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J.W. Klop , A. Middeldorp, Y. Toyama, R.C. de Vrijer

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A Simplified Proof of Toyama's Theorem

Jan Willem Klop[1,2,5,6] Aart Middeldorp[1,5,8] Yoshihito Toyama[1,3,7]

Roel de Vrijer[2,4]

[1] Department of Software Technology
CWI, Kruislaan 413, 1098 SJ Amsterdam

[2] Department of Mathematics and Computer Science
Vrije Universiteit, de Boelelaan 1081a, 1081 HV Amsterdam

[3] NTT Communication Science Laboratories
Sanpeidani, Inunidani, Seika-cho, Soraku-gun, Kyoto 619-02

[4] Department of Philosophy, University of Amsterdam
Nieuwe Doelenstraat 15, 1012 CP Amsterdam

ABSTRACT

In this note we present a simple proof of a result of Toyama which states that the disjoint union of confluent term rewriting systems is confluent.

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[8] Author's address after January 6, 1992: Advanced Research Laboratory, Hitachi Ltd, Hatoyama, Saitama 350-03, Japan.
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Introduction

The topic of modularity of properties of term rewriting systems has caught much attention recently. Several new results in this area have been obtained in Middeldorp [7]. For a survey one may consult Klop [6]. Moreover, the topic has received a fruitful offspring in the study of the conservation of properties when adding algebraic rewrite rules to various (typed) lambda calculi, see e.g. Breazu-Tannen and Gallier [1, 2] and Jouannaud and Okada [5].

This short paper goes back to the first important result in this area: the conservation of confluence under disjoint union of term rewriting systems. The original proof in Toyama [8] was rather complicated. The present proof is a considerable simplification.

The paper is organized as follows. In a preliminary section we briefly review the essential term rewriting background and introduce some specific notations concerning disjoint sums. Then the actual proof is divided over three very short sections. The division corresponds to a natural schematic representation of the proof in three distinct steps, each section focusing on one of these steps.

1. Preliminaries

We start by recapitulating some basic notions of term rewriting. Extensive surveys can be found in Dershowitz and Jouannaud [3] and Klop [6]. Our terminology is based on the latter.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{F}, \mathcal{R})$; here $\mathcal{F}$ is a set of *function symbols* and $\mathcal{R}$ a set of *rewrite rules.* Every rewrite rule has the form $l \to r$ with l, r terms built from $\mathcal{F}$ and a countably infinite set of *variables* $\mathcal{V}$, disjoint from $\mathcal{F}$, such that the following two conditions are satisfied:

- the left-hand side l is not a variable,
- the variables which occur in the right-hand side r also occur in l.

A rewrite rule $l \to r$ is called *collapsing* if r is a variable.

The set of all terms built from $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Identity of terms is denoted by $\equiv$. We introduce a fresh constant symbol $\Box$, named *hole*, and we abbreviate $\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ to $\mathcal{C}(\mathcal{F}, \mathcal{V})$. Terms in $\mathcal{C}(\mathcal{F}, \mathcal{V})$ will be called *contexts.* The designation *term* is restricted to members of $\mathcal{T}(\mathcal{F}, \mathcal{V})$. A context may contain zero, one or more holes. If C is a context with n holes and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ denotes the result of replacing from left to right the holes in C by $t_1, \ldots, t_n$. A term s is a *subterm* of a term t if there exists a context C such that $t \equiv C[s]$. A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We call $\sigma(t)$, which from now on we will write as t^σ, an *instance* of t.

An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression). The *rewrite relation* $\to_\mathcal{R}$ associated with a TRS $(\mathcal{F}, \mathcal{R})$ is defined as follows: $s \to_\mathcal{R} t$ if there exists a rewrite rule $l \to r$ in $\mathcal{R}$, a substitution σ and a context C such that $s \equiv C[l^\sigma]$ and $t \equiv C[r^\sigma]$. We say that s rewrites to t by *contracting* redex l^σ. We call $s \to_\mathcal{R} t$ a *rewrite step.* The transitive-reflexive closure of $\to_\mathcal{R}$ is denoted by $\twoheadrightarrow_\mathcal{R}$. If $s \twoheadrightarrow_\mathcal{R} t$ we say that s *reduces* to t and we call t a *reduct* of s. We write $s \leftarrow_\mathcal{R} t$ if $t \to_\mathcal{R} s$; likewise for $s \twoheadleftarrow_\mathcal{R} t$. The transitive-reflexive-symmetric closure of $\to_\mathcal{R}$ is called *conversion* and denoted by $=_\mathcal{R}$. If $s =_\mathcal{R} t$ then s and t are *convertible.* Two terms t_1, t_2 are *joinable*, denoted by $t_1 \downarrow_\mathcal{R} t_2$, if there exists a term t_3 such that $t_1 \twoheadrightarrow_\mathcal{R} t_3 \twoheadleftarrow_\mathcal{R} t_2$. A TRS is *confluent* or has the *Church-Rosser* property if t_1 and t_2 are joinable whenever $t_1 \twoheadleftarrow_\mathcal{R} s \twoheadrightarrow_\mathcal{R} t_2$, for all terms s, t_1, t_2. This notion specializes to terms in the obvious way. A well-known equivalent formulation of confluence states that conversion coincides with joinability.

DEFINITION 1.1. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be TRS's with disjoint alphabets (i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$). The *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ of $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ is the TRS $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$.

NOTATION. We abbreviate $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ to $\mathcal{T}_\oplus$. We write $\mathcal{T}_i$ instead of $\mathcal{T}(\mathcal{F}_i, \mathcal{V})$ for $i = 1, 2$. In the sequel, $\rightarrow$ without further decoration denotes the rewrite relation of $\mathcal{R}_1 \oplus \mathcal{R}_2$. The same frugality applies to its derived relations.

DEFINITION 1.2. A property $\mathcal{P}$ of TRS's is called *modular* if for all disjoint TRS's $(\mathcal{F}_1, \mathcal{R}_1)$, $(\mathcal{F}_2, \mathcal{R}_2)$ the following equivalence holds:

$$\mathcal{R}_1 \oplus \mathcal{R}_2 \text{ has the property } \mathcal{P}$$
$$\Longleftrightarrow$$
$$\text{both } (\mathcal{F}_1, \mathcal{R}_1) \text{ and } (\mathcal{F}_2, \mathcal{R}_2) \text{ have the property } \mathcal{P}.$$

Our aim in this note is to present a proof of the modularity of confluence. That is, we will show that confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ follows from confluence of $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$; the other direction is trivial.

In the remainder of this section we introduce several notations for coping with disjoint unions of TRS's. To this end we assume that $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are disjoint TRS's.

DEFINITION 1.3.

(1) The *root symbol* of a term $t \in \mathcal{T}_\oplus$, notation *root*(t), is defined by

$$\mathit{root}(t) = \begin{cases} F & \text{if } t \equiv F(t_1, \ldots, t_n), \\ t & \text{if } t \in \mathcal{V}. \end{cases}$$

(2) Let $t \equiv C[t_1, \ldots, t_n]$ with $C \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if $C \in \mathcal{C}(\mathcal{F}_a, \mathcal{V})$ and $\mathit{root}(t_1), \ldots, \mathit{root}(t_n) \in \mathcal{F}_b$ for some $a, b \in \{1, 2\}$ with $a \neq b$. The t_i's are the *principal* subterms of t. Observe that we allow for the case $n = 0$.

(3) The *rank* of a term $t \in \mathcal{T}_\oplus$ is defined by

$$\mathit{rank}(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T}_1 \cup \mathcal{T}_2, \\ 1 + \max\{\mathit{rank}(t_i) \mid 1 \leqslant i \leqslant n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \text{ with } n \geqslant 1. \end{cases}$$

(4) The set $S(t)$ of *special* subterms of a term $t \in \mathcal{T}_\oplus$ is defined as follows:

$$S(t) = \begin{cases} \{t\} & \text{if } \mathit{rank}(t) = 1, \\ \{t\} \cup \bigcup_{i=1}^{n} S(t_i) & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \text{ with } n \geqslant 1. \end{cases}$$

To achieve better readability we will call the function symbols of $\mathcal{F}_1$ *black* and those of $\mathcal{F}_2$ *white*. A black (white) term does not contain white (black) function symbols, but may contain variables. A *top black* (*top white*) term has a black (white) root symbol. In examples, black symbols will be printed as capitals and white symbols in lower case.

DEFINITION 1.4. Let $s \rightarrow t$ by application of a rewrite rule $l \rightarrow r$. We write $s \rightarrow^i t$ if the rewrite rule is being applied inside one of the principal subterms of s and we write $s \rightarrow^o t$ otherwise. The relation $\rightarrow^i$ is called *inner* reduction and $\rightarrow^o$ is called *outer* reduction.

DEFINITION 1.5. We say that a rewrite step $s \rightarrow t$ is *destructive at level* 1 if t is a variable or the root symbols of s and t have different colours. The rewrite step $s \rightarrow t$ is *destructive at level* $n + 1$ if $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!] \rightarrow^i C[s_1, \ldots, t_j, \ldots, s_m] \equiv t$ with $s_j \rightarrow t_j$ destructive at level n. Clearly, if a rewrite step is destructive then the applied rewrite rule is collapsing.

Notice that $s \to t$ is destructive at level 1 if and only if $s \to^o t$ and either t is a variable occurring in s or t is a principal subterm of s.

DEFINITION 1.6. We write $t \equiv C\langle\langle t_1, \ldots, t_n\rangle\rangle$ if either $t \equiv C[\![t_1, \ldots, t_n]\!]$ or $C \equiv \Box$ and $t \equiv t_1$.

The next proposition is used in the sequel although this will rarely be made explicit.

PROPOSITION 1.7.
(1) *If $s \to^o t$ then $s \equiv C[\![s_1, \ldots, s_n]\!]$ and $t \equiv C^*\langle\langle s_{i_1}, \ldots, s_{i_m}\rangle\rangle$ for some contexts C and C^*, indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and terms $s_1, \ldots, s_n \in \mathcal{T}_\oplus$. If $s \to^o t$ is not destructive then we may write $t \equiv C^*[\![s_{i_1}, \ldots, s_{i_m}]\!]$.*
(2) *If $s \to^i t$ then $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!]$ and $t \equiv C[s_1, \ldots, t_j, \ldots, s_n]$ for some context C, index $j \in \{1, \ldots, n\}$ and terms $s_1, \ldots, s_n, t_j \in \mathcal{T}_\oplus$ with $s_j \to t_j$. If $s \to^i t$ is not destructive at level 2 then we may write $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$.*

PROOF. Straightforward. $\Box$

PROPOSITION 1.8. *If $s \twoheadrightarrow t$ then $rank(s) \geqslant rank(t)$.*
PROOF. Suppose $s \to t$. Using Proposition 1.7 we obtain $rank(s) \geqslant rank(t)$ by a straightforward induction on $rank(s)$. The result now follows by induction on the length of $s \twoheadrightarrow t$. $\Box$

DEFINITION 1.9. Let $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}_\oplus$. We write $\langle s_1, \ldots, s_n\rangle \propto \langle t_1, \ldots, t_n\rangle$ if $t_i \equiv t_j$ whenever $s_i \equiv s_j$, for all $1 \leqslant i, j \leqslant n$. The combination of $\langle s_1, \ldots, s_n\rangle \propto \langle t_1, \ldots, t_n\rangle$ and $\langle t_1, \ldots, t_n\rangle \propto \langle s_1, \ldots, s_n\rangle$ is abbreviated to $\langle s_1, \ldots, s_n\rangle \infty \langle t_1, \ldots, t_n\rangle$.

PROPOSITION 1.10. *If $C[\![s_1, \ldots, s_n]\!] \to^o C^*\langle\langle s_{i_1}, \ldots, s_{i_m}\rangle\rangle$ then $C[t_1, \ldots, t_n] \to^o C^*[t_{i_1}, \ldots, t_{i_m}]$ for all terms $t_1, \ldots, t_n$ with $\langle s_1, \ldots, s_n\rangle \propto \langle t_1, \ldots, t_n\rangle$.*
PROOF. Routine. $\Box$

2. Preservation

The main obstacle for giving a 'straightforward' proof for the modularity of confluence, is the fact that the black and white layer structure of a term need not be preserved under reduction. That is, by a destructive rewrite step a e.g. black layer may disappear, thus allowing two originally distinct white layers to coalesce. Terms with an invariant layer structure will be called preserved.

DEFINITION 2.1. A term s is *preserved* if there are no reduction sequences starting from s that contain a destructive rewrite step. We call s *inner preserved* if all its principal subterms are preserved.

Note that the properties preserved and inner preserved are both conserved under reduction. Moreover, a destructive rewrite step from an inner preserved term can only be of level 1, and the result will be preserved. The modularity proof of confluence makes use of the fact that every term can be reduced to a preserved one. In the remainder of this section we prove this fact.

DEFINITION 2.2. We write $s \to_c t$ if there exists a context C and terms s_1, t_1 such that $s \equiv C[s_1]$, $t \equiv C[t_1]$, s_1 is a special subterm of s, $s_1 \twoheadrightarrow t_1$ and either t_1 is a variable or the root symbols of s_1 and t_1 have different colours. The relation $\to_c$ is called *collapsing reduction* and s_1 is a *collapsing redex*. Note that every destructive rewrite step is collapsing.

PROPOSITION 2.3.
(1) *If* $s \to_c t$ *then* $s \twoheadrightarrow t$.
(2) *A term is preserved if and only if it contains no collapsing redexes.*

PROOF. Straightforward. □

EXAMPLE 2.4. Let

$$\mathcal{R}_1 = \begin{cases} F(x,y) & \to & y \\ G(x) & \to & C \end{cases}$$

and $\mathcal{R}_2 = \{e(x) \to x\}$. We have the following collapsing reduction sequence:

$$\begin{aligned} F(C, e(F(e(C), G(e(C))))) &\to_c F(C, e(F(C, G(e(C))))) \\ &\to_c e(F(C, G(e(C)))) \\ &\to_c F(C, G(e(C))) \\ &\to_c F(C, G(C)). \end{aligned}$$

PROPOSITION 2.5. *Every term has a preserved reduct.*

PROOF. We first show that there are no infinite collapsing reduction sequences. Assign to every term t the multiset $\|t\| = [rank(s) \mid s \in S(t)]$, provided t is not a variable. If $t \in \mathcal{V}$ then $\|t\| = [\,]$. Suppose that $s \to_c t$. Using Proposition 1.8, one easily shows that $\|s\| \gg \|t\|$ where $\gg$ is the multiset extension of the standard ordering $>$ on natural numbers. The relation $\gg$ is well-founded (see Dershowitz and Manna [4]) and hence there can be no infinite collapsing reduction sequences. Proposition 2.3 now yields the desired result. □

As matter of fact we showed a little too much. We obtained strong normalization of collapsing reduction, where weak normalization would have sufficed. A simple proof of weak normalization, avoiding the multiset ordering machinery, is not hard to find.

3. Confluence of Inner Preserved Terms

From now on we assume that $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are disjoint and confluent TRS's. In this section we establish confluence for the inner preserved terms of the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$. This result will be extended to the whole of $\mathcal{R}_1 \oplus \mathcal{R}_2$ in the next section.

First we show that monochrome outer reduction is confluent.

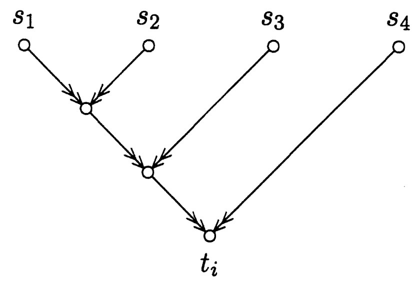
PROPOSITION 3.1. *The relations* $\to^o_{\mathcal{R}_1}$ *and* $\to^o_{\mathcal{R}_2}$ *are confluent.*

PROOF. We pick $\to^o_{\mathcal{R}_1}$. Suppose $t_1 \twoheadleftarrow^o_{\mathcal{R}_1} t \twoheadrightarrow^o_{\mathcal{R}_1} t_2$. We may write $t \equiv C[\![s_1, \ldots, s_n]\!]$, $t_1 \equiv C_1\langle\langle s_{i_1}, \ldots, s_{i_m}\rangle\rangle$ and $t_2 \equiv C_2\langle\langle s_{j_1}, \ldots, s_{j_p}\rangle\rangle$. Choose fresh variables $x_1, \ldots, x_n$ with $\langle s_1, \ldots, s_n\rangle \propto \langle x_1, \ldots, x_n\rangle$ and let $t' \equiv C[x_1, \ldots, x_n]$, $t_1' \equiv C_1[x_{i_1}, \ldots, x_{i_m}]$ and $t_2' \equiv C_2[x_{j_1}, \ldots, x_{j_p}]$. Repeated application of Proposition 1.10 yields $t_1' \twoheadleftarrow_{\mathcal{R}_1} t_1 \twoheadrightarrow_{\mathcal{R}_1} t_2'$. Since this is a conversion in $(\mathcal{F}_1, \mathcal{R}_1)$ there exists a common reduct $C^*[x_{k_1}, \ldots, x_{k_l}]$ of t_1' and t_2'. Instantiating the valley $t_1' \twoheadrightarrow_{\mathcal{R}_1} C^*[x_{k_1}, \ldots, x_{k_l}] \twoheadleftarrow_{\mathcal{R}_1} t_2'$ yields $t_1 \twoheadrightarrow^o_{\mathcal{R}_1} C^*\langle\langle s_{k_1}, \ldots, s_{k_l}\rangle\rangle \twoheadleftarrow^o_{\mathcal{R}_1} t_2$. □

DEFINITION 3.2. Let S be a set of confluent terms. A set $\hat{S}$ of terms *represents* S if the following two conditions are satisfied:
(1) every term in S has a unique reduct $\hat{s}$ in $\hat{S}$, which will be called the *representative* of s,
(2) joinable terms in S have the same representative in $\hat{S}$.

Proposition 3.3. *Every finite set S of confluent terms can be represented.*

Proof. Since S consists of confluent terms, joinability is an equivalence relation on S. Hence we can partition S into equivalence classes $C_1, \ldots, C_n$ of joinable terms. Because these classes are finite, we may associate with every C_i a 'common reduct' t_i as suggested in Figure 1. It is



$$C_i \;=\; \{s_1, s_2, s_3, s_4\}$$

Figure 1.

not difficult to see that the set $\{t_1, \ldots, t_n\}$ represents S. □

Lemma 3.4. *Inner preserved terms are confluent.*

Proof. By induction on $\mathit{rank}(t)$ we will show that every preserved term t is confluent. If $\mathit{rank}(t) = 1$ then t is a black or white term and the confluence of t is ensured by the confluence of $(\mathcal{F}_1, \mathcal{R}_1)$ or $(\mathcal{F}_2, \mathcal{R}_2)$, respectively. Suppose $\mathit{rank}(t) = n$ with $n > 1$ and consider a conversion $t_1 \twoheadleftarrow t \twoheadrightarrow t_2$. We have to show that t_1 and t_2 are joinable. Without loss of generality we assume that t is top black. Let S be the set of all maximal special subterms occurring in this conversion that are not top black. So if u is a top black term in the conversion $t_1 \twoheadleftarrow t \twoheadrightarrow t_2$ then the principal subterms of u belong to S, otherwise u itself is a member of S. Because every element of S has rank less than n, by the induction hypothesis S consists of confluent terms. From Proposition 3.3 it follows then that S can be represented by a set $\hat{S}$. Let u be a term in the conversion $t_1 \twoheadleftarrow t \twoheadrightarrow t_2$. The result of replacing in u every maximal special subterm that is not top black by its representative is denoted by $\tilde{u}$. Notice that $u \twoheadrightarrow \tilde{u}$.

We will show that $\tilde{t}_1 \twoheadleftarrow^{o}_{\mathcal{R}_1} \tilde{t} \twoheadrightarrow^{o}_{\mathcal{R}_1} \tilde{t}_2$. Let $u_1 \to u_2$ be a step in the conversion $t_1 \twoheadleftarrow t \twoheadrightarrow t_2$. Distinguish three cases.

(1) Suppose u_1 is top black and u_2 is either top black or a variable. If $u_1 \to^{o} u_2$ then we may write $u_1 \equiv C_1[\![s_1, \ldots, s_n]\!]$ and $u_2 \equiv C_2[\![s_{i_1}, \ldots, s_{i_m}]\!]$. Clearly $\tilde{u}_1 \equiv C_1[\hat{s}_1, \ldots, \hat{s}_n] \to^{o} C_2[\hat{s}_{i_1}, \ldots, \hat{s}_{i_m}] \equiv \tilde{u}_2$. Because u_1 is top black we have $\tilde{u}_1 \to^{o}_{\mathcal{R}_1} \tilde{u}_2$. Otherwise $u_1 \to^{i} u_2$ and because u_1 is inner preserved we may write $u_1 \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C[\![s_1, \ldots, s'_j, \ldots, s_n]\!] \equiv u_2$ with $s_j \to s'_j$. Since s_j and s'_j are trivially joinable, we have $\hat{s}_j \equiv \hat{s}'_j$ and hence $\tilde{u}_1 \equiv C[\hat{s}_1, \ldots, \hat{s}_j, \ldots, \hat{s}_n] \equiv \tilde{u}_2$.

(2) Suppose u_1 is top black and u_2 is top white. Then we have $u_1 \equiv C_1[\![s_1, \ldots, s_n]\!]$ and $u_2 \equiv s_i$ for some i, $1 \le i \le n$. Again $\tilde{u}_1 \equiv C_1[\hat{s}_1, \ldots, \hat{s}_n] \to^{o}_{\mathcal{R}_1} \hat{s}_i \equiv \tilde{u}_2$. Note that now, since u_1 is inner preserved, u_2 will be preserved.

(3) Suppose u_1 is top white. Then the step $u_1 \to u_2$ must in the reduction $t \twoheadrightarrow t_i$ be preceded by a, destructive, step of type (2). So u_1 is preserved and also u_2 will be top white and preserved. Hence u_1 and u_2 are both in S. Of course, they must have the same representative. So $\tilde{u}_1 \equiv \hat{u}_1 \equiv \hat{u}_2 \equiv \tilde{u}_2$.

It may be concluded that $\tilde{t}_1 \twoheadleftarrow^{o}_{\mathcal{R}_1} \tilde{t} \twoheadrightarrow^{o}_{\mathcal{R}_1} \tilde{t}_2$. Since $\to^{o}_{\mathcal{R}_1}$ is confluent, the terms $\tilde{t}_1$ and $\tilde{t}_2$ have a common reduct, which at the same time is a common reduct of t_1 and t_2, see Figure 2. □

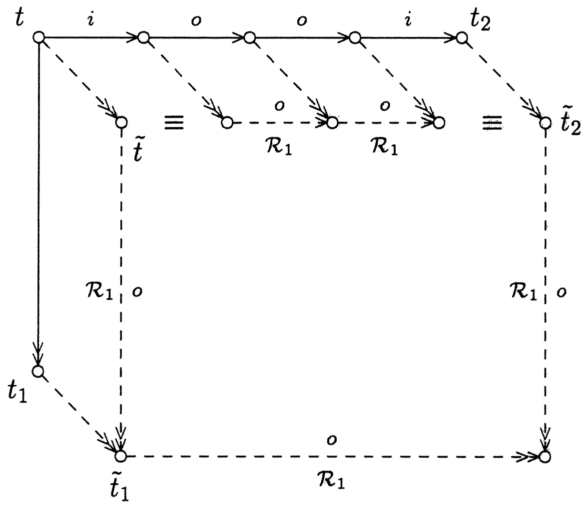


FIGURE 2.

4. Modularity of Confluence

Now the idea of the full modularity proof is to project divergent reductions $t_1 \twoheadleftarrow t \twoheadrightarrow t_2$ to a conversion involving only inner preserved terms, in order to be able to use Lemma 3.4. The projection consists of choosing an appropriate witness, according to the following definition.

DEFINITION 4.1. Let $s \equiv C[\![s_1, \ldots, s_n]\!]$. A *witness* of s is an inner preserved term $t \equiv C[t_1, \ldots, t_n]$ which satisfies the following two properties:
(1) $s_i \twoheadrightarrow t_i$ for $i = 1, \ldots, n$,
(2) $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$.

PROPOSITION 4.2. *Every term has a witness.*

PROOF. Let $s \equiv C[\![s_1, \ldots, s_n]\!]$. According to Proposition 2.5 every s_i has a preserved reduct t_i. We may of course assume that $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$. The term $t \equiv C[t_1, \ldots, t_n]$ clearly is inner preserved. □

In the following $\dot{s}$ denotes an arbitrary witness of s.

LEMMA 4.3. *Let $s \rightarrow t$. If all principal subterms of s are confluent then $\dot{s} \downarrow \dot{t}$.*

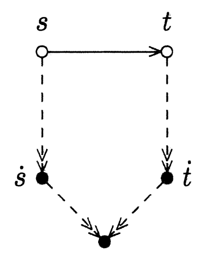


FIGURE 3.

PROOF. Suppose $s \equiv C[\![s_1, \ldots, s_n]\!]$ and $\dot{s} \equiv C[t_1, \ldots, t_n]$. We distinguish two cases:
(1) If $s \rightarrow^o t$ then we may write $t \equiv C^*\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$. We have $\dot{t} \equiv C^*[u_{i_1}, \ldots, u_{i_m}]$ for respective reducts $u_{i_1}, \ldots, u_{i_m}$ of $s_{i_1}, \ldots, s_{i_m}$. Since $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ we obtain $\dot{s} \rightarrow C^*[t_{i_1}, \ldots, t_{i_m}]$ from Proposition 1.10. We have $t_j \twoheadleftarrow s_j \twoheadrightarrow u_j$ for all $j \in \{i_1, \ldots, i_m\}$. Confluence of s_j yields the joinability of t_j and u_j, for all $j \in \{i_1, \ldots, i_m\}$. Therefore $\dot{s} \downarrow \dot{t}$.

(2) If $s \rightarrow^i t$ then $t \equiv C[s_1, \ldots, s_j', \ldots, s_n]$ with $s_j \rightarrow s_j'$. Since C is monochrome black or white, we have $\dot{t} \equiv C[u_1, \ldots, u_n]$ for some respective reducts $u_1, \ldots, u_j, \ldots, u_n$ of $s_1, \ldots, s_j', \ldots, s_n$. We obtain the joinability of t_k and u_k for $k = 1, \ldots, n$ as in the previous case. We conclude that $\dot{s} \downarrow \dot{t}$.

□

THEOREM 4.4. *Confluence is a modular property of TRS's.*

PROOF. By induction on $rank(t)$ we will show that every term t is confluent. If $rank(t) = 1$ then t is inner preserved and we can use Lemma 3.4. Suppose $rank(t) > 1$ and consider a conversion $t_1 \twoheadleftarrow t \twoheadrightarrow t_2$. The proof for this case is illustrated in Figure 4. First we reduce every term in this conversion to a witness. Since all principal subterms occurring in the conversion $t_1 \twoheadleftarrow t \twoheadrightarrow t_2$

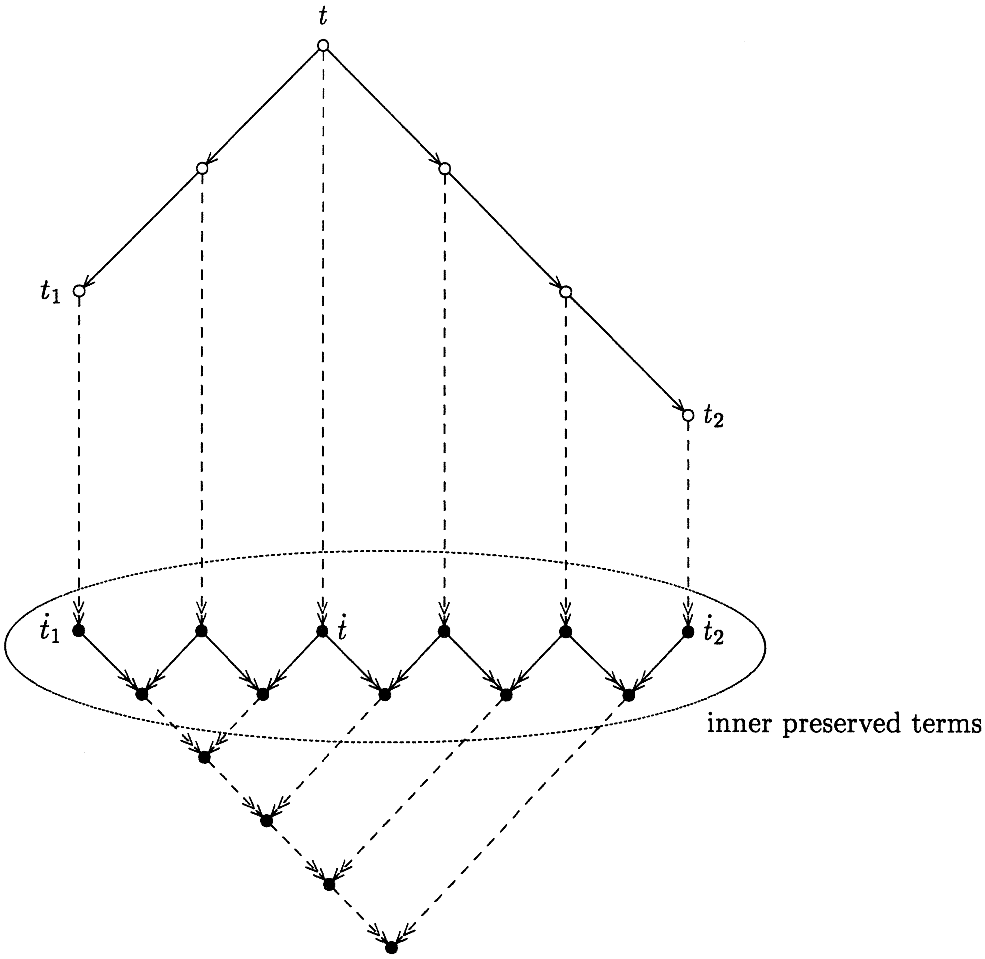


FIGURE 4.

have rank less than $rank(t)$, we may assume them to be confluent. Repeated application of Lemma 4.3 yields a conversion between the witnesses in which all terms are inner preserved. Then Lemma 3.4 yields a common reduct of t_1 and t_2. □

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