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Strong Sequentiality of Left-Linear Overlapping Term Rewriting Systems

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Abstract

Huet and Lévy showed that for every strongly sequential orthogonal (i.e., left-linear and non-overlapping) term rewriting system, the index reduction strategy is normalizing. This paper extends their result to *overlapping* term rewriting systems. We show that index reduction is normalizing for the class of strongly sequential left-linear term rewriting systems in which every critical pair can be joined with *root balanced reductions*.

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1 Introduction

The normalizing reduction strategies of reduction systems, such as leftmost-outermost evaluation of lambda calculus, combinatory logic, ordinal recursive program schemata and left-normal term rewriting systems, guarantee a *safe* evaluation which reduces a given expression to its normal form whenever possible [1, 2, 5, 9, 10, 11, 13, 14, 17]. Hence, the normalizing reduction strategies play an important role in the implementation of functional programming languages based on reduction systems [14].

Strong sequentiality formalized by Huet and Lévy [6] is a well-known practical criterion guaranteeing an efficiently computable normalizing reduction strategy for orthogonal (i.e., left-linear and non-overlapping) term rewriting systems. They showed that for every strongly sequential orthogonal term rewriting system R, index reduction is a normalizing strategy, that is, by rewriting a redex called an *index* at each step, every reduction starting with a term having a normal form eventually terminates at the normal from. Here, the index is defined as a *needed redex* concerning an approximation of R which is obtained by ignoring the information about the right-hand side of the rewriting rules of R. However, this result is restricted to orthogonal systems; hence, it cannot be applied to term rewriting systems with overlapping rules such as

$$\begin{cases} \mathrm{pred}(\mathrm{succ}(x)) \to x \\ \mathrm{succ}(\mathrm{pred}(x)) \to x \end{cases}$$

In this paper, we extend the result by Huet and Lévy to *overlapping* term rewriting systems. The definitions of index and strong sequentiality are naturally extended to the overlapping situation. Under these extended definitions, we show that index reduction is normalizing for the class of strongly sequential *balanced ambiguous* term rewriting systems. The balanced ambiguous term rewriting system is defined as a left-linear term rewriting systems in which every critical pair can be joined with *the root balanced reductions.* We also show that this class includes all weakly orthogonal left-normal systems, for which the leftmost-outermost reduction strategy is normalizing. For example, the leftmost-outermost reduction strategy is normalizing for combinatory logic CL + $\{\mathrm{pred} \cdot (\mathrm{succ} \cdot x) \to x,\ \mathrm{succ} \cdot (\mathrm{pred} \cdot x) \to x\}$.

The approach presented here is more direct than that in Huet and Lévy [6]. The key idea behind our approach is the *balanced weak Church-Rosser property.* This new concept gives a useful abstract framework for analyzing reduction strategies. Applying this framework to term rewriting systems, we can easily prove that index reduction is normalizing; our proof does not need the complicated notions and algorithms used in Huet and Lévy [6].

Other approaches extending the work of Huet and Lévy can be found in [3, 9, 11, 12, 15, 16].

2 Reduction Systems

Assuming that the reader is familiar with the basic concepts and notations concerning reduction systems in [7, 11], we briefly explain notations and definitions.

A reduction system (or an abstract reduction system) is a structure $A = \langle D, \to \rangle$ consisting of some set D and some binary relation $\to$ on D (i.e., $\to \subseteq D \times D$), called a reduction relation. A reduction (starting with x_0) in A is a finite or infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$. The identity of elements x, y of D is denoted by $x \equiv y$. $\stackrel{\equiv}{\to}$ is the reflexive closure of $\to$, $\leftrightarrow$ is the symmetric closure of $\to$, $\stackrel{+}{\to}$ is the transitive closure of $\to$, $\stackrel{*}{\to}$ is the transitive reflexive closure of $\to$, and $=$ is the equivalence relation generated by $\to$ (i.e., the transitive reflexive symmetric closure of $\to$). $\stackrel{m}{\to}$ denotes a reduction of m ($m \geq 0$) steps. If $x \in D$ is minimal with respect to $\to$, i.e., $\neg \exists y \in D[x \to y]$, then we say that x is a normal form; let NF be the set of normal forms. If $x \stackrel{*}{\to} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x.

Definition 2.1 *$A = \langle D, \to \rangle$ is strongly normalizing (or terminating) iff every reduction in A terminates, i.e., there is no infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$.*

Definition 2.2 *$A = \langle D, \to \rangle$ is Church-Rosser (or confluent) iff*
$\forall x, y, z \in D[x \stackrel{*}{\to} y \wedge x \stackrel{*}{\to} z \Rightarrow \exists w \in A, y \stackrel{*}{\to} w \wedge z \stackrel{*}{\to} w]$.

Definition 2.3 $A = \langle D, \rightarrow \rangle$ *is weakly Church-Rosser (or weakly confluent) iff* $\forall x, y, z \in D[x \rightarrow y \wedge x \rightarrow z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.

The following propositions are well known [1, 7, 11].

Proposition 2.4 *Let A be Church-Rosser, then,*

(1) $\forall x, y \in D[x = y \Rightarrow \exists w \in D, x \xrightarrow{*} w \wedge y \xrightarrow{*} w]$,

(2) $\forall x, y \in NF[x = y \Rightarrow x \equiv y]$,

(3) $\forall x \in D \forall y \in NF[x = y \Rightarrow x \xrightarrow{*} y]$.

Proposition 2.5 (Newman's Lemma) *If A is weakly Church-Rosser and strongly normalizing, then A is Church-Rosser.*

Definition 2.6 (Reduction Strategy) *Let* $A = \langle D, \rightarrow \rangle$ *and let* $\underset{s}{\rightarrow}$ *be a subrelation of* $\xrightarrow{+}$ *(i.e., if* $x \underset{s}{\rightarrow} y$ *then* $x \xrightarrow{+} y$*) such that a normal form concerning* $\underset{s}{\rightarrow}$ *is also a normal form concerning* $\rightarrow$ *(i.e., two binary relations* $\underset{s}{\rightarrow}$ *and* $\rightarrow$ *have the same domain). Then, we say that* $\underset{s}{\rightarrow}$ *is a reduction strategy for A (or for* $\rightarrow$*). If* $\underset{s}{\rightarrow}$ *is a subrelation of* $\rightarrow$ *then we call it a one step reduction strategy; otherwise* $\rightarrow$ *is called a many step reduction strategy.*

Definition 2.7 (Normalizing Strategy) *The reduction strategy* $\underset{s}{\rightarrow}$ *is normalizing iff for each x having a normal form concerning* $\rightarrow$*, there exists no infinite sequence* $x \equiv x_0 \underset{s}{\rightarrow} x_1 \underset{s}{\rightarrow} x_2 \underset{s}{\rightarrow} \cdots$ *(i.e., every* $\underset{s}{\rightarrow}$ *reduction starting with x must eventually terminate at a normal form of x).*

3 Balanced Weak Church-Rosser Property

This section introduces the new concept of balanced weak Church-Rosser property. Though in the later sections this concept will play an important role for analyzing the normalizing strategy of term rewriting systems, our results concerning the balanced weak Church-Rosser property can be presented in an abstract framework depending solely on the reduction relation.

Let $A = \langle D, \rightarrow \rangle$ be an abstract reduction system.

Definition 3.1 $A = \langle D, \rightarrow \rangle$ *is balanced weakly Church-Rosser (BWCR) iff*
$\forall x, y, z \in D[x \rightarrow y \wedge x \rightarrow z \Rightarrow \exists w \in D, \exists k \geq 0, y \xrightarrow{k} w \wedge z \xrightarrow{k} w]$.

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d, dashed, "k"'] \\ z \arrow[r, dashed, "k"] & w \end{tikzcd} \quad (k \geq 0)$$

We define the *local* Church-Rosser property and the *local* strong normalizing properties for an element $x \in D$. x is normalizing if every reduction starting with x terminates. x is Church-Rosser if
$\forall y, z \in D[x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.
x is complete if x is Church-Rosser and strong normalizing.

Lemma 3.2 (BWCR Lemma) *Let $A = \langle D, \rightarrow \rangle$ be BWCR. Let $x = y$ and $y \in NF$. Then,*

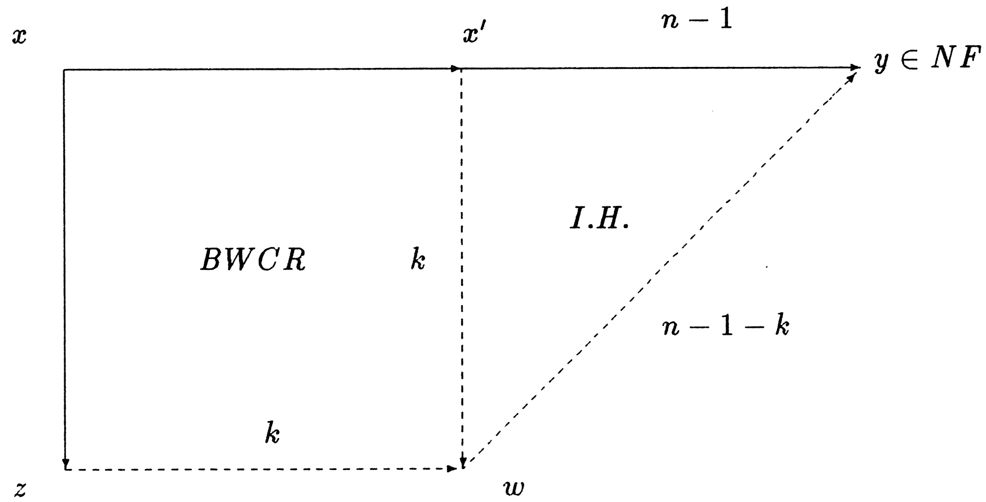
(1) *x is complete.*

(2) *All the reductions from x to y have the same length (i.e., the same number of reduction steps).*

Proof. We first prove the following claim: if $x \xrightarrow{n} y$ and $y \in NF$ $(n \geq 0)$ then x satisfies the properties (1) and (2).

Proof of the claim. We show the claim by induction on n. The case $n = 0$ is trivial. Let $x \rightarrow x' \xrightarrow{n-1} y \in NF$. Take any one step reduction $x \rightarrow z$ starting with x.

By the balanced weak Church-Rosser property, there exists some w and $k \geq 0$ such that $z \xrightarrow{k} w$ and $x' \xrightarrow{k} w$. By induction hypothesis, the properties (1) and (2) hold at x'; hence $x' \xrightarrow{k} w \xrightarrow{*} y$ must have $n-1$ steps in length. Thus, $w \xrightarrow{n-1-k} y$. Since $z \xrightarrow{k} w$, we obtain $z \xrightarrow{n-1} y$. By induction hypothesis, z satisfies the properties (1) and (2). Therefore, the claim follows.



We next show that if $x \overset{n}{\leftrightarrow} y$ and $y \in NF$ $(n \geq 0)$ then $x \xrightarrow{*} y$. Here, $\overset{n}{\leftrightarrow}$ denotes the n-step connection of $\leftrightarrow$. The proof is by induction on n. The case $n = 0$ is trivial. Let $x \leftrightarrow x' \overset{n-1}{\leftrightarrow} y$. By induction hypothesis, we have $x' \xrightarrow{*} y$. The case $x \rightarrow x'$ is trivial. Let $x \leftarrow x'$. By applying the claim to $x' \xrightarrow{*} y \in NF$, it is obtained that x' is complete. Thus, $x \xrightarrow{*} y$.

Therefore, from the claim it follows that if $x = y$ and $y \in NF$ then x satisfies the properties (1) and (2). □

We say that A has the unique normal form property (UN) iff
$\forall x, y \in NF[x = y \Rightarrow x \equiv y]$.

Corollary 3.3 *If an abstract reduction system A is BWCR then A is UN.*

Proof. Let $x, y \in NF$ and $x = y$. From the BWCR lemma we get $x \xrightarrow{*} y$ and $y \xrightarrow{*} x$; hence $x \equiv y$. □

The following lemma is essential for connecting the two concepts of balanced weakly Church-Rosser and normalizing reduction strategies.

Lemma 3.4 *Let* $\underset{\alpha}{\rightarrow}$ *and* $\underset{\beta}{\rightarrow}$ *be two reduction relations on* D *such that:*

(1) $\underset{\alpha}{\rightarrow}$ *is balanced weak Church-Rosser,*

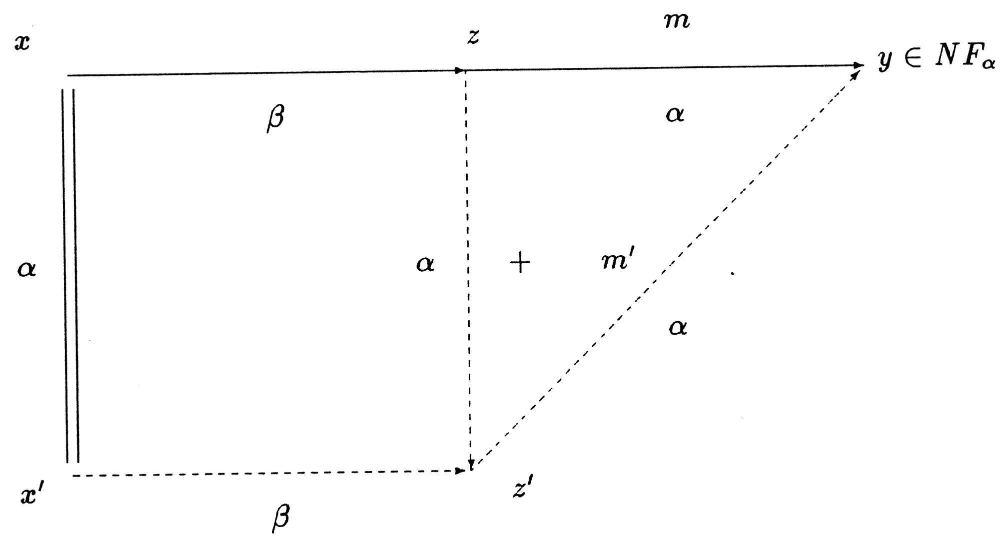
(2) *If* $x \underset{\beta}{\rightarrow} y$ *then;*

- **(i)** $x \underset{\alpha}{=} y$ *or,*
- **(ii)** $x \underset{\alpha}{=} \cdot \underset{\beta}{\rightarrow} \cdot \overset{+}{\underset{\alpha}{\leftarrow}} y.$

If $x \overset{*}{\underset{\beta}{\rightarrow}} y$ *and* $y \in NF_\alpha$, *then we have* $x \overset{*}{\underset{\alpha}{\rightarrow}} y$. *Here,* NF_α *is the set of the normal forms concerning* $\underset{\alpha}{\rightarrow}$.

Proof. We first show the claim: If $x \underset{\beta}{\rightarrow} \cdot \overset{m}{\underset{\alpha}{\rightarrow}} y$ and $y \in NF_\alpha$ $(m \geq 0)$, then we have $x \underset{\alpha}{=} y$.

The proof is by induction on m. The case $m = 0$ is trivial from $y \in NF_\alpha$ and (i) (since (ii) does not hold). Let $x \underset{\beta}{\rightarrow} z \overset{m}{\underset{\alpha}{\rightarrow}} y \in NF_\alpha$ $(m > 0)$. Then $x \underset{\beta}{\rightarrow} z$ must satisfy (i) or (ii). If $x \underset{\alpha}{=} z$ then $x \underset{\alpha}{=} y$ is trivial. Assume that $x \underset{\alpha}{=} x' \underset{\beta}{\rightarrow} z' \overset{+}{\underset{\alpha}{\leftarrow}} z$. By applying the BWCR lemma to $z \overset{m}{\underset{\alpha}{\rightarrow}} y \in NF_\alpha$, we have $z \overset{+}{\underset{\alpha}{\rightarrow}} z' \overset{m'}{\underset{\alpha}{\rightarrow}} y$ with $m' < m$. Applying induction hypothesis of the claim to x', we obtain that $x \underset{\alpha}{=} x' \underset{\alpha}{=} y$.



We next prove that if $x \xrightarrow[\beta]{n} y$ and $y \in NF_\alpha$ $(n \geq 0)$, then $x \xrightarrow[\alpha]{*} y$. The proof is by induction on n. The case $n = 0$ is trivial. Let $x \xrightarrow[\beta]{} x' \xrightarrow[\beta]{n-1} y \in NF_\alpha$. From induction hypothesis, we have $x' \xrightarrow[\alpha]{*} y$. Thus, form the claim, $x \underset{\alpha}{=} y$. From the BWCR lemma, it follows that $x \xrightarrow[\alpha]{*} y$. $\square$

The next corollary explains how the above lemma implies the normalizing property of a reduction strategy $\xrightarrow[s]{} \subseteq \rightarrow$.

Corollary 3.5 *Let* $\xrightarrow[s]{} \subseteq \rightarrow$ *such that:*

(1) $\xrightarrow[s]{}$ *is balanced weakly Church-Rosser,*

(2) *If* $x \rightarrow y$ *then;*

> **(i)** $x \underset{s}{=} y$ *or,*
>
> **(ii)** $x \underset{s}{=} \cdot \rightarrow \cdot \xleftarrow[s]{+} y$.

Then $\xrightarrow[s]{}$ *is a normalizing strategy for* $\rightarrow$.

Proof. Let $x \xrightarrow{*} y$ and $y \in NF$. Since $NF \subseteq NF_s$, we have $x \xrightarrow{*} y \in NF_s$. Take $\xrightarrow[s]{}$ as $\xrightarrow[\alpha]{}$ and $\rightarrow$ as $\xrightarrow[\beta]{}$ respectively, and apply Lemma 3.4. Then we obtain $x \xrightarrow[s]{*} y \in NF_s$. Form the BWCR lemma, it follows that $\xrightarrow[s]{}$ is a normalizing strategy. $\square$

Remark. One might think that by taking the index reduction $\xrightarrow[I]{}$ (which is defined later) as $\xrightarrow[s]{}$ in the above corollary, it might be shown directly that the index reduction is normalizing. However, this is not the case since the property (2) in the corollary does not hold.

4 Term Rewriting Systems

In the following sections, we will explain how to apply the BWCR lemma to term rewriting systems. We briefly explain the basic notions and definitions concerning term rewriting systems [4, 7, 8, 11].

Let $\mathcal{F}$ be an enumerable set of function symbols denoted by $f, g, h, \cdots$, and let $\mathcal{V}$ be an enumerable set of variable symbols denoted by $x, y, z, \cdots$ where $\mathcal{F} \cap \mathcal{V} = \phi$. By $T(\mathcal{F}, \mathcal{V})$, we denote the set of terms constructed from $\mathcal{F}$ and $\mathcal{V}$. The term set $T(\mathcal{F}, \mathcal{V})$ is sometimes denoted by T.

A substitution θ is a mapping from a term set $T(\mathcal{F}, \mathcal{V})$ to $T(\mathcal{F}, \mathcal{V})$ such that for a term t, $\theta(t)$ is completely determined by its values on the variable symbols occurring in t. Following common usage, we write this as $t\theta$ instead of $\theta(t)$.

Consider an extra constant $\square$ called a hole and the set $T(\mathcal{F} \cup \{\square\}, \mathcal{V})$. Then $C \in T(\mathcal{F} \cup \{\square\}, \mathcal{V})$ is called a context on $\mathcal{F}$. We use the notation $C[\ ,\ldots,\ ]$ for the context containing n holes ($n \geq 0$), and if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{V})$, then $C[t_1, \ldots, t_n]$ denotes the result of placing $t_1, \ldots, t_n$ in the holes of $C[\ ,\ldots,\ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole. s is called a subterm of $t \equiv C[s]$. If s is a subterm occurrence of t, then we write $s \subseteq t$. If a term t has an occurrence of some (function or variable) symbol e, we write $e \in t$. The variable occurrences $z_1, \cdots, z_n$ of $C[z_1, \cdots, z_n]$ are fresh if $z_1, \cdots, z_n \notin C[\ ,\cdots,\ ]$ and $z_i \not\equiv z_j$ $(i \neq j)$.

A rewriting rule is a pair $\langle l, r \rangle$ of terms such that $l \notin \mathcal{V}$ and any variable in r also occurs in l. We write $l \rightarrow r$ for $\langle l, r \rangle$. A redex is a term $l\theta$, where $l \rightarrow r$. In this case $r\theta$ is called a contractum of $l\theta$. The set of rewriting rules defines a reduction relation $\rightarrow$ on T as follows:

$$t \rightarrow s \text{ iff } t \equiv C[l\theta],\ s \equiv C[r\theta] \quad \text{for some rule } l \rightarrow r, \text{ and some } C[\ ], \theta.$$

When we want to specify the redex occurrence $\Delta \equiv l\theta$ of t in this reduction, we write $t \xrightarrow{\Delta} s$.

Definition 4.1 *A term rewriting system R is a reduction system $R = \langle T(\mathcal{F}, \mathcal{V}).,\rightarrow \rangle$ such that the reduction relation $\rightarrow$ on $T(\mathcal{F}, \mathcal{V})$ is defined by a set of rewriting rules. If R has $l \rightarrow r$ as a rewriting rule, we write $l \rightarrow r \in R$.*

We say that R is left-linear if for any $l \rightarrow r \in R$, l is linear (i.e., every variable in l occurs only once).

Let $l \to r$ and $l' \to r'$ be two rules in R. Assume that we have renamed the variables appropriately, so that l and l' share no variables. Assume $s \notin V$ is a subterm occurrence in t, i.e., $t \equiv C[s]$, such that s and l' are unifiable, i.e., $s\theta \equiv l'\theta$, with a minimal unifier θ [7, 11] . Since $l\theta \equiv C[s]\theta \equiv C\theta[l'\theta]$, two reductions starting with $l\theta$, i.e., $l\theta \to C\theta[r'\theta] \equiv C[r']\theta$ and $l\theta \to r\theta$, can be obtained by using $l' \to r'$ and $l \to r$. Then we say that $l \to r$ and $l' \to r'$ are overlapping, and that the pair $\langle C[r']\theta, r\theta\rangle$ of terms is critical in R [7, 8] . We may choose $l \to r$ and $l' \to r'$ to be the same rule, but in this case we shall not consider the case $s \equiv l$, which gives the trivial pair $\langle r, r\rangle$. If R has no critical pair, then we say that R is non-overlapping. If every critical pair $\langle s, t\rangle$ is trivial, i.e., $s \equiv t$, then R is weakly overlapping [7, 8, 11].

R is orthogonal if R is left-linear and non-overlapping. R is weakly orthogonal if R is left-linear and weakly overlapping. The following result is well known [7, 11].

Proposition 4.2 *Let R be orthogonal (or weakly orthogonal). Then R is Church-Rosser.*

5 Strong Sequentiality

The fundamental concept of strong sequentiality for orthogonal term rewriting systems was introduced by Huet and Lévy [6]. In this section we explain the basic notions and properties related to strong sequentiality, according to Huet and Lévy [6], and Klop and Middeldorp [12]. Instead of the orthogonality, we assume only left-linearity for term rewriting systems. Thus, strong sequentiality defined here is an extension of the original one in [6].

Note. From here on we assume that R is a left-linear term rewriting system which may have *overlapping* rules.

Consider an extra constant Ω and the set $T(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$, denoted by T_Ω. The element of T_Ω is called a Ω-term.

Definition 5.1 *The preordering $\geq$ on T_Ω is defined as follows:*

$t \geq \Omega$ *for all* $t \in T_\Omega$,

$$f(t_1, \cdots, t_n) \geq f(s_1, \cdots, s_n) \quad (n \geq 0) \quad \text{if } t_i \geq s_i \text{ for } i = 1, \cdots, n.$$

We write $t > s$ if $t \geq s$ and $t \not\equiv s$.

Definition 5.2 (Compatibility) *Two Ω-terms t and s are compatible, denoted by $t \uparrow s$, if there exists some Ω-term r such that $r \geq t$ and $r \geq s$; otherwise, t and s are incompatible, denoted by $t \# s$. Let S be a set of Ω-term. Then $t \uparrow S$ if there exists some $s \in S$ such that $t \uparrow s$; otherwise, $t \# S$.*

Let t_Ω denote the Ω-term obtained from a term t by replacing each variable in t with Ω. The set of redex schemata of R is $Red = \{ l_\Omega \mid l \to r \in R\}$. The Ω-reduction $\to_\Omega$ is defined on T_Ω as $C[s] \to_\Omega C[\Omega]$ where $s \uparrow Red$ and $s \not\equiv \Omega$. Since $\to_\Omega$ is strongly normalizing, the following lemma can be easily proven [12].

Lemma 5.3 *$\to_\Omega$ is complete (i.e., Church-Rosser and strongly normalizing).*

Proof. It is clear that $\to_\Omega$ is weakly Church-Rosser. Apply Newman's Lemma. □

$\omega(t)$ denotes the normal form of t concerning $\to_\Omega$. Note that $\omega(t)$ is well-defined according to the completeness of $\to_\Omega$.

Lemma 5.4 *If $t \geq s$ then $\omega(t) \geq \omega(s)$.*

Proof. By induction on the size of t, it can be easily proven. □

Example 5.5 *Let R be the (left-linear) term rewriting system with the following rewriting rules:*

$$R \quad \begin{cases} f(c, h(x)) \to c \\ g(h(x), y) \to f(x, y) \\ d \to c. \end{cases}$$

Then $Red = \{f(c, h(\Omega)), g(h(\Omega), \Omega), d\}$ and $\omega(f(g(d, c), c)) \equiv f(\Omega, c)$.

Definition 5.6 (Index) *Let Δ be a redex occurrence in $C[\Delta]$ such that $z \in \omega(C[z])$ where z is a fresh variable. Then the redex occurrence Δ is called an index of t. If Δ is an index of $C[\Delta]$ then we write $C[\Delta_I]$; otherwise $C[\Delta_{NI}]$.*

The original definition of index in Huet and Lévy [6] is restricted to orthogonal term rewriting systems; hence, any two indexes occurring in a term must be disjoint. On the other hand we assume only left-linearity for term rewriting systems. Hence, if a term rewriting system is overlapping then two indexes may be overlapping as follows.

Example 5.7 *Let $Red = \{p(s(\Omega)), s(p(\Omega))\}$. Then we have the overlapping indexes $f(s(p(s(x))_I)_I)$ since $\omega(f(z)) \equiv f(z)$ and $\omega(f(s(z))) \equiv f(s(z))$.*

One might think that overlapping redex occurrences always make overlapping indexes, but this is not the case from the following example.

Example 5.8 *Let $Red = \{0, f(0), g(f(\Omega), 1)\}$. Then we have $g(f(0_{NI})_I, 0_I)$. Note that two redex occurrences $f(0)$ and 0 are overlapping but 0 occurring in $f(0)$ is not an index.*

$t \xrightarrow{\Delta} s$ is the index reduction if Δ is an index of t. We indicate the index reduction with $t \underset{I}{\rightarrow} s$; otherwise $t \underset{NI}{\rightarrow} s$.

We say that R is strongly sequential if for each term $t \notin NF$, t has an index [6, 12, 11]. Note that index reduction of a strongly sequential system R is a reduction strategy because we can always apply an index reduction to a term being not a normal form.

The decidability of strong sequentiality for orthogonal term rewriting systems was first proven by Huet and Lévy [6], through a complicated decision procedure. A simple proof by Klop and Middeldorp can be found in [12]. This result can be immediately generalized to left-linear term rewriting systems.

Theorem 5.9 *Strong sequentiality of left-linear term rewriting systems (which may have overlapping rules) is decidable.*

Proof. The proof of the decidability by Klop and Middeldorp [12] does not use the non-overlapping limitation. Thus, their proof can be applied to the above theorem without modification. □

6 Index Reduction of Overlapping Systems

We will now explain how to prove the normalizing property of index reduction for *balanced ambiguous* term rewriting systems by using the BWCR lemma. We first define balanced ambiguous term rewriting systems.

Let R be a term rewriting system. The root reduction $t \xrightarrow[r]{} s$ is defined as $t \xrightarrow{\Delta} s$ and $\Delta \equiv t$.

Definition 6.1 *A critical pair $\langle s, t \rangle$ is root balanced joinable if $s \xrightarrow[r]{k} t'$ and $t \xrightarrow[r]{k} t'$ for some t' and $k \geq 0$. A term rewriting system R is root balanced joinable if every critical pair is root balanced joinable.*

$$\begin{tikzcd} & t \arrow[d, dashed, "k"', "r"] \\ s \arrow[r, dashed, "k", "r"'] & t' \end{tikzcd} \qquad (k \geq 0)$$

Definition 6.2 *A term rewriting system R is balanced ambiguous if R is left-linear and root balanced joinable.*

Note that every weakly orthogonal term rewriting system is trivially balanced ambiguous since every critical pair is root balanced joinable with $k = 0$.

Definition 6.3 *Let Δ and Δ' be two redex occurrences in a term t, and let $\Delta \equiv C[s_1, \cdots, s_n]$ and $C[\Omega, \cdots, \Omega] \in Red$. Then Δ and Δ' (or Δ' and Δ) are overlapping if $\Delta' \subseteq \Delta$ and $\Delta' \not\subseteq s_i$ for any i.*

Lemma 6.4 *Let R be balanced ambiguous. Let $t \xrightarrow[I]{\Delta} t'$ and $t \xrightarrow{\Delta'} t''$, where $\Delta' \subseteq \Delta$ and Δ and Δ' are overlapping. Then, we have $t' \xrightarrow[I]{k} s$ and $t'' \xrightarrow[I]{k} s$ for some s and $k \geq 0$.*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
t \arrow[r, "\Delta'"] \arrow[d, "\Delta"', "I"] & t'' \arrow[d, dashed, "k"', "I"] \\
t' \arrow[r, dashed, "k", "I"'] & s
\end{tikzcd} \qquad (k \geq 0)$$

Proof. Let $t \equiv C[\Delta]$, $t' \equiv C[p]$, $t'' \equiv C[q]$. From the root balanced joinability of the critical pair concerning Δ and Δ', we have $p \xrightarrow[r]{k} s'$ and $q \xrightarrow[r]{k} s'$ for some s' and $k \geq 0$. Since $C[\Delta_I]$, for any redex $\tilde{\Delta}$ we have $C[\tilde{\Delta}_I]$. Thus, it follows that $C[p] \xrightarrow[I]{k} C[s']$ and $C[q] \xrightarrow[I]{k} C[s']$. □

Lemma 6.5 *Let $C[\Delta_I, \Delta']$. Then $C[\Delta_I, t]$ for any t.*

Proof. $z \in \omega(C[z, \Delta']) \equiv \omega(C[z, \Omega]) \leq \omega(C[z, t])$. Thus, we have $z \in \omega(C[z, t])$.
□

.

Lemma 6.6 *Let R be balanced ambiguous. Let $t \xrightarrow[I]{\Delta} t'$, $t \xrightarrow[I]{\Delta'} t''$. Then, we have $t' \xrightarrow[I]{k} s$ and $t'' \xrightarrow[I]{k} s$ for some s and $k \geq 0$ (i.e., $\xrightarrow[I]{}$ is BWCR).*

$$\begin{tikzcd}[column sep=huge, row sep=huge] t \arrow[r, "\Delta'", "I"'] \arrow[d, "\Delta"', "I"] & t'' \arrow[d, dashed, "k"', "I"] \\ t' \arrow[r, dashed, "k", "I"'] & s \end{tikzcd} \quad (k \geq 0)$$

Proof. If Δ and Δ' are disjoint, then from Lemma 6.5 the theorem clearly holds with $k = 1$. Assume that Δ and Δ' are not disjoint, say $\Delta' \subseteq \Delta$. Then, Δ and Δ' must be overlapping as they both are the indexes. Apply Lemma 6.4. □

The parallel reduction $t \mathrel{+\!\!\!+\!\!\!\!\rightarrow} s$ is defined with $t \equiv C[\Delta_1, \cdots, \Delta_n] \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_n} s$ $(n \geq 0)$.

Lemma 6.7 *Let R be strongly sequential and balanced ambiguous, and let $t \mathrel{+\!\!\!+\!\!\!\!\rightarrow} s$. Then $t \underset{I}{=} s$ or $t \underset{I}{=} \cdot \mathrel{+\!\!\!+\!\!\!\!\rightarrow} \cdot \overset{+}{\underset{I}{\leftarrow}} s$.*

Proof. Let $t \overset{\Delta_1 \cdots \Delta_n}{\mathrel{+\!\!\!+\!\!\!\!\rightarrow}} s$ $(n \geq 0)$. The proof is by induction on n. The case $n = 0$ is trivial. Induction Step:

Case 1. Some Δ_i, say Δ_1, is an index.

We have $t \underset{I}{\xrightarrow{\Delta_1}} t' \overset{\Delta_2 \cdots \Delta_n}{\mathrel{+\!\!\!+\!\!\!\!\rightarrow}} s$. By applying induction hypothesis to $t' \overset{\Delta_2 \cdots \Delta_n}{\mathrel{+\!\!\!+\!\!\!\!\rightarrow}} s$, we obtain the lemma.

Case 2. No Δ_i is an index.

From the strong sequentiality there must exist an index, say Δ, in t. Let $t \underset{I}{\xrightarrow{\Delta}} t''$ and consider the following two cases.

Case 2-1. Δ and Δ_i $(i = 1 \cdots n)$ are non-overlapping.

By using the left-linearity of R, we can easily show that $t'' \mathrel{-\!\!\Vert\!\!\!\rightarrow} s'$ and $s \underset{I}{\rightarrow} s'$ for some s'.

$$
\begin{tikzcd}
t \arrow[rr, "\Delta_1, \cdots, \Delta_n", "\Vert"'] \arrow[d, "\Delta", "I"'] & & s \arrow[d, dashed, "I"'] \\
t'' \arrow[rr, dashed, "\Vert"] & & s'
\end{tikzcd}
$$

Case 2-2. Δ and some Δ_i, say Δ_1, are overlapping.

Let $t \underset{NI}{\overset{\Delta_1}{\rightarrow}} t' \overset{\Delta_2 \cdots \Delta_n}{\mathrel{-\!\!\Vert\!\!\!\rightarrow}} s$. Note that $\Delta_1 \subseteq \Delta$. From Lemma 6.4, it follows that $t'' \underset{I}{\overset{k}{\rightarrow}} s'$ and $t' \underset{I}{\overset{k}{\rightarrow}} s'$ for some $k \geq 0$. Thus, we can obtain $t \underset{I}{=} t'$. Apply induction hypothesis to $t' \overset{\Delta_2 \cdots \Delta_n}{\mathrel{-\!\!\Vert\!\!\!\rightarrow}} s$. $\Box$

$$
\begin{tikzcd}
t \arrow[r, "\Delta_1", "NI"'] \arrow[d, "\Delta"', "I"] & t' \arrow[r, "\Delta_2, \cdots, \Delta_n", "\Vert"'] \arrow[d, dashed, "k"', "I"] & s \\
t'' \arrow[r, dashed, "k", "I"'] & s' &
\end{tikzcd}
$$

Theorem 6.8 *Let R be strongly sequential and balanced ambiguous. Then index reduction $\xrightarrow[I]{}$ is normalizing.*

Proof. Note that from the strong sequentiality, we have $NF = NF_I$ (NF_I denotes the set of the normal forms concerning $\xrightarrow[I]{}$). Let $t \xrightarrow{*} s \in NF$. Since $\xrightarrow{*} = \overset{*}{\mathrel{+\!\!\!\!\!\!\longrightarrow}}$, we have $t \overset{*}{\mathrel{+\!\!\!\!\!\!\longrightarrow}} s \in NF_I$. Taking $\xrightarrow[I]{}$ as $\xrightarrow[\alpha]{}$ and $\mathrel{+\!\!\!\!\!\!\longrightarrow}$ as $\xrightarrow[\beta]{}$ respectively, from Lemma 6.7, we can apply Lemma 3.4. Hence, it is proven that $t \xrightarrow[I]{*} s \in NF_I$. From the BWCR Lemma, it follows that index reduction $\xrightarrow[I]{}$ is normalizing. □

Corollary 6.9 *If R is strongly sequential and balanced ambiguous, then R is UN.*

Proof. By Corollary 3.3 and Theorem 6.8, it is trivial. □

Quasi-index reduction (or the hyper-index reduction) is defined as $\xrightarrow[NI]{*} \cdot \xrightarrow[I]{}$. We show that in Theorem 6.8 index reduction can be relaxed into quasi-index reduction. We first prove the next lemma.

Lemma 6.10 *Let R be strongly sequential and balanced ambiguous. Let $t \xrightarrow[I]{n} s \in NF$ for some $n \geq 0$ and $t \xrightarrow{*} t'$. Then, we have $t' \xrightarrow[I]{m} s$ for some $m \leq n$.*

$$\begin{tikzcd}
t \arrow[rr, "n"', "I"] \arrow[d, "*"'] & & s \in NF \\
t' \arrow[urr, dashed, "m", "I"'] & &
\end{tikzcd}$$

Proof. The proof is by induction on n. The case $n = 0$ is trivial. Induction Step: To prove the lemma, we only have to show that if $t \underset{I}{\overset{\Delta}{\rightarrow}} t'' \underset{I}{\overset{n-1}{\rightarrow}} s \in NF$ and $t \overset{\Delta'}{\rightarrow} t'$, then $t' \underset{I}{\overset{m}{\rightarrow}} s \in NF$ for some $m \leq n$.

$$\begin{tikzcd}
t \arrow[r, "\Delta", "I"'] \arrow[d, "\Delta'"'] & t'' \arrow[r, "n-1", "I"'] & s \in NF \\
t' \arrow[urr, dashed, "m", "I"'] & &
\end{tikzcd}$$

Case 1. Δ and Δ' are non-overlapping.

By the left-linearity of R, we can easily show that $t' \underset{I}{\rightarrow} s'$ and $t'' \overset{*}{\rightarrow} s'$ for some s'. From induction hypothesis, it follows that $s' \underset{I}{\overset{m'}{\rightarrow}} s$ for some $m' \leq n-1$. Thus, we obtain $t' \underset{I}{\overset{m'+1}{\rightarrow}} s$.

Case 2. Δ and Δ' are overlapping.

By using Lemma 6.4, we have $t' \underset{I}{\overset{k}{\rightarrow}} s'$ and $t'' \underset{I}{\overset{k}{\rightarrow}} s'$ for some s' and $k \geq 0$. Applying the BWCR lemma to t'', we have $s' \underset{I}{\overset{n-1-k}{\rightarrow}} s$. Thus, it follows that $t' \underset{I}{\overset{n-1}{\rightarrow}} s$. □

Theorem 6.11 *Let R be strongly sequential and balanced ambiguous. Then quasi-index reduction $\underset{NI}{\overset{*}{\rightarrow}} \cdot \underset{I}{\rightarrow}$ is normalizing.*

Proof. Let t have a normal form s. Then we have $t \underset{I}{\overset{n}{\rightarrow}} s$ for some n. By using induction on n we prove that every quasi-index reduction starting with t is normalizing. The case $n = 0$ is trivial as t is the normal from. Let $t \underset{I}{\overset{n}{\rightarrow}} s$ $(n > 0)$. Take any one-step quasi-index reduction starting with t, say $t \underset{NI}{\overset{*}{\rightarrow}} t' \underset{I}{\rightarrow} t''$. From Lemma 6.10 we have $t' \underset{I}{\overset{m}{\rightarrow}} s$ for some $m \leq n$. Thus, by applying the BWCR lemma to t' we obtain

$t'' \xrightarrow[I]{n'} s$ for some $n' < n$ as $m = n' + 1 \leq n$. From induction hypothesis it follows that the quasi-index reduction starting with t'' is normalizing. Therefore the theorem holds. □

7 Balanced Ambiguous Left-Normal Systems

In this section we discuss the question: what kind of term rewriting systems are strongly sequential. An answer concerning orthogonal term rewriting systems was found by O'Donnell [13]. He proved that if an orthogonal term rewriting system R is left-normal then R is strongly sequential and leftmost-outermost reduction is normalizing. We show that his result can be naturally extended to balanced ambiguous term rewriting systems.

Definition 7.1 *The set T_L of the left-normal terms is inductively defined as follows:*

1. *$x \in T_L$ if x is a variable,*
2. *$f(t_1, \cdots, t_{p-1}, t_p, t_{p+1} \cdots, t_n) \quad (0 \leq p \leq n)$*
   if $t_1, \cdots, t_{p-1}$ are ground terms (i.e., no variable occurs in $t_1, \cdots, t_{p-1}$), $t_p \in T_L$, and $t_{p+1}, \cdots, t_n$ are variables.

The set of the left-normal schemata is $T_{L\Omega} = \{t_\Omega \mid t \in T_{L\Omega}\}$. We say that R is left-normal [13, 6, 11] iff for any rule $l \to r$ in R, l is a left-normal term, i.e., $Red \subseteq T_{L\Omega}$.

Let Δ be the leftmost-outermost redex in $t \equiv C[\Delta]$. Then we write $t^z \equiv C[z]$, where z is a fresh variable.

Lemma 7.2 *Let R be left-linear and left-normal (note that R may be overlapping). If t is not a normal form, then:*

(1) *$z \in \omega(t^z)$ (i.e., the leftmost-outermost redex is an index),*

(2) *For any $s \in T_{L\Omega}$, if $\omega(t^z) \uparrow s$ then $t \uparrow s$.*

Proof. Let Δ be the leftmost-outermost redex in $t \equiv C[\Delta]$. By induction on the size of $C[\,]$, we will prove the lemma.

Basic step: $C[\,] \equiv \Box$. Then $t \equiv \Delta$ and $t^z \equiv z$. Thus (1) is trivial. If $\omega(t^z) \uparrow s$ then $s \equiv \Omega$. Hence, (2) is also trivial.

Induction step: Since $C[\,] \not\equiv \Box$, we can write $t \equiv f(t_1, \cdots, t_{p-1}, t_p, t_{p+1}, \cdots, t_n)$, where $t_1, \cdots, t_{p-1}$ are normal forms and the leftmost-outermost redex Δ occurs in t_p. Note that Δ is the leftmost-outermost redex not only of t but also of t_p. Thus we can write $t^z \equiv f(t_1, \cdots, t_{p-1}, t_p^z, t_{p+1}, \cdots, t_n)$. Let us consider
$\tilde{t}^z \equiv f(\omega(t_1), \cdots, \omega(t_{p-1}), \omega(t_p^z), \omega(t_{p+1}), \cdots, \omega(t_n))$. Then
$\tilde{t}^z \equiv f(t_1, \cdots, t_{p-1}, \omega(t_p^z), \omega(t_{p+1}), \cdots, \omega(t_n))$ as $t_1, \cdots, t_{p-1}$ are normal forms. From induction hypothesis, $z \in \omega(t_p^z)$.

We will show the claim: For any $s \in T_{L\Omega}$, if $\tilde{t}^z \uparrow s$ then $t \uparrow s$.

Proof of the claim. Let $\tilde{t}^z \uparrow s$ and $s \equiv f(s_1, \cdots, s_p, \cdots, s_n)$. Then $\omega(t_p^z) \uparrow s_p$. By induction hypothesis, we have $t_p \uparrow s_p$; hence, $\Omega \in s_p$ as z occurs in $\omega(t_p^z)$. From $s \in T_{L\Omega}$ it follows that $s_{p+1} \equiv, \cdots, \equiv s_n \equiv \Omega$. Hence, it is clear that $t \uparrow s$.

Assume that $\tilde{t}^z \uparrow Red$. Then, from the claim, we have $t \uparrow Red$; contradiction to $\Delta \not\equiv t$. Thus, $\tilde{t}^z \# Red$. Hence $\omega(t^z) \equiv \tilde{t}^z$. Therefore, $z \in \omega(t^z)$. From the claim, it is clear that (2) also holds. $\Box$

Theorem 7.3 *Let R be balanced ambiguous left-normal. Then, leftmost-outermost reduction is normalizing.*

Proof. This follows from Theorem 6.8 and Lemma 7.2. $\Box$

Corollary 7.4 *Every balanced ambiguous left-normal term rewriting system is UN.*

Proof. From Corollary 6.9 and the above theorem, it is trivial. $\Box$

Note that every left-normal weakly orthogonal term rewriting system is balanced ambiguous left-normal. Thus the following corollary follows.

Corollary 7.5 *Let R be left-normal weakly orthogonal. Then, leftmost-outermost reduction is normalizing.*

Example 7.6 *Combinatory logic CL [1] is the orthogonal term rewriting system having the following rewriting rules:*

$$CL \quad \begin{cases} ((S \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z) \\ (K \cdot x) \cdot y \to x. \end{cases}$$

Let R be combinatory logic CL +

$$\begin{cases} pred \cdot (succ \cdot x) \to x \\ succ \cdot (pred \cdot x) \to x \end{cases}$$

It is clear that R is left-normal weakly orthogonal. Thus, leftmost-outermost reduction strategy is normalizing for R.

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