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$u$-statistics of Increasing Degrees with Asymptotically Poisson Distributions

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This paper is devoted to the limit theorem for $U$-statistics with increasing degrees provided that the counting measure of sampling variables has asymptotically Poisson distribution.

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Let $X_{i1}, X_{i2}, \ldots$ be a sequence of independent identically distributed random variables taking values in a measurable space $(X, \mathcal{F})$ having a common distribution $\pi_n$ on this space. Consider an $U$-statistic

$$U_n = \left[\frac{n}{m}\right]^{-1} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \Phi(X_{i_1}, \ldots, X_{i_m})$$

(1)

where $\Phi : X^m \to \mathbb{R}$ is a symmetric function, $m$ is integer number, $1 \leq m \leq n$, which is called the degree of the $U$-statistic. The asymptotic theory, as $n \to \infty$, $m$ being fixed, for $U$-statistics is constructed in [1]. The case when $m$ depends on $n$, i.e. $m = m(n)$, so that $m \to \infty$ as $n \to \infty$ is studied very little. Some results (see [2],[3],[4]) give a hint that in this situation the asymptotics is essentially different to that in the case of fixed $m$.

The aim of this paper is to study the asymptotics of the $U$-statistics (1) under the condition

$$mn^{-1} \to \beta, \quad 0 < \beta < 1$$

(2)

as $n \to \infty$.

The General Theorem
Let $\lambda$ stand for the Radon measure (i.e. $\lambda(A) < \infty$ for all $A \in \mathcal{G}$) having no atoms on the measurable space $(X, \mathcal{F})$ and $P_\lambda(A) = P_\lambda(\omega, A)$ denotes the Poisson measure with intensity $\lambda$, i.e.

$$P[P_\lambda(A) = k] = [k!]^{-1}\lambda^k e^{-\lambda(A)}, \quad k = 0, 1, 2, \ldots$$

and also events $\{\omega : P_\lambda(A_1) = q_1, \ldots, \omega : P_\lambda(A_p) = q_p\}$ are independent for any positive integer $p$, any disjoint subsets $A_1, \ldots, A_p \in \mathcal{G}$ and any non-negative numbers $q_1, \ldots, q_p$. Further, put $\lambda_n = \pi_n(A), A \in \mathcal{G}$ and define a sequence of point measures

$$Q_n(A) = \delta_{X_{n_1}}(A) + \delta_{X_{n_2}}(A) + \ldots + \delta_{X_{n_m}}(A)$$

(3)

based on the random variables $X_{n_1}, X_{n_2}, \ldots, X_{n_m}$ for any $A \in \mathcal{G}$, with $\delta_A$ standing for the Dirac measure.

The method of analysis proposed is based under condition (2) on the canonical Hoeffding decomposition ([1], p. 21)

$$U_n - \Theta(\lambda_n) = \sum_{c=1}^{m} \left[\frac{m}{c}\right] \left[\frac{n}{c}\right]^{-1} S_c(g_{m_c})$$

(4)

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where
\[
\Theta(\lambda_n) = \int \cdots \int \Phi(y_1, \ldots, y_m) \prod_{j=1}^m \lambda_n(dy_j),
\]
\[
S_{nc}(g_{mc}) = \sum_{1 \leq i_1 < \cdots < i_c \leq n} g_{mc}(X_{n_{i_1}}, \ldots, X_{n_{i_c}}),
\]
\[
g_{mc}(x_1, \ldots, x_c) = \int \cdots \int \Phi(\lambda_1, \ldots, \lambda_m) \prod_{j=1}^c \left[ \delta_{x_j} (dy_j) - \lambda_n(dy_j) \right] \prod_{j=c+1}^m \lambda_n(dy_j).
\]

**Theorem 1.** Suppose that the weak convergence of random measures
\[
Q_n \xrightarrow{d} P_\lambda
\]
takes place. Let the limit
\[
\sigma^{-1}(m, n) g_{mc}(x_1, \ldots, x_c) \to g_\infty(x_1, \ldots, x_c) \quad \text{as} \quad n \to \infty
\]
exist for some \(\sigma(m, n) > 0\), the functions \(g_\infty(x_1, \ldots, x_c)\) being continuous, \(|g_\infty| \leq M^c\) (\(M\) is a constant), and there exists a compact in \(\mathbb{R}^c\) outside of which \(g_\infty\) equal zero; and also
\[
|\sigma^{-1}(m, n) g_{mc}(x_1, \ldots, x_c)| \leq M^c, \quad c = 0, 1, 2, \ldots
\]
If condition (2) takes place then the weak convergence
\[
\sigma^{-1}(m, n)[U_r - \Theta] \equiv \sum_{c=1}^\infty \frac{F^c}{c!} W_c(\lambda)
\]
where
\[
W_c(\lambda) = \int \cdots \int g_\infty(x_1, \ldots, x_c) P_\lambda(dx_1) \cdots P_\lambda(dx_c)
\]
denotes \(c\)-dimensional stochastic Poisson-Wiener-Ito integral [5].

To prove (6) the representation (4) is rewritten into the form
\[
\sigma^{-1}(m, n)[U_r - \Theta] = \sum_{c=1}^m \left[ \frac{m}{n} \right]^c S_{nc}(g_{\infty}) + r_{nm} + \rho_{nm}
\]
where
\[
\rho_{nm} = \sum_{c=1}^m \left[ \frac{m}{n} \right]^c S_{nc}(g_{\infty} - g_\infty),
\]
\[
r_{nm} = \sum_{c=1}^m \left[ \frac{m}{n} \right]^c \delta(c) S_{nc}(g_{\infty}),
\]
in addition \(\delta(1) = 0, \quad \delta(c) = \prod_{j=1}^{c-1} [1 - jm^{-1}] [1 - ja^{-1}]^{-1} - 1, \quad c = 2, 3, \ldots\)
The detailed analysis shows that in (7)
\[
r_{nm} \xrightarrow{p} 0, \quad \rho_{nm} \xrightarrow{p} 0
\]
as \(n \to \infty\). It follows from [4] and [6] that in (7)
\[
\sum_{c=1}^m \left[ \frac{m}{n} \right]^c S_{nc}(g_{\infty}) \xrightarrow{d} \sum_{c=1}^\infty \frac{F^c}{c!} W_c(\lambda)
\]
Le's illustrate the more detailed estimates of this proof by the following example.
THE PARTICULAR CASE

Let in (1)
\[ \Phi(x_1, \ldots, x_m) = h(x_1) \cdots h(x_m), \]
where $h: \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support. Denote
\[ \mu_n = \int h(x) \lambda_n(dx), \quad \mu = \int h(x) \lambda(dx) \]
and let $\mu_n \neq 0$, $\mu \neq 0$. The assumption (5) implies that $\mu_n \to \mu$ as $n \to \infty$.

**Theorem 2.** If condition (5) holds then the weak convergence
\[ \mu_n^{-m} U_n \overset{d}{\to} \exp \left\{ \int \ln (1 - \beta + \frac{\beta}{\mu} h(x)) P_\lambda(dx) \right\} \tag{8} \]

or, on the other form,
\[ \mu_n^{-m} [U_n - \mu_n^m] \overset{d}{\to} \sum_{c=1}^{\infty} \left[ \frac{\beta}{\mu} \right]^c \frac{1}{c!} \int \cdots \int \left( \prod_{j=1}^{c} h(x_j) - \mu \right) \left( \prod_{j=1}^{c} P_\lambda(dx_j) \right) \tag{9} \]

**Proof.** Under the above conditions
\[ g_\infty (x_1, \ldots, x_c) = \mu_n^{-m} \left( h(x_1) - \mu_n \right) \cdots \left( h(x_c) - \mu_n \right), \]
\[ \Theta = \mu_n^m \]
Then one can choose $\sigma(n, n) = \mu_n^m$ and hence
\[ g_\infty (x_1, \ldots, x_c) = \mu^{-m} \left( h(x_1) - \mu \right) \cdots \left( h(x_c) - \mu \right) \]
Reduce then representation (4) to the form
\[ \mu_n^{-m} U_n = \sum_{c=0}^{m} \left[ \frac{m}{c} \right] \left[ \frac{n}{c} \right]^{-1} \mu_n^{-c} \sum_{1 \leq i_1 < \cdots < i_c \leq n} \left( h(X_{ni_1}) - \mu_n \right) \cdots \left( h(X_{ni_c}) - \mu_n \right) \]
Whence we derive as $n \to \infty$
\[ \mu_n^{-m} U_n = \sum_{c=0}^{m} \left[ \frac{m}{n \mu_n} \right]^c \sum_{1 \leq i_1 < \cdots < i_c \leq n} \left( h(X_{ni_1}) - \mu_n \right) \cdots \left( h(X_{ni_c}) - \mu_n \right) + R_n(\omega) \]
where $R_n(\omega) \overset{p}{\to} 0$. Then making use of the generating function for a symmetric polynomial we have
\[ \mu_n^{-m} U_n = \prod_{j=1}^{n} \left[ 1 + \frac{m}{n \mu_n} \left( h(X_{nj}) - \mu_n \right) \right] + R_n(\omega) \tag{10} \]

In (10)
\[ = \prod_{j=1}^{n} \left[ 1 + \frac{m}{n \mu_n} \left( h(X_{nj}) - \mu_n \right) \right] \]
\[ = \exp \left\{ \sum_{j=1}^{n} \ln \left[ 1 + \frac{m}{n \mu_n} \left( h(X_{nj}) - \mu_n \right) \right] \right\} \]
\[ = \exp \left\{ \int \ln \left[ 1 + \frac{m}{n n \mu_n} h(x) \right] \mathcal{Q}_n(dx) \right\} \]

Under the condition (5) the weak convergence
\[
\int \ln \left[ 1 - \frac{m}{n} \right] h(x) \, Q_n(dx) \xrightarrow{d} \int \ln \left[ 1 - \beta + \frac{\beta h(x)}{\mu} \right] P_\lambda(dx)
\]
takes place. This relation taken together with (10) implies (8).

Note that
\[
\mathbb{E} \exp \left\{ i s \int \ln \left[ 1 - \beta + \frac{\beta h(x)}{\mu} \right] P_\lambda(dx) \right\} = \exp \left\{ \int \left[ e^{i s \ln \left[ 1 - \beta + \frac{\beta h(x)}{\mu} \right]} - 1 \right] P_\lambda(dx) \right\}
\]
for all \( s \in \mathbb{R} \).

The second statement from (9) follows directly from (6).

**The Weak Convergence of von Mises Functional.**

Consider the von Mises functional
\[
V_n = n^{-m} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \ldots \sum_{i_n=1}^{m} \phi(x_{i_1}, \ldots, x_{i_n})
\]
and denote
\[
EV_n = \int \cdots \int V_n(x_1, \ldots, x_n) \lambda_1(dx_1) \cdots \lambda_n(dx_n),
\]
where \( V_n(x_1, \ldots, x_n) = V_n \). We suppose that \( |EV_n| < \infty \). \( V_n \) satisfies decomposition similar to (4)
\[
V_n - EV_n = \sum_{c=1}^{m} \left[ \frac{m}{c} \right] n^{-c} \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} g_{mc}(x_{i_1}, \ldots, x_{i_n})
\]
Further, by means of measure \( Q_n \) the expression in the right-hand side of the latter relation can be represented in the form
\[
V_n - EV_n = \sum_{c=1}^{m} \left[ \frac{m}{c} \right] n^{-c} \int \cdots \int g_{mc}(x_1, \ldots, x_c) \prod_{j=1}^{c} Q_n(dx_j)
\]
Thus, under the conditions of Theorem 1 one can state the weak convergence
\[
\sigma^{-1}(m,n)[V_n - EV_n] \xrightarrow{d} \sum_{c=1}^{m} \frac{m}{c!} \int \cdots \int g_{mc}(x_1, \ldots, x_c) \prod_{j=1}^{c} P_\lambda(dx_j)
\]
as \( n \to \infty \) holds, i.e. the weak limits for \( U_n \) and \( V_n \) coincide.

**References**