also be small, and similarly, interchanging top with left and bottom with right. The following axiom tries to formalise this.

DEFINITION. MT (metric tiling) is the following property. Let $a \twoheadrightarrow^{\infty} b$ and $a \twoheadrightarrow^{\infty} c$ be two reduction sequences. Let there exist d and reduction sequences $b \twoheadrightarrow^{\infty} d$ and $c \twoheadrightarrow^{\infty} d$. Then d and these two sequences can be chosen in such a way that the following inequalities hold:

$$\mathbf{d}(a \twoheadrightarrow^{\infty} b, c \twoheadrightarrow^{\infty} d) \leq |a \twoheadrightarrow^{\infty} c|$$

$$\mathbf{d}(a \twoheadrightarrow^{\infty} c, b \twoheadrightarrow^{\infty} d) \leq |a \twoheadrightarrow^{\infty} b|$$

$$|b \twoheadrightarrow^{\infty} d| \leq K|a \twoheadrightarrow^{\infty} c|$$

$$|c \twoheadrightarrow^{\infty} d| \leq K|a \twoheadrightarrow^{\infty} b|$$

where K is a constant, fixed for the whole system.

However, this property does not yet establish $WCR^{\leq 1} \Rightarrow CR^{\infty}$. The counterexample to $WCR^{\leq 1} \wedge CS \Rightarrow CR^{\infty}$ is also a counterexample to $WCR^{\leq 1} \wedge MT \Rightarrow CR^{\infty}$.

DEFINITION. CP (the compression property) is the property that if $a \twoheadrightarrow^{\infty} b$, then $a \twoheadrightarrow^{\leq\omega} b$.

THEOREM. $WCR^{\leq 1} \wedge MT \wedge CS \wedge CP \Rightarrow CR^{\infty}$.

PROOF. For ordinals α and β, let $CR^{\alpha,\beta}$ be the property that for any two sequences $a \twoheadrightarrow^{\leq\alpha} b$ and $a \twoheadrightarrow^{\leq\beta} c$ there exist d and reduction sequences $b \twoheadrightarrow^{\infty} d$ and $c \twoheadrightarrow^{\infty} d$.

By CP, CR^{∞} is equivalent to $CR^{\omega,\omega}$.

Even without MT and CP, $WCR^{\leq 1}$ already implies $CR^{\alpha,\beta}$ for all finite α and β.

Consider two sequences $a \twoheadrightarrow^{\omega} b$ and $a \twoheadrightarrow^{\omega} c$. By $WCR^{\leq 1}$ we can form a tiling diagram as in Figure 9, except that the arrows may represent empty steps. The MT axiom ensures that each tile can be chosen to have dimensions roughly as drawn. In particular, since the top and left side are Cauchy sequences, so is each row and column of terms. Indeed, each row and column sequence must be strongly continuous, hence have a limit, and the limits of the sequences of limit points down the right or along the bottom must be the same. In addition, the sequence of vertical sequences must be Cauchy, hence by CS have a limit. The same is true of the sequence of horizontal sequences. □

QUESTION. Is there any interesting class of TARSs satisfying the stringent conditions of the above theorem? Orthogonal TGRSs satisfy $WCR^{\leq 1}$. We believe, however, that orthogonal TRSs and TGRSs do satisfy MT, but we do not have a proof of this. [Ken91] shows that orthogonal TRSs satisfy CP. However, we have seen that some quite ordinary TGRSs fail condition CS. If, as conjectured above, non-decreasing TGRSs satisfy CS, then such systems may be examples of the theorem. However, we already know from [Ken91] that they are CR^{∞}. It remains to be seen whether the theorem has any more useful application.

Conclusion

TARS's are full of pathologies. We have not found useful axioms which will eliminate them, while still admitting a wider class of systems than the orthogonal TRSs and TGRSs.

References

[Ken91] Kennaway, J.R., Klop, J.W., Sleep, M.R., and de Vries, F.J., Transfinite reductions in orthogonal term rewriting systems, Proc. Conference on Rewriting Techniques and Applications, LNCS vol. 488, (Springer-Verlag, 1991).

[Klo80] Klop, J.W. *Combinatory Reduction Systems*, Mathematical Centre Tracts no. 127, (Mathematical Centre, Amsterdam, 1980).