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Non-Interfering Dipaths in Planar Digraphs

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Abstract. We give a min-max theorem for the problem of finding the maximum number of dipaths connecting two given vertices in a planar graph so that the internal vertices of distinct dipaths are not adjacent. The theorem also yields a polynomial-time algorithm for finding such a collection of dipaths. We also show that the result can be extended to the case where the internal vertices of distinct dipaths are to be separated by a larger (directed) distance $d > 1$. This is an extension of the work done in [5].

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Let $D = (V, E)$ be a directed graph without loops, and let $s, t \in V$. We denote by $G = (V, E)$ the underlying undirected graph of D. We identify G and D with their embeddings in the 2-sphere S_2. A *d-path* is a simple dipath of length at most d. We call two $s - t$ dipaths P', P'' *d-separate* if there are no d-paths in $G\backslash\{s,t\}$ connecting an internal vertex of P' with an internal vertex P''. Note that P' and P'' being 0-separate is equivalent to their being internally disjoint. We consider the problem of finding a maximum number of pairwise d-separate $s-t$ dipaths. Note that if each arc of D is contained in a digon, then this is equivalent to disallowing undirected paths of length d between paths in the collection. In particular for $d = 1$, this becomes the problem of finding a maximum number of $s-t$ paths (in G) with no chords between them. Fellows [2] proved that for general graphs, deciding if there exists a chordless circuit containing s and t is NP-complete. This chordless circuit problem is equivalent to determining whether there is a pair of 1-separate $s-t$ paths. Hence the d-separation problem is NP-complete for $d = 1$ (and in fact for $d \geq 1$).

We show that for planar graphs the problem can be solved in polynomial time. Moreover, we give a good characterization based on the following concepts. We give somewhat precise definitions, however the reader may choose to omit this for the time being. Recall that a (closed) curve is any continuous function $C : S_1 \to S_2$, where S_1 denotes a closed interval $[a, b]$ with $a \leq b$ (respectively, unit circle in the complex plane). For any pair of curves C, D a pair (x, y) is said to give an *intersection* if $C(x) = D(y)$. We sometimes say that C and D *intersect* at $C(x)$. A pair of connected, closed sets (I, J) (in the domains) is said to give a *crossing* if $C(I) = D(J)$ and there are open neighbourhoods N_I,N_J of I and J such

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that the curves $C|N_I$ and $D|N_J$ only intersect in $I \times J$ and $C(N_I)$ contains points on either side of D (relative to some fixed orientation of D). Intuitively, we say that the curves *cross along* $C(I)$ or $D(J)$. When it is clear from the context we do not distinguish between a curve and its image.

Let C be a closed curve not traversing s or t. The *winding number* $w(C)$ of C is, roughly speaking, the number of times that C separates s and t. More precisely, consider any curve P from s to t, crossing C only a finite number of times. Let λ be the number of times C crosses P from left to right, and let ρ be the number of times C crosses P from right to left (giving C a clockwise orientation with respect to s and orienting P from s to t). Then $w(C) = \lambda - \rho$. This number can be seen to be positive and independent of the choice of P. We say a curve C *traverses* a path p if C follows p from one end vertex to the other.

We call a closed curve C (with clockwise orientation relative to s) *d-alternate* if C does not traverse s or t, and there exists a sequence

(1) $$(C_0, p_1, C_1, p_2, C_2, \ldots, p_l, C_l)$$

such that

(2)

(i) p_i is a d-path of $D \setminus \{s,t\}$ with endpoints s_i, t_i $(i = 1, \ldots, l)$;

(ii) C_i is a (noncrossing) curve of positive length from t_i to s_{i+1} (if $l = 0$ we take $s_0 = t_1$ as point on C_0) and these are the only vertices of D that C_i intersects ($i = 1, \ldots, l$ and $C_0 = C_l$);

(iii) C traverses the paths and curves given in (1) in the described order;

(iv) C_i may only cross arcs from right to left (relative to the orientation derived from C) and none of these arcs corresponds to an arc of any p_i.

Informally, condition (iv) requires that any arcs crossed by C_i must be directed towards s.

We prove the following theorem in which the alternating curves form the analogue of an edge cut in Menger's Theorem.

Theorem A.

(i) *There exist k pairwise d-separate $s - t$ dipaths, if and only if $l(C) \geq k \cdot w(C)$ for each d-alternate closed curve C.*

(ii) *A maximum number of pairwise d-separate $s - t$ dipaths can be found in polynomial time.*

(iii) *The curves C in* (i) *can be restricted to those with $l(C) < |V|$.*

Before continuing, we point out the significance of forbidding an alternate curve to cross arcs which are contained in the p_i's (see 2 (iv)). For convenience, we say that such a curve *does not cross its graphical components.* If this were not the case, then Figure 1 displays a graph which has an $s - t$ dipath, however the curve C has length one which is less than

$w(C) = 2$. The curve C is not a proof, however, that there is no collection of size one. This is because we allow the dipaths in our collections to have chords themselves. Thus considering curves which **do** cross their graphical components gives a condition which is too strong.

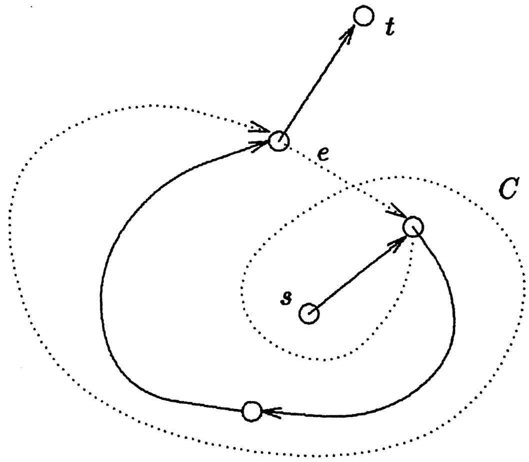

Figure 1: The curve C traverses a single arc e.

We will need the following notion in our proof. Let X be a subset of $S_2 \setminus \{s,t\}$. A dipath or a curve p is called a *singel (relative to X)*, if every curve from s to t either intersects X or p.

Proof of Theorem. Consider the universal covering surface (ref. [3]) U of $S_2 \setminus \{s,t\}$, with projection mapping $\pi : U \longrightarrow S_2 \setminus \{s,t\}$. The inverse $\pi^{-1}[D \setminus \{s,t\}]$ of $D \setminus \{s,t\}$ is an infinite graph on U. (The universal covering surface is obtained from S_2 by puncturing holes at s and t and then cutting between the holes to form a rectangular surface. Copies of this rectangle are glued together to form the covering surface which then contains an infinite number of copies of $S_2 \setminus \{s,t\}$.) For a point $v \in U$, we denote by v' the lifting of $\pi(v)$ which occurs next to the right of v. If Q is a lifting of a path P in S_2, then Q' denotes the next lifting of P to the right of Q.

I. **Necessity in** (i). Without loss of generality $k > 0$. Let C be a d-alternate closed curve and let $P_1, \ldots, P_k$ be pairwise d-separate $s-t$ dipaths. Without loss of generality we assume that the dipaths are simple. First suppose that $k = 1$. It is clear that G_C can be decomposed into circuits $R_1, R_2, \ldots R_m$ such that the first $w(C)$ correspond to curves with winding number one. Furthermore, any such curve must have length at least one since P_1 must cross it in the *forward direction* (i.e., cross the curve from right to left). Hence $l(C) \geq w(C)$.

Now suppose that $k > 1$. Clearly each P_i crosses C in the forward direction at least $w(C)$ times. It is not the case that any dipath must intersect at least $w(C)$ paths p_j in (1) (an example can be obtained for $d = 2$ by modifying Figure 1 by adding a vertex of D at the non-simple point in the image of C). We shall see this *is* true for dipaths which are not singels relative to a any orange segment. Note that since $k > 1$, no P_i can be such a singel and hence each path in the collection must intersect $w(C)$ of the d-paths p_j. Since distinct P_i and $P_{i'}$ are d-separate, this shows that there are at least $k \cdot w(C)\, p_j$'s, i.e., $l(C) \geq k \cdot w(C)$. So it is enough to show that any dipath which is not a singel must intersect C sufficiently many times. We must introduce some tedious concepts before stating a more general result.

A *bicolouring* (O, B) of a closed curve C is any partition of its range into two sets: an orange set whose inverse image consists of a finite number of closed sets (of the unit circle in R^2) and a blue set whose inverse image consists of a finite number of open sets. An *orange segment* of C is any maximal subcurve of C whose image is contained in O. Here we mean maximal by set inclusion of the image. We call a simple curve Q from s to t *regular* (with respect to the bicolouring) if it only crosses C in the forward direction along orange segments of C.

Claim. Let C be a closed curve not traversing s and t and let (O, B) be a bicolouring of C. Then each regular curve Q from s to t either intersects at least $w(C)$ distinct orange segments or is a singel relative to some orange segment.

Proof of Claim: Let $C = C_1 \circ C_2 \ldots \circ C_{\omega(C)}$, where $C_1, \ldots C_{\omega(C)}$ are closed curves each of winding number at least one. Consider any lifitng $L_1 \circ L_2^{(1)} \ldots \circ L_{\omega(C)}^{(\omega(C)-1)}$ of C such that L_i is a lifting of C_i and $L_{i+1}^{(i)}$ denotes the i^{th} lifting to the right of L_{i+1}. Then any lifting L of Q must cross each L_i at least once in the forward direction. Hence L crosses liftings (not necessarily distinct) $O_1, \ldots, O_{\omega(C)}$ of orange segments of C such that O_i intersects L_i. Hence if $O_i = O_j$, then it follows that there is a completely orange subcurve of C which has winding number one and so Q is clearly a singel. Thus we assume that the O_i's are distinct. Hence either Q crosses at least $\omega(C)$ distinct orange segments of C, or we have, say, that O_1 and O_2 are distinct liftings of the same orange segment. But then there is a point v in the image of O_1 such that there is a curve P from v to v' (the lifting to the right) which follows O_1 and then a subcurve of L and then a subcurve of O_2. Therefore, Q is a singel with respect to $\pi(O_1)$. The proof of the claim and necessity in the theorem are now complete.

II.**Algorithm.** We next describe an algorithm finding for any k, either k pairwise d-separate $s - t$ dipaths or a d-alternate closed curve C with $l(C) < k \cdot w(C)$. We assume, without loss of generality, that there is no arc connecting s and t.

First we introduce some notation and terminology. We think of D being embedded on the 2-sphere S_2. For a noncrossing, closed curve C, $R(C)$ denotes the region encircled by C in a clockwise orientation. Let P', P'' be two arc-disjoint $s - t$ dipaths without crossings. We denote by $R(P', P'')$ the region $R(P' \cdot (P'')^{-1})$. We call the pair (P', P'') *internally d-separate* if $R(P', P'')$ is an open disc not containing a dipath which is a singel, relative to $R(P', P'')$. Note that even if (P', P'') is internally d-separate, P' and P'' can have a vertex $v \neq s, t$ in common. Moreover, P' and P'' are d-separate if and only if both (P', P'') and (P'', P') are internally d-separate.

The proof of sufficiency (i.e., the algorithm) works by induction on d. The case $d = 0$ follows from the directed vertex version of Menger's Theorem. So suppose that d is positive.

For $k = 1$ the algorithm is trivial: either there exists an $s - t$ dipath, or there exists a closed curve C with $w(C) = 1$ such that C does not intersect any vertices of D and only crosses arcs from right to left. Thus implying $l(C) = 0 < 1 \cdot w(C)$.

Suppose now that $k > 1$, and that we have found $k-1$ pairwise d-separate $s-t$ dipaths $P_1, \ldots, P_{k-1}$. In the special case that $k = 2$ we assume that there exist two $(d-1)$-separate $s-t$ dipaths P, Q. If no such pair exists, then (by induction) we can find an appropriate $(d-1)$-alternate curve. Hence for $k = 2$ we may choose P_1 to be P.

We may assume that the first arcs of $P_1, \ldots, P_{k-1}$ occur in this order clockwise at s. Let P_k be a dipath 'parallel' to the left of P_1. That is, we add to each arc traversed by P_1 a parallel arc at the left hand side (with respect to the orientation of P_1), and P_k follows these new arcs. Note that we may (implicitly) add parallel arcs without changing our problem. Note also that in the case $k = 2$ we have chosen P_1 so that (P_1, P_2) is internally d-separate. Then the first arcs of $P_1, \ldots, P_k$ occur in this order clockwise at s, and each pair (P_{i-1}, P_i) is internally d-separate $(i = 2, \ldots, k)$.

Now for $n = k, k+1, k+2, \ldots$ we do the following. We have pairwise arc-disjoint $s-t$ dipaths $P_{n-k+1}, \ldots, P_n$, without crossings, so that the first arcs of $P_{n-k+1}, \ldots, P_n$ occur in this order clockwise at s, and each pair (P_{i-1}, P_i) is internally d-separate $(i = n-k+2, \ldots, n)$.

If also the pair (P_n, P_{n-k+1}) is internally d-separate, then $P_{n-k+1}, \ldots, P_n$ are pairwise d-separate, and hence we have our desired set of k $s-t$ dipaths. If (P_n, P_{n-k+1}) is not internally d-separate, let P_{n+1} be the dipath in $\overline{R}(P_{n-k+1}, P_{n-k+2})$ (the closure) such that (P_n, P_{n+1}) is internally d-separate and such that $R(P_{n+1}, P_{n-k+2})$ is as large as possible. Then reset $n := n+1$, and repeat.

III. **Correctness and running time.** Suppose we do $|V|$ iterations without finding k d-separate paths. Let $l := k + |V|$. Consider the universal covering surface (ref. [3]) U of $S_2 \setminus \{s,t\}$, with projection mapping $\pi : U \longrightarrow S_2 \setminus \{s,t\}$. The inverse $\pi^{-1}[D \setminus \{s,t\}]$ of $D \setminus \{s,t\}$ is an infinite graph on U.

For noncrossing dipaths (or paths) Q_a and Q_b in U with Q_a to the left of Q_b, we denote by $R(Q_a, Q_b)$ the bounded region *between* the two paths. Hence $R(Q, Q')$ contains no other lifting of $\pi(Q)$.

By our construction, there exist liftings $Q_1, \ldots, Q_l$ of $P_1, \ldots, P_l$, respectively, so that Q_n is to the right of Q_{n-1} (possibly touching) for $n = 2, \ldots, l$, and such that $Q_{n-k+2}, \ldots, Q_n$ are contained in $\overline{R}(Q_{n-k+1}, Q'_{n-k+1})$ for $n = k, k+1, \ldots, l$.

For each $n = k+1, \ldots, l$ let $T_n \neq Q_{n+1}$ be an $s-t$ dipath of $\pi^{-1}(D)$ such that T_n is contained in $\overline{R}(Q'_{n-k+1}, Q_{n+1})$, $(P_n, \pi(T_n))$ is not internally d-separate and subject to this $R(Q'_{n-k+1}, T_n)$ is maximized.

Consider D', the digraph obtained by restricting $\pi^{-1}(D)$ to the region $\overline{R}(T_n, Q_{n+1})$. It follows that D' contains precisely two $s-t$ dipaths: T_n and Q_{n+1}. (Otherwise we contradict either the definition of T_n or Q_{n+1}.) This implies that $R(T_n, Q_{n+1})$ is an open disc F. For each $n \geq k$, let V_{n+1} denote those internal vertices of Q_{n+1} which lie on the boundary of this disk and let V_k denote the internal vertices of Q_k. For each $v \in V_{n+1}$, there is some $u_1 \notin V_{n+1}$ which lies on the boundary of F and some u_2 of Q_n such that there is a d-path p_v (in $\pi^{-1}(D)$) from u_1 to u_2. For any such v we choose u_1, u_2 so that p_v is of minimum length. Furthermore, we can choose our T_i's so that $u_2 \in V_n$.

Now consider C_1 the vertices which occur before u_1 and v on T_n and Q_{n+1}, and C_2 the vertices appearing after u_1 and v, then we have:

(3) there is no dipath in $D' - \{u_1, v\}$ from C_1 to C_2.

It follows that there is a a curve C_v oriented from u_1 to v which does not intersect $\pi^{-1}(V) - \{u_1, v\}$ and only traverses arcs of $\pi^{-1}(D)$ from right to left. If $u_1 = v$, we simply choose C_v to be a loop - so that (3) will hold.

Now choose $v_m \in V_m$ and for each $n = m-1, m-2, \ldots k$, let v_n be the starting vertex of $p_{v_{n+1}}$. Since $l = k + |V|$, there exist n', n'' with $l \geq n'' > n' \geq k$ such that $\pi(v_{n''}) = \pi(v_{n'})$. Let R be the curve

$$p_{v_{n'+1}} \cdot C_{v_{n'+1}} \cdot p_{v_{n'+2}} \cdot C_{v_{n'+2}} \cdot \ldots \cdot p_{v_{n''}} \cdot C_{v_{n''}},$$

and let C be the projection $\pi \circ R$ of R to S_2 (where each p_{v_j} starts at some v_{j-1} on Q_{j-1}). Thus C is a d-alternate curve as long as no $\pi(C_{v_{n+1}})$ crosses an arc of some $\pi(p_{v_{i+1}})$. Suppose that such an arc is crossed. and let Q_i and T_i be the liftings (of P_i and $\pi(T_i)$) which intersect $p_{v_{i+1}}$. Note that (3) implies that neither Q_i nor T_i intersects the region $R(T_n, Q_{n+1})$. By the same reasoning, we also have that $p_{v_{i+1}}$ intersects at most one of T_n or Q_{n+1}, say it intersects T_n. This is eqivalent to saying $i < n$ (see Figure 2).

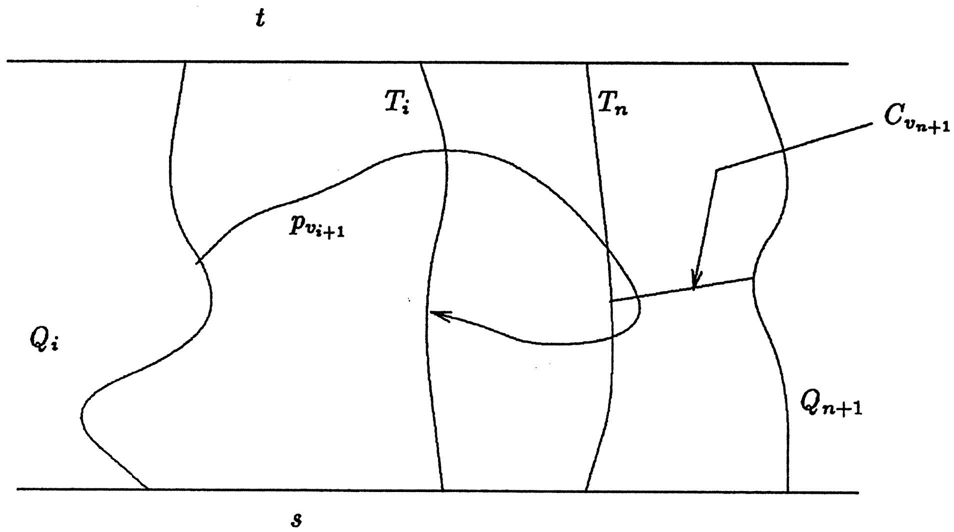

Figure 2:

It follows that T_i is contained in $\overline{R}(Q_i, T_n)$ and hence T_i intersects an internal vertex of $p_{v_{i+1}}$. But then we could have chosen p_{v_i} to have shorter length, a contradiction. Hence C is a d-alternate closed curve with $l(C) = n'' - n'$. We now show that $k \cdot w(C) > n'' - n'$, proving sufficiency in (i).

For any lifting Q of any simple $s-t$ dipath P and any $i \geq 0$, let $Q^{(i)}$ be the ith lifting to the right of Q. That is, $Q^{(0)} = Q$ and $Q^{(i+1)} = (Q^{(i)})'$.

Let $u := \lfloor \frac{n''-n'}{k} \rfloor$. We must show $w(C) > u$. If $u = 0$, then $w(C) > u = 0$ since $v_{n''} \neq v_{n'}$. If $u > 0$, then $v_{n''}$ is strictly to the right of $Q'_{n''-k}$ and $Q'_{n''-k}$ is to the right of $Q^{(u)}_{n'}$ (since $Q_{n''-k}$ is to the right of $Q^{(u-1)}_{n'}$, as $n'' - k \geq n' + (u-1)k$). So $v_{n''}$ is strictly to the right of $Q^{(u)}_{n'}$. Therefore, $w(C) > u$.

□

This algorithm can also be extended to show that for any fixed surface S and any fixed k, there exists a polynomial-time algorithm for the problem of finding k pairwise d-separate $s-t$ paths in any graph embedded on S.

We consider the following problem: given a planar undirected graph G and collection $\mathcal{T} = \{T_1, \ldots, T_m\}$, of subsets of the edges each of which induces a connected subgraph of G, find a maximum collection of vertex-disjoint paths such that there is no path which is contained in some T_i and connects internal vertices of distinct paths in the collection. Such a collection we call *$\mathcal{T}$-separate*. We define a curve to be $\mathcal{T}$-alternate in a fashion similar to before except that now each p_i is simply a subpath of one of the T_j and that each C_i is a curve passing through a face of G. Using the previous theorem and the above method of proof, one can show the following.

Theorem B.

(i) *There exist k pairwise $\mathcal{T}$-separate $s-t$ dipaths, if and only if $l(C) \geq k \cdot w(C)$ for each $\mathcal{T}$-alternate closed curve C.*

(ii) *A maximum number of pairwise $\mathcal{T}$-separate $s-t$ dipaths can be found in polynomial time.*

(iii) *The curves C in* (i) *can be restricted to those with $l(C) < |V|$.*

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