1991

J. Leslie, C. van Eeden

On a characterization of the exponential distribution based on a type 2 right censored sample

Department of Operations Research, Statistics, and System Theory Report BS-R9117 July

CWI, nationaal instituut voor onderzoek op het gebied van wiskunde en informatica

CWI is the research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a non-profit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands organization for scientific research (NWO).

On a Characterization of the Exponential Distribution Based on a Type 2 Right Censored Sample

Julian Leslie
Macquarie University, Sydney, Australia

Constance van Eeden
University of British Columbia, Vancouver, Canada
Université du Québec à Montréal, Canada

1985 Mathematics Subject Classification: Primary: 62E10. Secondary: 62E15.
Key Words & Phrases: Characterization, exponential distribution.

Note: This research was supported by the Natural Sciences and Engineering Research Council of Canada, by F.C.A.R. of Québec and by Birkbeck College, London, England.
The major component of J. Leslie's contribution was carried out whilst he was at Birkbeck College, University of London, England.
Also published as Technical report of the Department of Statistics at the University of British Columbia, Vancouver and as "Rapport de recherche" of the Département de mathématiques et d'informatique of the Université du Québec à Montréal, Montréal.
This report contains full details of the proof outlined in the article *On a conjecture concerning a characterization of the exponential distribution* by C. van Eeden, which will appear in the forthcoming CWI Quarterly Volume on Statistics.

Abstract

Dufour (1982) gives a conjecture concerning a characterization of the exponential distribution based on type 2 right censored samples. This conjecture, if true, generalizes the characterization based on complete samples of Seshadri, Csörgö, Stephens (1969) and Dufour, Maag, van Eeden (1984). In this paper it is shown that Dufour's conjecture is true if the number of censored observations is no larger than $\frac{1}{3}n - 1$, where n is the sample size.

Report BS-R9117
ISSN 0924-0659
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

1. Introduction

Dufour, in his 1982 Ph.D. thesis, presents the following conjecture concerning a characterization of the exponential distribution. Let $X_1, X_2, \dots, X_n$ be independent, identically distributed, non-negative random variables and let r be an integer satisfying $2 \le r < n$. Write

$Y_{1,n} \le Y_{2,n} \le \dots \le Y_{n,n}$ for the order statistics of $X_1, X_2, \dots, X_n$, with $Y_{0,n} = 0$ and define

$$(1.1) \qquad \begin{cases} D_{i,n} = (n-i+1)(Y_{i,n} - Y_{i-1,n}) \\ S_{i,n} = \sum_{j=1}^{i} D_{j,n}. \end{cases} \qquad i = 1,2, \dots ,n$$

The conjecture then states that, if

$$(1.2) \qquad W_{r,n} = \left(\frac{S_{1,n}}{S_{r,n}}, \frac{S_{2,n}}{S_{r,n}}, \dots, \frac{S_{r-1,n}}{S_{r,n}}\right)$$

is distributed as the vector of order statistics of a sample of size $r-1$ from a U (0, 1) - distribution (i.e. a uniform distribution on the interval (0,1)), then X_1 has an exponential distribution. That this result, if true, characterizes the exponential distribution follows from the fact that $W_{r,n}$ has this uniform-order-statistics distribution when X_1 is exponentially distributed.

For the uncensored case $(n = r)$ a proof of the characterization for the case where $n \ge 3$ was given by Dufour, Maag, van Eeden (1984) (see also Seshadri, Csörgõ, Stephens (1969)); Menon and Seshadri (1975) show that for the case $n = 2$, the uniform distribution of $W_{2,2}$ does not characterize the exponential distribution.

The problem of whether the uniform-order-statistics distribution of $W_{r,n}$ characterizes the exponential distribution arises, for example, when one bases a test of the hypothesis that X_1 is exponentially distributed on the statistic $W_{r,n}$. If the conjecture is true then the hypothesis that X_1 is exponential is equivalent to the hypothesis that $W_{r,n}$ is distributed as the vector of order statistics of a sample of size $r-1$ from a $U(0, 1)$ – distribution. There are of course numerous tests of uniformity available and this result offers a natural way of testing the exponentiality of a sequence of ordered observations before the complete set of observations has been collected. However, if the conjecture is false then there exists at least one alternative distribution for X_1 for which the power of the test equals the size.

In this paper it will be shown that Dufour's conjecture is true if $r \geq \frac{2n}{3} + 1$. We do not know whether it is true for the case where $n \geq 3$ and $r < \frac{2n}{3} + 1$.

Section 2 contains the main result and its proof. Some lemmas, needed for the proof in section 2, are given in section 3. The vector of order statistics of a sample of size j from a $U(0, 1)$ – distribution will be denoted by $U_{(.)}(j) = \left(U_{(1)}, U_{(2)}, \dots, U_{(j)}\right)$ and $Z_1 \sim Z_2$ will be used to denote that the random vectors Z_1 and Z_2 have the same distribution. Finally, $\bar{F}(x) = 1 - F(x)$, $-\infty < x < \infty$, where F is the distribution function of X_1.

Under the assumption that the density of X_1 exists, the proof of the main theorem is relatively straightforward. In avoiding that assumption the proof has had to become somewhat more complicated.

2. The main result

This section contains the proof of the following theorem.

Theorem 2.1. Let $r \geq \frac{2n}{3} + 1$. Then X_1 is exponentially distributed if and only if

$$W_{r,n} \sim U_{(.)}(r-1).$$

□

Proof. As already mentioned above, $W_{r,n} \sim U_{(.)}(r-1)$ when X_1 is exponential.

We show that

$$W_{r,n} \sim U_{(.)}(r-1) \Rightarrow X_1 \text{ is exponential.} \tag{2.1}$$

By Lemma 3.8, $W_{r,n} \sim U_{(.)}(r-1)$ implies

$$P\left(\frac{Z_1}{Z} > s_1, \frac{Z_2}{Z} > s_2\right) = \frac{1}{1+s_1+s_2}, \quad s_1, s_2 \geq 0, \tag{2.2}$$

where Z, Z_1 and Z_2 are independent random variables each distributed as $\min(X_1, X_2, \dots, X_{n-r+1})$.

We apply the following key theorem from Kotlarski (1967).

Theorem A (Kotlarski): Let W_1, W_2, W_3 be three independent random variables, and set $Y_1 = W_1/W_3$, $Y_2 = W_2/W_3$. The necessary and sufficient condition for W_k $(k = 1,2,3)$ to be identically and exponentially distributed is that the joint distribution of (Y_1,Y_2) has the following density

$$g(y_1,y_2) = 2(1+y_1+y_2)^{-3}, \quad y_1 > 0, \quad y_2 > 0.$$

□

From (2.2) and Theorem A it follows that Z_1 has an exponential distribution and from

$$P(Z_1 > t) = \bar{F}^{n-r+1}(t) \qquad t \geq 0 \tag{2.3}$$

it then follows that X_1 has an exponential distribution.

□

Remark 2.2. Note that the above proof includes the case where $n = r$. In that case the condition $r \geq \frac{2n}{3} + 1$ becomes $n \geq 3$. The proof given by Dufour, Maag, van Eeden (1984) for the case $n = r \geq 3$ is of a totally different nature.

□

3. Some lemmas

In this section we develop a sequence of lemmas which lead ultimately to a proof of Lemma 3.8.

Lemma 3.1. (Dufour 1982, p 146-151). For $2 \leq j \leq n$, let

$$V_{1,j,n} = Y_{1,n}/ Y_{j,n},\ V_{2,j,n} = Y_{2,n}/ Y_{j,n},\ \dots,\ V_{j-1,j,n} = Y_{j-1,n}/Y_{j,n}$$

then $W_{j,n} \sim U_{(.)}(j-1)$ if and only if

$(V_{1,j,n}, V_{2,j,n}, \dots, V_{j-1,j,n})$ has density

$$(3.1)\quad \begin{cases} \frac{n!\,(j-1)!}{(n-j)!} \left(\sum_{i=1}^{j-1} v_i + n - j + 1 \right)^{-j}, & 0 < v_1 \leq v_2 \leq \dots \leq v_{j-1} < 1 \\ 0 & \text{otherwise}. \end{cases}$$

□

Remark 3.2. Note that if $(V_{1,j,n}, V_{2,j,n}, \dots, V_{j-1,j,n})$ has a density, then F must be a continuous function. For if F has a jump at x_0 (say) of size p then $(V_{1,j,n}, V_{2,j,n}, \dots, V_{j-1,j,n})$ can take the value (1,1, ... ,1) with probability at least p^n.

□

For $2 \le j \le n$ let, $X_{1,j,n}, X_{2,j,n}, \ldots, X_{j-1,j,n}$ be random variables which are, conditionally on $Y_{j,n} = y$, independent and identically distributed with distribution function $F(x)/F(y)$, $0 < x \le y$. (Note that the $X_{i,j,n}$ should, in fact, carry an extra subscript, namely $Y_{j,n}$).

Let $\quad T_{i,j,n} = X_{i,j,n}/Y_{j,n}, i = 1, 2, \ldots, j-1$ and set

$$T_{j,n} = (T_{1,j,n}, T_{2,j,n}, \ldots, T_{j-1,j,n}).$$

The following is an immediate consequence of Lemma 3.1.

Lemma 3.3. For $2 \le j \le n$, $W_{j,n} \sim U_{(.)}(j-1)$ if and only if $T_{j,n}$ has density

$$[n!/(n-j)!]\left(\sum_{i=1}^{j-1} t_i + n-j+1\right)^{-j}, \quad 0 < t_i < 1, \; i = 1, 2, \ldots, j-1. \tag{3.2}$$

□

We can further simplify the joint distribution of $T_{j,n}$ by representing it in terms of the original independent X_i's.

Lemma 3.4 For $2 \le j \le n$ and $\underset{\sim}{t} = (t_1, t_2, \ldots, t_{j-1})$

$$P(T_{j,n} \le \underset{\sim}{t}) = \binom{n}{j-1} P(X_1 \le t_1 Z, X_2 \le t_2 Z, \ldots, X_{j-1} \le t_{j-1} Z),$$

where Z is independent of $X_1, X_2, \ldots, X_{j-1}$ and distributed as $\min(X_j, X_{j+1}, \ldots, X_n)$.

□

Proof: From the definition of $T_{j,n}$

$$P\{T_{j,n} \le \underset{\sim}{t}\} = P\{X_{1,j,n} \le t_1 Y_{j,n}, X_{2,j,n} \le t_2 Y_{j,n}, \ldots, X_{j-1,j,n} \le t_{j-1} Y_{j,n}\}$$

$$= \int_0^\infty \prod_{i=1}^{j-1} \left[\frac{F(xt_i)}{F(x)}\right] \frac{n!}{(j-1)!\,(n-j)!} \cdot \left[F(x)\right]^{j-1} \cdot \left[\bar{F}(x)\right]^{n-j} . \, dF(x)$$

$$= \binom{n}{j-1} \cdot \int_0^\infty \left[\prod_{i=1}^{j-1} F(xt_i) \right] (n-j+1)\left[\bar{F}(x)\right]^{n-j} . \, dF(x).$$

The second equality follows from the fact that, for $j = 1,2, \dots ,n$,

$$F_{Y_{j,n}}(x) = \sum_{i=j}^{n} \binom{n}{i} \left[F(x)\right]^{i} \left[\bar{F}(x)\right]^{n-i}$$

and that F is uniformly continuous, so that for any measurable function h,

$$\int h \, dF_{Y_{j,n}}(x) = \int h.\binom{n}{j-1} (n-j+1)[F(x)]^{j-1} \left[\bar{F}(x)\right]^{n-j} . \, dF(x) \tag{3.3}$$

The lemma now follows.

□

The next result will be useful in an induction argument needed later on.

Lemma 3.5: For $0 \leq k \leq r-2 \leq n-2$,

(3.4) if $W_{r,n} \sim U_{(.)}(r-1)$ then $W_{r-k,n-k} \sim U_{(.)}(r-k-1)$.

□

Proof: The result is trivial when $r = 2$, so assume $r > 2$.

Using standard properties of uniform order statistics we have that for $2 \leq j \leq r$,

$$W_{r,n} \sim U_{(.)}(r-1) \Rightarrow W_{j,n} \sim U_{(.)}(j-1). \tag{3.5}$$

Further we have that for any $A \in \mathfrak{B}_{j-2}$, the σ-algebra of Borel sets of $\mathbb{R}^{j-2}$,

$$\begin{cases} P(T_{j-1,n-1} \in A) = (n-j+1)n^{-1} P(T_{j-1,n} \in A) + \\ \qquad + (j-1)n^{-1} P(T_{j,n} \in A \times [0, 1]). \end{cases} \tag{3.6}$$

This can be seen as follows. From Lemma 3.4 one obtains

$$P(T_{j-1,n-1} \in A) = \binom{n-1}{j-2} P((X_1, X_2, \dots , X_{j-2}) \in AZ, X_{j-1} > Z)$$
$$+ \binom{n-1}{j-2} P((X_1, X_2, \dots , X_{j-2}) \in AZ, X_{j-1} \leq Z),$$

where $Z \sim \min(X_j, X_{j+1}, \ldots, X_n)$ and independent of $(X_1, X_2, \ldots, X_{j-1})$. Using (3.3) and noting that $Z \sim Y_{1,n-j+1}$

$$\begin{aligned} &P(X_1, X_2, \ldots, X_{j-2}) \in AZ, X_{j-1} > Z) \\ &= E\left\{[P((X_1, X_2, \ldots, X_{j-2}) \in AZ)]\, \bar{F}(Z)|Z\right\} \\ &= \{(n-j+1)/(n-j+2)\}\, P((X_1, X_2, \ldots, X_{j-2}) \in AZ^*) \end{aligned}$$

where $Z^* \sim Y_{1,n-j+2}$ and is independent of $(X_1, X_2, \ldots, X_{j-2})$. A second application of Lemma 3.4 yields the first term on the right of (3.6).

The second term follows as

$$P\Big((X_1, X_2, \ldots, X_{j-2}) \in AZ, X_{j-1} \le Z\Big) = P\Big((X_1, X_2, \ldots, X_{j-1}) \in (A \times [0,1])Z\Big)$$

and $\binom{n-1}{j-2} = \left(\frac{j-1}{n}\right)\binom{n}{j-1}$.

From (3.5) with $j = r-1$, (3.6) with $j = r$ and Lemma 3.3 with $j = r$ and with $j = r-1$, we have that $W_{r,n} \sim U_{(.)}(r-1)$ implies that for all $A \in \mathcal{B}_{r-2}$

(3.7) $\quad P(T_{r-1,n-1} \in A) =$

$$(n-r+1)\, n^{-1} \int_{\underset{\sim}{t} \in B}\!\!\cdots\!\int n!\, [(n-r+1)!]^{-1} \left(\sum_{i=1}^{r-2} t_i + n-r+2\right)^{-r+1} dt_1 \ldots dt_{r-2}$$

$$+ \; (r-1)n^{-1} \int_{\underset{\sim}{t} \in B}\!\!\cdots\!\int \int_{t_{r-1} \in [0,1]} n!\, [(n-r)!]^{-1} \left(\sum_{i=1}^{r-1} t_i + n-r+1\right)^{-r} dt_1 \ldots dt_{r-1}$$

$$= \int_{\underset{\sim}{t} \in B}\!\!\cdots\!\int (n-1)![(n-r)!]^{-1} \left(\sum_{i=1}^{r-2} t_i + n-r+2\right)^{-r+1} dt_1 \ldots dt_{r-2} \; +$$

$$\int_{\underset{\sim}{t} \in B}\!\!\cdots\!\int (n-1)![(n-r)!]^{-1} \left\{\left(\sum_{i=1}^{r-2} t_i + n-r+1\right)^{-r+1} - \left(\sum_{i=1}^{r-2} t_i + n-r+2\right)^{-r+1}\right\} dt_1 \ldots dt_{r-2}$$

$$= \int_{\underset{\sim}{t} \in B}\!\!\cdots\!\int (n-1)![(n-r)!]^{-1} \left(\sum_{i=1}^{r-2} t_i + n-r+1\right)^{-r+1} dt_1 \ldots dt_{r-2},$$

where $\underset{\sim}{t} = (t_1, t_2, \dots, t_{r-2})$ and $B = A \cap [0, 1]^{r-2}$.

This in turn implies, via Lemma 3.3 with $j = r-1$ and n replaced by $n-1$, that $W_{r-1,n-1} \sim U_{(.)}(r-2)$. This establishes the lemma for the case $k = 1$. The other cases follow by repeated application of the $k = 1$ case.

□

From Lemma 3.3 and Lemma 3.5 one obtains immediately

<u>Lemma 3.6</u>: For $2 \le j \le r \le n$, $W_{r,n} \sim U_{(.)}(r-1)$ implies that $T_{j,n-r+j}$ has density

$$(3.8)\quad (n-r+j)!\,[(n-r)!]^{-1}\left(\sum_{i=1}^{j-1} t_i + n-r+1\right)^{-j},\ 0 < t_i < 1,\ i = 1, 2, \dots, j-1.$$

□

The following is an immediate consequence of Lemmas 3.4 and 3.6.

<u>Lemma 3.7</u>: For $2 \le j \le r \le n$, $W_{r,n} \sim U_{(.)}(r-1)$ implies

$$P(X_1 \le s_1 Z, X_2 \le s_2 Z, \dots, X_{j-1} \le s_{j-1} Z) =$$
$$(n-r+1)(j-1)! \int_0^{s_1} \dots \int_0^{s_{j-1}} \left(\sum_{i=1}^{j-1} t_i + n-r+1\right)^{-j} dt_1 \dots dt_{j-1},$$

$0 \le s_i \le 1,\ i = 1, 2, \dots, j-1,$

where $Z \sim \min(X_1, X_2, \dots, X_{n-r+1})$ and independent of $(X_1, X_2, \dots, X_{j-1})$.

□

We now give a key lemma whose proof will have to wait till near the end of the paper but which motivates a number of the subsequent lemmas. It is the lemma used in the proof of Theorem 2.1.

Lemma 3.8: If $r \geq \frac{2}{3}n + 1$, then $W_{r,n} \sim U_{(.)}(r-1)$ implies

$$P(Z_1 > s_1 Z, Z_2 > s_2 Z) = (s_1 + s_2 + 1)^{-1}, s_1, s_2 \geq 0,$$

where Z_1, Z_2, Z are independent and identically distributed as $\min(X_1, X_2, \dots, X_{n-r+1})$.

□

For the proof of this result the following lemmas are needed. In each of these lemmas

$$Z \sim \min(X_1, X_2, \dots, X_{n-r+1}) \text{ independently of } (X_1, X_2, \dots, X_{j-1}).$$

Lemma 3.9. For $2 \leq j \leq r \leq n$ and $0 \leq l < j-1$, $W_{r,n} \sim U_{(.)}(r-1)$ implies

$$P\Big(X_i \leq s_i Z, i = 1, 2, \dots, j-l-1, \min_{j-l \leq i \leq j-1} X_i > sZ\Big) \tag{3.9}$$

$$= (n-r+1)(j-l-1)! \int_0^{s_1} \dots \int_0^{s_{j-l-1}} \left[\sum_{i=1}^{j-l-1} t_i + n - r + 1 + ls\right]^{-j+l} dt_1 \dots dt_{j-l-1},$$

for $0 \leq s_i \leq 1, i = 1, 2, \dots, j-l-1$, and $s \in \{s_1, s_2, \dots, s_{j-l-1}\}$.

□

Proof: We use an induction argument on l and j but not on r and n.

The trivial case of $l = 0$ and $2 \leq j \leq r$ is simply a statement of Lemma 3.7.

Further, the case $l = 1$, $3 \leq j \leq r$ follows from the fact that we can write

$$P(X_i \leq s_i Z, i = 1, 2, \dots, j-2, X_{j-1} > sZ)$$
$$= P(X_i \leq s_i Z, i = 1, 2, \dots, j-2) - P(X_i \leq s_i Z, i = 1, 2, \dots, j-2, X_{j-1} < sZ).$$

Two applications of Lemma 3.7 now establish the required result. Next assume (3.9) is true for some fixed $l \geq 1$ with $l+2 \leq j \leq r$ then it holds for $l+1$, and $l+3 \leq j \leq r$, because

$$P\Big(X_i \leq s_i Z, i = 1, 2, \dots, j-l-2; \min_{j-l-1 \leq i \leq j-1} X_i > sZ\Big) \tag{3.10}$$

$$= P\Big(X_i \le s_i Z,\ i = 1, 2, \dots, j-l-2;\ \min_{j-l \le i \le j-1} X_i > sZ\Big)$$

$$- P\Big(X_i \le s_i Z,\ i = 1, 2, \dots, j-l-2;\ X_{j-l-i} \le sZ;\ \min_{j-l \le i \le j-1} X_i > sZ\Big)$$

$$= (n-r+1)(j-l-2)! \int_0^{s_1} \cdots \int_0^{s_{j-l-2}} \left(\sum_{i=1}^{j-l-2} t_i + n - r + 1 + ls \right)^{-j+l+1} dt_1 \dots dt_{j-l-2}$$

$$- (n-r+1)(j-l-1)! \int_0^{s_1} \cdots \int_0^{s_{j-l-2}} dt_1 \dots dt_{j-l-2} \int_0^{s} \left(\sum_{i=1}^{j-l-1} t_i + n - r + 1 + ls \right)^{-j+l} dt_{j-l-1}$$

$$= (n-r+1)(j-l-2)! \int_0^{s_1} \cdots \int_0^{s_{j-l-2}} \left(\sum_{i=1}^{j-l-2} t_i + n - r + 1 + (l+1)s \right)^{-j+l+1} dt_1 \dots dt_{j-l-2},$$

which is (3.9) with l replaced by l + 1.

□

The case where $l = j-1$ in Lemma 3.9 needs special attention:

<u>Lemma 3.10</u>. For $2 \le j \le r \le n$, $W_{r,n} \sim U_{(.)}(r-1)$ implies

(3.11) $$P\Big(\min_{1 \le i \le j-1} X_i > sZ\Big) = (n-r+1)/(n-r+1+(j-1)s),\ 0 \le s \le 1.$$

□

<u>Proof</u>: Induction on j will be used in this proof. For $j = 2$ it follows from Lemma 3.7 that, for $0 \le s \le 1$

$$P(X_1 > sZ) = 1-(n-r+1) \int_0^{s} (t+n-r+1)^{-2}\, dt$$
$$= (n-r+1)/(n-r+1+s).$$

Now suppose the lemma is true for some $j \in \{2, 3, \dots, r-1\}$, then for $0 \le s \le 1$,

$$P\Big(\min_{1 \le i \le j} X_i > sZ\Big) = P\Big(X_1 > sZ,\ \min_{2 \le i \le j} X_i > sZ\Big).$$

From the induction hypothesis and Lemma 3.9 it then follows that for

$0 \le s \le 1$,

$$P\Big(\min_{1 \le i \le j} X_i > sZ\Big)$$
$$= (n-r+1)[n-r+1+(j-1)s]^{-1} - (n-r+1)\int_0^s \{t+n-r+1+(j-1)s\}^{-2}\,dt$$
$$= (n-r+1)/(n-r+1+js),$$

which is (3.11) with j replaced by j + 1.

□

Next we introduce a second minimum term into the probability expression in (3.9) with the ultimate objective: an expression like (3.11) but with two minimum terms.

Lemma 3.11. For $3 \le j \le r \le n$, $l \ge 0$, $k \ge 0$, $l+k \le j-3$, $\alpha = j-l-k$,

$W_{r,n} \sim U_{(.)}(r-1)$ implies

$$\text{(3.12)} \quad P\Big(X_i \le s_i Z,\ i=1,2,\dots,\alpha-1,\ \min_{\alpha \le i \le j-l-1} X_i > sZ,\ \min_{j-l \le i \le j-1} X_i > s'Z\Big)$$

$$= (n-r+1)(\alpha-1)!\int_0^{s_1}\cdots\int_0^{s_{\alpha-1}}\left(\sum_{i=1}^{\alpha-1} t_i + n-r+1+ks+ls'\right)^{-\alpha} dt_1 \dots dt_{\alpha-1},$$

$0 \le s_i \le 1$, $i = 1, 2, \dots, \alpha-1$; $s, s' \in \{s_1, s_2, \dots, s_{\alpha-1}\}$.

□

Proof: We use an induction argument on l, j and k but not on r and n.

For $k = 0$ (and $l \le j-3$, $3 \le j \le r$) the first minimum term disappears and the result is a particular case of Lemma 3.9. Further, in the spirit of (3.10) the case $k = 1$ (and $l \le j-4$, $3 \le j \le r$) also follows from Lemma 3.9.

Now we assume the lemma true for some $k \ge 1$ ($l \le j-k-3$, $k+3 \le j \le r$) and show it holds for $k+1$ ($l \le j-k-4$, $k+4 \le j \le r$).

The idea of the rest of the proof is essentially the same as that of the proof of Lemma 3.9. We start with the left side of (3.12) with k replaced by $k+1$, i.e. α replaced by $\alpha - 1$. We leave the second minimum term untouched, but take the variable $X_{\alpha-1}$ out of the first minimum term. We then represent the probability (i.e. left-side of (3.12) with $\alpha \to \alpha - 1$) as a difference of two probabilities in a way directly analogous to (3.10). The resulting probabilities are covered by the results assumed true under the induction hypothesis and can be replaced by appropriate integrals. These are manipulated in a way similar to the treatment given to the corresponding integrals in the proof of Lemma 3.9.

□

Lemma 3.12. For $2 \le j \le r \le n$, $0 \le l \le j-2$

$W_{r,n} \sim U_{(.)}(r-1)$ implies

$$(3.13) \quad P\Big(X_1 \le s_1 Z;\ \min_{2 \le i \le l+1} X_i > s_2 Z,\ \min_{l+2 \le i \le j-1} X_i > s_1 Z\Big)$$

$$= (n-r+1)\Big\{\big(n-r+1+ls_2+(j-l-2)s_1\big)^{-1} - \big(n-r+1+ls_2+(j-l-1)s_1\big)^{-1}\Big\}$$

$0 \le s_i \le 1$, $i = 1, 2$.

□

Proof: The case $l = 0$ is just a special case of Lemma 3.9. Also when $l = 1$, we represent, as in (3.10), the probability in the left-side of (3.13) as the difference between two probabilities the first of which involves an event placing no restriction on X_2 and the second includes the condition $X_2 \le s_2 Z$. Lemma 3.9 can be applied and the resulting integrals evaluated. We now assume the lemma true for some $l \geq 1$ and show it holds for $l+1$. The general case is essentially the same as the $l = 1$ case, i.e. we represent the probability on the left of (3.13) as the difference between two probabilities. The first term is the probability of the event:

$$X_1 \leq s_1 Z, \quad \min_{3 \leq i \leq l+2} X_i > s_2 Z, \quad \min_{l+3 \leq i \leq j-1} X_i > s_1 Z,$$

whereas for the 2nd term the associated event is:

$$X_1 \leq s_1 Z, \; X_2 \leq s_2 Z, \quad \min_{3 \leq i \leq l+2} X_i > s_2 Z, \quad \min_{l+3 \leq i \leq j-1} X_i > s_1 Z.$$

The first term is covered by the induction hypothesis whereas the second can be evaluated using Lemma 3.11, the latter involving a double integral which is easily evaluated.

□

We are now in a position to state the key lemma on which Lemma 3.8 depends and hence on which the main result of the paper rests.

Lemma 3.13. For $2 \leq j \leq r \leq n, \quad 0 \leq l \leq j-2$

(3.14) $$P\Big(\min_{1 \leq i \leq l} X_i > s_1 Z, \quad \min_{l+1 \leq i \leq j-1} X_i > s_2 Z\Big)$$

$$= (n-r+1)/[n-r+1+ls_1+(j-l-1)s_2], \qquad 0 \leq s_i \leq 1, i=1, 2.$$

□

Proof: For $l = 0$ the result reduces to Lemma 3.10. The case $l = 1$ is covered by Lemmas 3.9 and 3.10 after using a representation analogous to (3.10).

Using an induction argument on l, we represent the left side of (3.14), with l replaced by $l+1$, as a difference in two probabilities obtained by splitting off one of the variables from one of the minimum terms, for example X_{l+1}. The split being based, as usual, on the fact that

$$X_{l+1} \in R \Leftrightarrow \{X_{l+1} \le s_1 Z\} \cup \{X_{l+1} > s_1 Z\}.$$

□

Proof of Lemma 3.8. In Lemma 3.13 we make sure that the *number* of variables involved in the two minimum terms as well as in the minimum expression defining the distribution of Z, are all equal. We thus set $l = j - l - 1 = n - r + 1$. Clearly $j = 2l + 1$ and as $j \le r$ so $2l + 1 \le r$. But l also equals $n - r + 1$ so we need $2(n - r + 1) + 1 \le r$. This is equivalent to $r \ge \frac{2}{3}n + 1$. Lemma 3.13 then states that

$$P(Z_1 > s_1 Z, Z_2 > s_2 Z) = (s_1 + s_2 + 1)^{-1}, \quad 0 \le s_i \le 1, \quad i = 1, 2. \tag{3.15}$$

where Z, Z_1, Z_2, are i.i.d. as $\min(X_1, X_2, \ldots, X_{n-r+1})$.

We now show that (3.15) holds for all $s_1, s_2 \ge 0$. Let $V_1 = Z_1/Z$, $V_2 = Z_2/Z$ and let $g_{V_1V_2}(v_1, v_2)$ denote the density of (V_1, V_2) on $[0, 1]^2$. Then, by (3.15)

$$g_{V_1V_2}(v_1, v_2) = \frac{1}{2}(1 + v_1 + v_2)^{-3}, \quad 0 \le v_i \le 1, \quad i = 1, 2. \tag{3.16}$$

Now set $W_1 = V_1^{-1}$, $W_2 = V_2/V_1$, then $W_1 = Z/Z_1$, $W_2 = Z_2/Z_1$ and from (3.16) it follows that the density of (W_1, W_2) is

$$h_{W_1W_2}(w_1, w_2) = \frac{1}{2} w_1^{-3}\left(1 + w_1^{-1} + w_2 w_1^{-1}\right)^{-3}$$

$$= \tfrac{1}{2}(1+w_1+w_2)^{-3}, \quad w_1 > 1,\ 0 < w_2 < w_1.$$

The lemma then follows from the fact that W_1 and W_2 are exchangeable and that V_1, V_2 have the same distribution.

□

References

Dufour, R. (1982). Tests d'ajustement pour des échantillons tronqués ou censurés, Ph.D. thesis, Université de Montréal.

Dufour, R., Maag, U.R. and van Eeden, C. (1984). Correcting a proof of a characterization of the exponential distribution, J.R. Statist. Soc. B, 46, 238-241.

Kotlarski, I. (1967). On characterizing the gamma and the normal distribution, Pacif.J.Math., 20, 69-76.

Menon, M.V. and Seshadri, V. (1975). A characterization theorem useful in hypothesis testing, Contributions libres, 40$^{\text{iéme}}$ Session de l'Institut International de Statistique, 586-590.

Seshadri, V., Csörgö, M. and Stephens, M.A. (1969). Tests for the exponential distribution using Kolmogorov-type statistics, J.R. Statist. Soc. B, 31, 499-509.

School of Economic and Financial Studies
Macquarie University
Sydney 2109
Australia

Moerland 19
1151 BH Broek in Waterland

The Netherlands