1991

H.J.J. te Riele

On the history of the function $M(x)/\sqrt{x}$
since Stieltjes

Department of Numerical Mathematics   Report NM-R9120 December

CWI, nationaal instituut voor onderzoek op het gebied van wiskunde en informatica

CWI is the research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a non-profit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands organization for scientific research (NWO).

On the History of the function $M(x)/\sqrt{x}$ since Stieltjes

H.J.J. te Riele
CWI
Kruislaan 413, 1098 SJ Amsterdam
The Netherlands

Abstract

By $M(x)$ one usually denotes the function which counts the difference between the number of squarefree positive integers $\leq x$ with an *even* and those with an *odd* number of prime factors. It is known that the boundedness of $M(x)/\sqrt{x}$ would imply the truth of the Riemann hypothesis. Stieltjes was one of the very first who studied $M(x)$ and tried to prove the boundedness of $M(x)/\sqrt{x}$.
This paper sketches the historical developments since Stieltjes which now point in the direction of the *unboundedness* of this function. The best results *proved* so far are that there exists an x for which $M(x)/\sqrt{x} > 1.06$ and another x for which $M(x)/\sqrt{x} < -1.009$.

1991 Mathematics Subject Classification: 11M06, 11Yxx, 11-03, 01A55, 01A60.
Keywords & Phrases: Squarefree numbers, Möbius function, Riemann hypothesis, Lattice basis reduction.

Note: This paper will appear in an Appendix to a new edition of the Collected Works of T.J. Stieltjes, which will commemorate the 100th anniversary of his death, in 1994.

Report NM-R9120
ISSN 0169-0388
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

1. Introduction

The Möbius function $\mu(n)$ is defined as follows:

$$\mu(n) := \begin{cases} 1, & n = 1, \\ 0, & \text{if } n \text{ is divisible by the square of a prime number,} \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases}$$

Taking the sum of the values of $\mu(n)$ for all $n \leq x$, we obtain the function

$$M(x) = \sum_{1 \leq n \leq x} \mu(n),$$

which is the difference between the number of squarefree positive integers $n \leq x$ with an even number of prime factors and those with an odd number of prime factors.

In the "Comptes Rendues de l'Académie des Sciences de Paris" of 13 July 1885, Stieltjes published a two-page note (Paper # XLIV) under the rather vague title: "Sur une fonction uniforme". In this note he announced a proof of the (now famous) Riemann hypothesis as follows: "I have succeeded to put this proposition beyond doubt by a rigorous proof". The only explanation Stieltjes gave for this remarkable assertion was that he was able to prove that the series

$$\frac{1}{\zeta(z)} = 1 - \frac{1}{2^z} - \frac{1}{3^z} - \frac{1}{5^z} + \frac{1}{6^z} - \cdots = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$$

"converges and defines an analytic function as long as the real part of z exceeds $\frac{1}{2}$" (here, $\zeta(z)$ is the well-known Riemann zeta function). This indeed would imply that all the complex zeros of $\zeta(z)$ have real part $\frac{1}{2}$.

Stieltjes never published his "proof". In *Section 2* we will quote from his correspondence with his friend Hermite and with Mittag-Leffler [BB]. From this we learn that Stieltjes believed that he could prove that the function $M(x)/\sqrt{x}$ always stays within two fixed limits (possibly $+1$ and -1). This was probably based on a table of values of $M(n)$, for $1 \leq n \leq 1200$, $2000 \leq n \leq 2100$, and $6000 \leq n \leq 6100$, which was found in the inheritance of Stieltjes. The Riemann hypothesis can be derived from the boundedness of $M(x)/\sqrt{x}$ as follows. For $\sigma = \Re z > 1$, we have (by partial summation)

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \sum_{n=1}^{\infty} \frac{M(n) - M(n-1)}{n^z}$$
$$= \sum_{n=1}^{\infty} M(n) \left\{ \frac{1}{n^z} - \frac{1}{(n+1)^z} \right\} = \sum_{n=1}^{\infty} M(n) \int_n^{n+1} \frac{z dx}{x^{z+1}}$$
$$= z \sum_{n=1}^{\infty} \int_n^{n+1} \frac{M(x)dx}{x^{z+1}} = z \int_1^{\infty} \frac{M(x)dx}{x^{z+1}},$$

since $M(x)$ is constant on each interval $[n, n+1)$. The boundedness of $M(x)/\sqrt{x}$ would imply that the last integral in the above formula defines a function analytic in $\sigma > \frac{1}{2}$, and this would give an analytic continuation of $1/\zeta(z)$ from $\sigma > 1$ to $\sigma > \frac{1}{2}$. In particular, this would imply that $\zeta(z)$ has no zeros in $\sigma > \frac{1}{2}$, which is, by the functional equation for $\zeta(z)$, equivalent to the Riemann hypothesis. In addition, it is not difficult to derive from the above formula that all the complex zeros of $\zeta(z)$ are simple (see, e.g., [OR, p. 141]).

After Stieltjes, many other researchers have computed tables of $M(x)$, in order to collect more numerical data about the behaviour of $M(x)/\sqrt{x}$. In *Section 3* we will briefly survey these computations. The first one after Stieltjes was Mertens [Mer] who, in 1897, published a paper with a 50-page table of $\mu(n)$ and $M(n)$ for $n = 1, 2, \cdots, 10000$. Based on his table, Mertens concluded that the inequality

$$|M(x)| < \sqrt{x}, \quad x > 1,$$

is "very probable". This is now known as the *Mertens conjecture.* Some historical notes about Mertens and his conjecture may be found in [Ri2].

In 1942, Ingham [Ing] published a paper which raised the first serious doubts about the validity of the Mertens conjecture. Ingham's paper showed that it is possible to prove the existence of certain large values of $|M(x)|/\sqrt{x}$ without the need to explicitly compute $M(x)$. This stimulated a series of subsequent papers until, in 1985, Odlyzko and Te Riele [OR] published a disproof of the Mertens conjecture. This disproof is indirect, and does not produce any single value of x for which $|M(x)|/\sqrt{x} > 1$. In 1987, Pintz [Pin] was able to show, on the basis of certain computations carried out by Te Riele, that

$$\max_{1 \le x \le X} |M(x)|/\sqrt{x} > 1 \text{ for } X = \exp(3.21 \times 10^{64}).$$

These developments will be sketched briefly in *Section 4*. There it will become clear that since 1942 the evidence for the *unboundedness* of the function $M(x)/\sqrt{x}$ has increased considerably, as opposed to what was believed about 100 years ago by skilled researchers like Stieltjes and Mertens.

The function $M(x)$ is known to change sign infinitely often. Various rigorous results about sign changes of $M(x)$ can be found in [Pi3] and the references given there. Dress [Dre] has written an interesting historical survey on oscillating properties of $M(x)$.

2. Some correspondence between Stieltjes and Hermite, and between Stieltjes and Mittag-Leffler, on the boundedness of $M(x)/\sqrt{x}$

In this section we shall quote some correspondence between Stieltjes and Hermite, and between Stieltjes and Mittag-Leffler, in order to clarify Stieltjes' role in the history of the function $M(x)/\sqrt{x}$.

Two days before the appearance of Stieltjes' announcement in the "Comptes Rendues", Stieltjes wrote a letter to Hermite (Lettre # 79 in [BB]) in which he claimed to have a proof of the boundedness of the function $M(x)/\sqrt{x}$. After some preliminary remarks, Stieltjes writes[1] (we translate into English):

Indeed, if, instead of $1 : \zeta(s) = \prod(1 - p^{-s})$, *I consider*

$$1 : \zeta(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{6^s} \cdots = \sum_{1}^{\infty} \frac{f(n)}{n^s},$$

there is this main difference, between the infinite product and the series, that the latter converges for $s > \frac{1}{2}$, *while, in the product, one must assume that* $s > 1$. *Look how I demonstrate it: The function* $f(n)$ *is equal to zero when* n *is divisible by a square and for other values of* n, *it is equal to* $(-1)^k$, k *being the number of prime factors of* n. *Now, I find that in the sum*

$$g(n) = f(1) + f(2) + \cdots + f(n),$$

the terms ± 1 *compensate sufficiently well, so that* $g(n)/\sqrt{n}$ *always stays within two fixed limits, no matter how large* n *is (probably one can take* $+1$ *and* -1 *for these limits).*

After deriving from the last statement that $\sum f(n)n^{-s}$ converges for $s > \frac{1}{2}$, Stieltjes remarks:

You see that everything depends on an arithmetical study of that sum $f(1) + f(2) + \cdots + f(n)$. *My proof is very painful; I will try, as soon as I will resume these studies, to simplify it.*

Two days later, Hermite presented Stieltjes' note, mentioned in the introduction, to the French Academy of Sciences. No doubt, it must have fascinated many mathematicians. Mittag-Leffler immediately asked for details. This appears from four letters of Stieltjes to Mittag-Leffler given in an Appendix to [BB]. In the fourth letter, dated April 15, 1887 [BB, pp. 449-452], Stieltjes still claims the boundedness of $M(x)/\sqrt{x}$:

Denoting by[2]

---

[1] Stieltjes writes $f(n)$ for the Möbius function $\mu(n)$, and $g(n)$ for $M(n)$.
[2] Here, Stieltjes writes $\lambda(n)$ for $\mu(n)$.

$$\sum_1^\infty \lambda(n) n^{-s}$$

the series which is obtained by expanding the infinite product

$$1 : \zeta(s) = \prod (1 - p^{-s}),$$

the convergence of the series for $s > \frac{1}{2}$ *is a consequence of the following* lemma.

The expression $\{\lambda(1)+\lambda(2)+\cdots+\lambda(n)\}/\sqrt{n}$ *alway stays between two fixed limits. (See the theory of the series of this kind in the* "Théorie des nombres" *of Lejeune-Dirichlet, Dedekind.)*
But the proof of this lemma is purely arithmetical and very difficult and I only can obtain it as a result of a whole series of preliminary propositions. I hope that this proof can still be simplified, but in 1885 I already have done my utmost both by looking at the problem in some other way and by replacing this lemma by another one which, however, is of the same nature.

3. Explicit computation of the function $M(x)$

In this section we give a concise survey of explicit computations of $M(x)$ carried out after Stieltjes. These computations were motivated by the wish to collect more numerical evidence for the possible boundedness - or unboundedness - of the function $M(x)/\sqrt{x}$. Here, one should distinguish between the systematic computation of $M(n)$ (and of $\mu(n)$) for all n in a given interval $[1, N]$, and the computation of selected, isolated values of $M(n)$. At first sight, it seems necessary to know all the values of $\mu(m)$, $1 \leq m \leq n$, for the computation of $M(n)$. However, below we will encounter formulas where $M(n)$ is expressed in terms of $M(j)$ with $j \leq n/k$, for some fixed $k \geq 2$ (these formulas become more complicated as k increases). In this way it is possible to compute $M(n)$ for large isolated values of n, in order to get an impression of the behaviour of $M(x)/\sqrt{x}$ in ranges where it is infeasible to compute *all* the values of $M(n)$.

As already mentioned in the introduction, Mertens was the first [Mer] to publish a table of $\mu(n)$ and $M(n)$ (for $n = 1, 2, \cdots, 10000$). He does not explain how he computed this table. From the well-known formula

$$\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor \mu(i) = \sum_{i=1}^{n} M\left(\frac{n}{i}\right) = 1 \tag{3.1}$$

(where by $\lfloor x \rfloor$ we mean the greatest integer $\leq x$) he derives the following relation, which expresses $M(n)$ in terms of $M(n/2)$, $M(n/3)$, $\cdots$, $M(n/k)$, $M(k)$ and $\mu(1)$, $\cdots$, $\mu(k)$, where $k = \lfloor \sqrt{n} \rfloor$:

$$\sum_{i=1}^{k} \left\lfloor \frac{n}{i} \right\rfloor \mu(i) + \sum_{i=1}^{k} M\left(\frac{n}{i}\right) - kM(k) = 1. \tag{3.2}$$

This served as a check, as Mertens writes on p. 763 of [Mer], during the computation of his table. Moreover, Mertens derives a second relation, viz.,

$$M(n) = 2M(k) - \sum_{r,s=1}^{k} \left\lfloor \frac{n}{rs} \right\rfloor \mu(r)\mu(s), \quad k = \lfloor \sqrt{n} \rfloor, \tag{3.3}$$

which expresses $M(n)$ in terms of $M(k)$ and $\mu(1), \cdots, \mu(k)$. This "allows to compute $M(n)$ without knowing the decomposition of the numbers $k+1$ up to n in their prime factors" [Mer, p. 764].

In the year that Mertens published his table, Von Sterneck started a series of four papers presenting tables of $M(n)$, for $n = 1, 2, \cdots, 150000$ [St1], for $n = 150000(50)500000$ [St2] and for 16 selected values of n between 5×10^5 and 5×10^6 [St3, St4]. The latter values were computed by means of a refined version of Mertens' formula (3.2), viz.,

$$\sum_{i=1}^{k} \omega_j \left(\frac{n}{i}\right) \mu(i) + \sum_{i' \leq k} M\left(\frac{n}{i'}\right) - \omega_j(k) M(k) = 0, \quad k = \lfloor \sqrt{n} \rfloor, \quad j = 0, 1, \cdots, \tag{3.4}$$

where $\omega_j(m)$ denotes the number of positive integers $\leq m$ which are not divisible by any of the first j primes and where i' runs through all such positive integers $\leq k$. For $j = 0$, (3.4) reduces to (3.2). Von Sterneck applied (3.4) for $j = 1, 2, 3$ and 4. For $j = 4$, e.g., i' runs through the integers $1, 11, 13, 17, \cdots$ so that it is possible to compute $M(n)$ from a table of M - values up to $\lfloor n/11 \rfloor$. From his results, Von Sterneck draws the conclusion [St4] that the inequality $|M(n)| < \frac{1}{2}\sqrt{n}$, for $n > 200$, "represents an unproved, but extremely probable number-theoretic law".

Fifty years after Von Sterneck, Neubauer [Neu] published an empirical study in which *all* the values of $M(n)$, $1 \leq n \leq 10^8$, were computed. Neubauer computed $\mu(m)$ for a series of 1000 values of m: $1000n < m \leq 1000(n+1)$, for $n = 0, 1, \cdots, 10^5 - 1$, by means of a sieving process which strongly resembles the well-known sieve of Eratosthenes for finding all the primes below a given limit. This is considerably cheaper than computing $\mu(m)$, $\mu(m+1)$, $\cdots$ by factoring m, $m+1$, $\cdots$. Neubauer checked the computations of Von Sterneck in [St2, St3] and he found several errors in [St2] and errors in 9 of the 16 sample values of $M(n)$ which Von Sterneck had published in [St3]. Neubauer also computed many sample values of $M(n)$ for several n between 10^8 and 10^{10}, by means of (3.4), $j = 4$. As a result, he found four values of n for which $M(n) > \frac{1}{2}\sqrt{n}$ (but none for which $M(n) < -\frac{1}{2}\sqrt{n}$), the smallest being $n_0 = 7,760,000,000$ with $M(n_0) = 47465$ and $M(n_0)/\sqrt{n_0} = 0.5388....$ The largest $M(n)/\sqrt{n}$ - value he found was 0.5572..., for $n = 7,770,000,000$.

Yorinaga [Yor] computed all the values of $M(n)$ for $n \leq 4 \times 10^8$, by factoring all $n \leq 4 \times 10^8$.

The most extensive systematic computations have been carried out by Cohen and Dress [Coh]. Their purpose was to find the smallest $n > 200$ for which $M(n)/\sqrt{n} > \frac{1}{2}$, knowing from Neubauer's computations that this n is smaller than 7.76×10^9. Without taking the trouble to mention their method, they state that they have carried out their computations in one week on a TI 980B mini-computer. They computed all the values of $M(n)$ for n up to 7.8×10^9 and saved a table of $M(n)$ for $n = 10^7(10^7)7.8 \times 10^9$. The smallest $n > 200$ for which $M(n)/\sqrt{n} > \frac{1}{2}$ turned out to be $n_0 = 7,725,038,629$ with $M(n_0) = 43947$.

J. Schröder ([Sc1], [Sc2], [Sc3]) has derived several rather complicated formulas for computing $M(x)$. As far as we know these formulas have never been used for the computation of extensive tables of $M(x)$.

Liouville's function $\lambda(n)$ is defined by the equation $\lambda(n) = (-1)^r$ where r is the number of prime factors of n, multiple factors counted according to their multiplicity. Lehman [Leh] has published a method to compute the function

$$L(x) = \sum_{n \le x} \lambda(n)$$

at isolated values of x in $\mathcal{O}(x^{2/3+\epsilon})$ bit operations. According to Lehman, a similar method (with the same amount of work) can be derived from (3.1) for the computation of $M(x)$. As far as we know, this method has never been implemented. An analytic method of Lagarias and Odlyzko [LO] for computing $\pi(x)$ (i.e., the number of primes $\le x$) can be adapted to obtain a method for computing $M(x)$ that requires on the order of $\mathcal{O}(x^{1/2+\epsilon})$ bit operations. However, this method is not likely to become practical in the near future [Opc].

4. Evidence for the unboundedness of $M(x)/\sqrt{x}$

In Section 2 of [OR] an extensive historical survey is given of the work on the Mertens conjecture. Various reasons are discussed why this, and the weaker conjecture

$$|M(x)| < C\sqrt{x} \quad \text{for any given } C > 0, \quad x > x_0(C), \tag{4.1}$$

are believed to be false. Here, we shall mainly discuss the developments which have led to the disproof of the Mertens conjecture, and to the belief that the function $M(x)/\sqrt{x}$ is unbounded.

We write $x = e^y$, $-\infty < y < \infty$, and we define

$$m(y) := M(x)x^{-1/2} = M(e^y)e^{-y/2},$$

and

$$\overline{m} := \limsup_{y \to \infty} m(y), \quad \underline{m} := \liminf_{y \to \infty} m(y).$$

Then we have the following ([Ing], [JP], [Jur], [OR])

Theorem 1. *Suppose that* $K(y) \in C^2(-\infty, \infty)$, $K(y) \ge 0$, $K(-y) = K(y)$, $K(y) = \mathcal{O}((1 + y^2)^{-1})$ *as* $y \to \infty$, *and that the function* $k(t)$ *defined by*

$$k(t) = \int_{-\infty}^{\infty} K(y) e^{-ity}\, dy,$$

satisfies $k(t) = 0$ *for* $|t| \geq T$ *for some* T*, and* $k(0) = 1$*. If the zeros* $\rho = \beta + i\gamma$ *of the zeta function with* $0 < \beta < 1$ *and* $|\gamma| < T$ *satisfy* $\beta = \frac{1}{2}$ *and are simple, then for any* y_0*,*

$$\underline{m} \leq h_K(y_0) \leq \overline{m},$$

where

$$h_K(y) = \sum_\rho k(\gamma) \frac{e^{i\gamma y}}{\rho\, \zeta'(\rho)}.$$

From an almost-periodicity argument it follows that any value $h_K(y)$ is approximated arbitrarily closely, infinitely often by $M(x)/\sqrt{x}$.

The simplest known function $k(t)$ that satisfies the conditions of Theorem 1 is the Fejer kernel used by Ingham:

$$k(t) = \begin{cases} 1 - |t|/T, & |t| \leq T, \\ 0, & |t| > T. \end{cases} \tag{4.2}$$

This yields

$$\begin{aligned} h_K(y) &= \sum_{|\gamma|<T} \left(1 - \frac{|\gamma|}{T}\right) \frac{e^{i\gamma y}}{\rho\, \zeta'(\rho)} \\ &= 2 \sum_{0<\gamma<T} \left(1 - \frac{\gamma}{T}\right) \frac{\cos(\gamma y - \psi_\gamma)}{|\rho\, \zeta'(\rho)|}, \end{aligned} \tag{4.3}$$

where

$$\psi_\gamma = \arg \rho\, \zeta'(\rho).$$

It is known that $\sum_\rho |\rho\, \zeta'(\rho)|^{-1}$ diverges, so that the sum of the cos-coefficients in (4.3) can be made arbitrarily large by choosing T large enough. If we could manage to find a value of y such that all of the $\gamma y - \psi_\gamma$ were close to integer multiples of 2π, then we could make $h_K(y)$ arbitrarily large. This would contradict, by Theorem 1, any conjecture of the form (4.1). If the γ's were linearly independent over the rationals, then by Kronecker's theorem (see, e.g., [HW], Theorem 442) there would indeed exist, for any $\epsilon > 0$, integer values of y satisfying

$$|\gamma y - \psi_\gamma - 2\pi m_\gamma| < \epsilon$$

for all $\gamma \in (0, T)$ and certain integers m_γ. That would show that $h_K(y)$, and hence $M(x)/\sqrt{x}$, can be made arbitrarily large. A similar argument can be given to imply that $h_K(y)$, and hence $M(x)/\sqrt{x}$, can be made arbitrarily large on the negative side, assuming again the linear independency of the γ's over the rationals.

No good reason is known why among the γ's there should exist any linear dependencies over the rationals. Bailey and Ferguson [BF] have shown that if there exists any linear relation of the form $\sum_{i=1}^{8} c_i \gamma_i = 0$ where $c_i \in \mathbb{Z}$ and γ_i is the imaginary part of the i-th complex zero of the Riemann zeta function, then the Euclidean norm of the vector (c_i) exceeds the value 5.1×10^{24}. Bateman et al. [Bat] have shown that if $m(y)$ is *bounded* then there exist *infinitely many* non-trivial relations of the form $\sum_{i=1}^{N} c_i\gamma_i = 0$, where $c_i = 0, \pm 1$, or ± 2, and at most one of the c_i satisfies $|c_i| = 2$. Bateman et al. [Bat] also showed that there are no such relations for $N \leq 20$. Their numerical results did not give any evidence for the possible existence of such relations for $N > 20$.

The method which actually led to a disproof of the Mertens conjecture is based on finding values of y for which $h_K(y)$ is large in absolute value. Spira [Spi] was the first to follow this approach. He started to compute $h_K(y)$ according to (4.3) for $T = 100$ for a fine grid of values of $y \in [0, 1000]$, and subsequently he computed $h_K(y)$ for $T = 200$, 500 and 1000 for a selection of y-values. In this way Spira showed that $\overline{m} \geq 0.5355$ and $\underline{m} \leq -0.6027$.
Jurkat et al. [JP] realized that the size of the sum $h_K(y)$ is determined largely by the first few terms, since, numerically, the numbers $(\rho\,\zeta'(\rho))^{-1}$ typically appear to be of order ρ^{-1}. Therefore, they looked for values of y such that

$$\cos(\gamma_1 y - \pi\psi_1) = 1$$

and

$$\cos(\gamma_i y - \pi\psi_i) > 0.9 \text{ (say), for } i = 2, \cdots, N+1,$$

N being as large as possible. This gives an inhomogeneous Diophantine approximation problem, for which Jurkat et al. [JP] devised an ingenious algorithm. Moreover, they used a somewhat better kernel than (4.2), viz., $k(t) = g(t/T)$ where

$$g(t) = \begin{cases} (1-|t|)\cos(\pi t) + \pi^{-1}\sin(\pi|t|), & |t| \leq 1, \\ 0, & |t| > 1. \end{cases} \tag{4.4}$$

By applying their algorithm with $N = 12$ they found that $\overline{m} \geq 0.779$.
Jurkat and Peyerimhoff used a programmable desk calculator to carry out their computations. Te Riele [Ri1] implemented the J.-P. algorithm (together with a few small improvements) on a high speed computer and proved that $\overline{m} \geq 0.860$ and $\underline{m} \leq -0.843$.
A remarkably efficient new algorithm of Lenstra, Lenstra and Lovász [LLL] for finding short vectors in lattices was applied by Odlyzko et al. [OR] to the above mentioned inhomogeneous Diophantine approximation problem. It was estimated that $N = 70$ would be sufficient, in order to disprove the Mertens conjecture. Any value of y that would come out was likely to be quite large, viz., of the order of 10^{70} in size. Therefore, it was necessary to compute the first 2000 γ's to a precision of about 75 decimal digits (actually, 100 decimal digits were used). The best lower and upper bounds found for $\overline{m}$ and $\underline{m}$ were 1.06 and -1.009, respectively.
Recently, Möller [Mol] carried out a numerical study on the Mertens conjecture along the lines of Jurkat et al [JP]. By means of a Z80-microcomputer system he found that $\overline{m} \geq 0.875$, which slightly improves Te Riele's result [Ri1].
Table 1 summarizes the results obtained by various authors for $M(x)/\sqrt{x}$.

The method used here is *ineffective* in that it does not give a precise value, or an upperbound for x where $M(x)/\sqrt{x}$ becomes large (resp. small). Recently, Pintz [Pin] gave an *effective* disproof of the Mertens conjecture, based on the following theorem. Let

$$h_1(y,T,\epsilon) := 2 \sum_{0<\gamma<T} e^{-\epsilon\gamma^2} \left[\frac{\cos(\gamma y - \pi\psi_\gamma)}{|\rho\,\zeta'(\rho)|}\right].$$

Theorem 2 (Pintz [Pin]). *If there exists a value of $y \in [e^7, e^{5\times 10^4}]$ with*

$$|h_1(y, 1.4\times 10^4, 1.5\times 10^{-6})| > 1 + e^{-40}$$

then

$$\max_{1\le x\le X} |M(x)|/\sqrt{x} > 1 \quad \textit{for} \quad X = e^{y+\sqrt{y}}.$$

Let

$$h_2(y,T) := 2 \sum_{0<\gamma<T} g\left(\frac{\gamma}{T}\right) \frac{\cos(\gamma y - \pi\psi_\gamma)}{|\rho\,\zeta'(\rho)|},$$

where g is defined by (4.4). A good candidate for y in Theorem 2 is naturally any positive value of y in the given range for which $|h_2(y,T)| > 1$. Such a value $y_0 \approx 3.2097 \times 10^{64}$ was given by Odlyzko and Te Riele on line 21 of Table 3 in [OR]. The present author verified that $h_1(y_0, 1.4\times 10^4, 1.5\times 10^{-6}) = -1.00223...$ so that from Theorem 2 it follows that $|M(x)/\sqrt{x}| > 1$ for some x with $1 \le x \le \exp(3.21 \times 10^{64})$.

Table 1. Results on $M(x)/\sqrt{x}$ obtained by means of Theorem 1

author(s)	$\overline{m} = \limsup_{x\to\infty} M(x)/\sqrt{x} \ge$	$\underline{m} = \liminf_{x\to\infty} M(x)/\sqrt{x} \le$
Spira [Spi]	0.535	−0.602
Jurkat et al. [JP]	0.779	−0.638
Te Riele [Ri1]	0.860	−0.843
Odlyzko et al. [OR]	1.060	−1.009
Möller [Mol]	0.875	

Acknowledgements

The author likes to thank Andrew Odlyzko for pointing out the work of J. Schröder ([Sc1], [Sc2], [Sc3]) and Jan van de Lune for his constructive comments on a draught version of this paper.

References

[BF] Bailey, D.H., Ferguson, H.R.P.: Numerical results on relations between fundamental constants using a new algorithm. Math. Comp. **53** (1989) 649-656

[BB] Baillaud, B., Bourget, H.: Correspondance d'Hermite et de Stieltjes. (With an appendix entitled: Lettres de Stieltjes à M. Mittag-Leffler sur la fonction $\zeta(s)$ de Riemann.) Gauthier-Villars, Paris, 1905

[Bat] Bateman, P.T., Brown, J.W., Hall, R.S., Kloss, K.E., Stemmler. R.M.: Linear relations connecting the imaginary parts of the zeros of the zeta function. In: Atkin, A.O.L., Birch, B.J. (eds.) Computers in Number Theory. New York, 1971, 11-19

[Coh] Cohen, H.: Arithmétique et Informatique. Astérisque **61** (1979) 57-61

[Dre] Dress, Fr.:. Théorèmes d'oscillations et fonction de Möbius. In: Séminaire de Théorie des Nombres de Bordeaux. 1983-1984, 33.01-33.33

[HW] Hardy, G.H., Wright, E.M.: An Introduction to the Theory of Numbers, 4th edn. Oxford Univ. Press, 1975

[Ing] Ingham, A.E.: On two conjectures in the theory of numbers. Amer. J. Math. **64** (1942) 313-319

[JP] Jurkat, W., Peyerimhoff, A.: A constructive approach to Kronecker approximations and its application to the Mertens conjecture. J. reine angew. Math. **286/287** (1976) 332-340

[Jur] Jurkat, W.B.: On the Mertens conjecture and related general Ω-theorems. In: Diamond, H. (ed.) Analytic Number Theory. Providence, 1973, 147-158

[LO] Lagarias, J.C., Odlyzko, A.M.: Computing $\pi(x)$: an analytic method. J. Algorithms **8** (1987) 173-191

[Leh] Lehman, R.S.: On Liouville's function. Math. Comp. **14** (1960) 311-320

[LLL] Lenstra, A.K., Lenstra Jr., H.W., Lovász, L.: Factoring polynomials with rational coefficients. Math. Ann. **261** (1982) 515-534

[Mer] Mertens, F.: Über eine zahlentheoretische Funktion. Sitzungsberichte Akad. Wiss. Wien IIa **106** (1897) 761–830

[Mol] Möller, H.: Zur Numerik der Mertens'schen Vermutung. PhD thesis, Universität Ulm, Fakultät für Naturwissenschaften und Mathematik, January 1987

[Neu] Neubauer, G.: Eine empirische Untersuchung zur Mertensschen Funktion. Numer. Math. **5** (1963) 1-13

[OR] Odlyzko, A.M., Te Riele, H.J.J.: Disproof of the Mertens conjecture. J. reine angew. Math. **357** (1985) 138-160

[Pi3] Pintz, J.: Oscillatory properties of $M(x) = \sum_{n \leq x} \mu(n)$. III. Acta Arith. **43** (1984) 105-113

[Pin] Pintz, J.: An effective disproof of the Mertens conjecture. (Journées arithmétiques, Besançon/France, 1985) Astérisque **147-148** (1987) 325-333

[Opc] Odlyzko, A.M.: private communication, Nov. 3, 1991

[Scl] Schröder, J.: Zur Berechnung von Teilsummen der summatorischen Funktion der Möbius'schen Funktion $\mu(x)$. Norsk Mat. Tidsskrift **14** (1932) 45-53

[Sc2] Schröder, J.: Beiträge zur Darstellung der Möbiusschen Funktion. Jahresberichte der Deutschen Math. Ver. **42** (1932) 223-237

[Sc3] Schröder, J.: Zur Auswertung der zur Möbius'schen Funktion gehörenden summatorischen Funktion. Mitt. Math. Ges. Hamburg **7**, no. 3 (1933) 148-163

[Spi] Spira, R.: Zeros of sections of the zeta function. II. Math. Comp. **22** (1966) 163-173

[Ri1] Te Riele, H.J.J.: Computations concerning the conjecture of Mertens. J. reine angew. Math. **311/312** (1979) 356-360

[Ri2] Te Riele, H.J.J.: Some historical and other notes about the Mertens conjecture and its recent disproof. Nieuw Archief voor Wiskunde (4) **3** (1985) 237-243

[St1] Von Sterneck, R.D.: Empirische Untersuchung über den Verlauf der zahlentheoretischen Funktion $\sigma(n) = \sum_{x=1}^{n} \mu(x)$ im Intervalle von 0 bis 150000. Sitzungsberichte Akad. Wiss. Wien IIa **106** (1897) 835-1024

[St2] Von Sterneck, R.D.: Empirische Untersuchung über den Verlauf der zahlentheoretischen Funktion $\sigma(n) = \sum_{x=1}^{n} \mu(x)$ im Intervalle 150000 bis 500000. Sitzungsberichte Akad. Wiss. Wien IIa **110** (1901) 1053-1102

[St3] Von Sterneck, R.D.: Die zahlentheoretischen Funktion $\sigma(n)$ bis zur Grenze 5000000. Sitzungsberichte Akad. Wiss. Wien IIa **121** (1912) 1083-1096

[St4] Von Sterneck, R.D.: Neue empirischen Daten über die zahlentheoretischen Funktion $\sigma(n)$. In: Proc. of the 5-th International Congress of Mathematicians, vol. I. Cambridge University Press, 1913, 341-343

[Yor] Yorinaga, M.: Numerical investigation of sums of the Möbius function. Math. J. Okayama Univ. **21** (1979) 41-47

This article was processed by the author using the TEX macro package from Springer-Verlag.