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Abstract: Hyperasymptotic expansions are recently introduced by Berry and Howls, and yield refined information by expanding remainders in asymptotic expansions. In a recent paper of Olde Daalhuis a method is given for obtaining hyperasymptotic expansions of integrals that represent the confluent hypergeometric U-function. This paper gives an extension of that method to neighbourhoods of the so-called Stokes lines. At each level, the remainder is exponentially small compared with the previous remainders. Two numerical illustrations confirm these exponential improvements.

1. Introduction

In this paper we give hyperasymptotic expansions for the confluent hypergeometric functions $U(a, c, z)$ for large $|z|$, where z is in a neighbourhood of a so-called Stokes line. An integral representation for the $U(a, c, z)$ is

$$U(a, c, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{-b} dt, \qquad (1.1)$$

where $b = a - c + 1$ and where the constant a fulfils $\Re a > 0$.

First we give a general description of hyperasymptotic expansions. Simple asymptotic expansions are of the form $f(z) = a_0 + a_1 z^{-1} + \cdots + a_{N_0-1} z^{1-N_0} + R_{N_0}(z)$, where $R_{N_0}(z) = \mathcal{O}(z^{-N_0})$ as $|z| \to \infty$. The optimal N_0 can be obtained by minimization of the remainder R_{N_0} when z is give... Generally, N_0 is a function of z. Considering R_{N_0} as a function depending on two large parameters (z and N_0), it is expanded in a new asymptotic expansion, which is truncated at N_1: $R_{N_0}(z) = B_0 + B_1 + \cdots + B_{N_1-1} + R_{N_1}(z)$, where the remainder $R_{N_1}(z)$ appears to be exponentially small compared with $R_{N_0}(z)$, as $|z| \to \infty$. The B_m are not of the form $const.z^{-M-m}$. Repeating this, we obtain

$$f(z) = a_0 + \cdots + a_{N_0-1} z^{1-N_0} + B_0 + \cdots + B_{N_1-1} + C_0 + C_1 + \cdots + R_{N_n}(z), \qquad (1.2)$$

where R_{N_n} appears to be exponentially small compared with $R_{N_{n-1}}$, i.e. $R_{N_n} = \mathcal{O}(e^{-\lambda_n |z|})$, as $|z| \to \infty$, where $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$.

A hyperasymptotic expansion for $U(a, c, z)$ is given in OLDE DAALHUIS [5], for $|z| \to \infty$. This expansion is limited to the sector $|\arg(z)| < \pi$. This is due to new exponentially small terms in the asymptotic expansion of $U(a, c, z)$, which appear in neighbourhoods of the Stokes lines $\arg(z) = \pm\pi$. These new terms are of the same order as $R_{N_0}(z)$. For hyperasymptotic expansions in a neighbourhood of these Stokes lines we need more complicated functions than the functions used in [5].

Two asymptotic expansions for $U(a, c, z)$, in a neighbourhood of $\arg(z) = \pi$, are

$$U(a, c, z) \sim z^{-a} \sum_{n=0}^\infty (-1)^n \frac{(a)_n (b)_n}{n! z^n}, \qquad |\arg(z)| \leq \tfrac{3}{2}\pi - \delta, \qquad (1.3)$$

$$U(a,c,z) \sim z^{-a} \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (b)_n}{n! z^n}$$
$$- 2\pi i \frac{e^{-(a+b)\pi i}}{\Gamma(a)\Gamma(b)} z^{b-1} e^z \sum_{n=0}^{\infty} \frac{(1-a)_n(1-b)_n}{n! z^n}, \qquad \tfrac{1}{2}\pi - \delta \le \arg(z) \le \tfrac{5}{2}\pi + \delta, \qquad (1.4)$$

where Pochhammer's symbol $(a)_n$ is defined by $(a)_n = \Gamma(a+n)/\Gamma(a)$, and where δ is a small positive constant. These expansions are combined into the following expansion given in OLVER [8]

$$U(a,c,z) = z^{-a} \sum_{n=0}^{N-1} (-1)^n \frac{(a)_n (b)_n}{n! z^n} + R_N(z),$$

$$R_N(z) = (-1)^N 2\pi \frac{z^{b-1} e^z}{\Gamma(a)\Gamma(b)} \sum_{m=0}^{M-1} (-1)^M \frac{(1-a)_m(1-b)_m}{m!} \frac{F_{N-m+a+b-1}(z)}{z^m} + R_M(z), \qquad (1.5)$$

where $N = |z| - a - b + 1 + \alpha$, $|\alpha|$ being bounded, and where M is an arbitrary fixed integer. This expansion holds for $|\arg(z)| \le \tfrac{5}{2}\pi - \delta$. The extra functions that we will use in hyperasymptotic expansions of $U(a,c,z)$, in neighbourhoods of the Stokes lines, are

$$F_p(z) = \frac{e^{-z}}{2\pi} \int_0^{\infty} \frac{e^{-zt} t^{p-1}}{1+t} dt. \qquad (1.6)$$

This function is closely related to one of the incomplete gamma functions in the following way: $F_p(z) = (2\pi)^{-1} \Gamma(p) \Gamma(1-p, z)$. In OLVER [7] it is shown that

$$F_{N+p}(z) = \tfrac{1}{2}(-1)^N i e^{-p\pi i} \operatorname{erfc}\{c(\theta)\sqrt{\rho}\} + \mathcal{O}\Big(\Big|\frac{\exp(-\rho[c(\theta)]^2)}{\sqrt{\rho}}\Big|\Big), \qquad \text{as } |z| \to \infty, \qquad (1.7)$$

where z is written as $z = \rho e^{i\theta}$, and where p is fixed and $|N - \rho|$ is bounded; furthermore $c(\theta) = \sqrt{e^{i\theta} + 1 + i(\theta - \pi)}$. This expansion combined with (1.5) shows the appearance of new exponentially small terms in the asymptotic expansion of $U(a,c,z)$, in a neighbourhood of the Stokes lines $\theta = \pm\pi$.

The hyperasymptotic expansion presented in this paper is of the form

$$U(a,c,z) = z^{-a} \sum_{n=0}^{N_0-1} (-1)^n \frac{(a)_n (b)_n}{n! z^n}$$
$$+ (-1)^{N_0} 2\pi \frac{e^z}{\Gamma(a)\Gamma(b)} U(1-b, c, \rho e^{i(\theta-\pi)}) F_{a+N_0}(z) \qquad \text{as } \rho \to \infty, \qquad (1.8)$$
$$+ N.H.E.,$$

and it holds for $\delta \le \theta \le 2\pi - \delta$, where, again, $z = \rho e^{i\theta}$ and δ is a small positive constant. The $N.H.E.$ stands for Normal Hyperasymptotic Expansion, which is of the form described at the beginning of this section, and in [5].

2. Hyperasymptotic expansions of $U(a,c,z)$, for $|\arg(z)| < \pi$

In this section we summarize the results and the method of [5]. We write $z = \rho e^{i\theta}$ and we take $t = \tau e^{-i\theta}$, where $-\pi + \delta \le \theta \le \pi - \delta$ and δ small and positive. In this way we obtain an analytic continuation with respect to this z-domain for $U(a,c,z)$, with

$$U(a,c,z) = \frac{e^{-ia\theta}}{\Gamma(a)} \int_0^{\infty} e^{-\rho\tau} \tau^{a-1} (1 + \tau e^{-i\theta})^{-b} d\tau. \qquad (2.1)$$

We define $f_0(\tau) = (1 + \tau e^{-i\theta})^{-b}$ and we expand $f_0(\tau)$ at $\tau = 0$

$$f_0(\tau) = \sum_{n=0}^{N_0-1} \frac{(b)_n}{n!} (-\tau e^{-i\theta})^n + \tau^{N_0} f_1(\tau). \tag{2.2}$$

Taylor's theorem gives for $f_1(\tau)$ the following integral representation

$$f_1(\tau) = \frac{1}{2\pi i} \int_{\Omega_0(0,\tau)} \frac{f_0(w)dw}{w^{N_0}(w-\tau)}, \tag{2.3}$$

where we choose $\Omega_0(0, \tau)$ the union of the circles $|w| = 1 - \varepsilon$ and $|w - \tau| = \varepsilon/2$, where $0 < \varepsilon < \frac{1}{2}$ and ε is bounded away from 0. In the case that $|\tau| \leq 1 - \varepsilon/2$ we have to adjust this contour. See Figure 2.1.
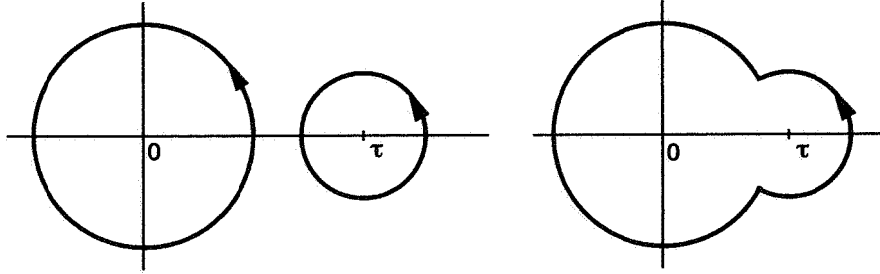

FIGURE 2.1. The normal and adjusted contour of integration of (2.3)

On $|\tau + e^{i\theta}| \geq \varepsilon$ we have $|f_0(\tau)| \leq \varepsilon^{-\Re b}$, and with (2.3) we obtain

$$|f_1(\tau)| \leq C_1(1-\varepsilon)^{-N_0}, \qquad \text{as } |\tau + e^{i\theta}| \geq \tfrac{3}{2}\varepsilon, \tag{2.4}$$

where C_1 does not depend on N_0.
We substitute (2.2) in (2.1) and obtain

$$U(a, c, z) = z^{-a} \sum_{n=0}^{N_0-1} (-1)^n \frac{(a)_n (b)_n}{n! z^n} + R_{N_0}(z), \tag{2.5}$$

with

$$R_{N_0}(z) = \frac{e^{-ia\theta}}{\Gamma(a)} \int_0^\infty e^{-\rho\tau} \tau^{a-1+N_0} f_1(\tau) d\tau. \tag{2.6}$$

With (2.4) we estimate

$$|R_{N_0}(z)| \leq \frac{C_1}{|\Gamma(a)|} \int_0^\infty e^{-\rho\tau} \tau^{\Re a - 1 + N_0} (1-\varepsilon)^{-N_0} d\tau. \tag{2.7}$$

The integrand has a maximum at $\tau = \gamma_1$, where γ_1 fulfils $\rho = (\Re a - 1 + N_0)/\gamma_1$. This maximum is $e^{-\rho\gamma_1} \gamma_1^{\rho\gamma_1} (1 - \varepsilon)^{\Re a - 1 - \rho\gamma_1}$, and it is the main factor in asymptotic estimations of the right-hand side of (2.7). This factor as a function of γ_1 is minimal at $\gamma_1 = 1 - \varepsilon$. So, we choose $N_0 = [\rho(1 - \varepsilon) - \Re a + 1]$ and afterwards we take $\gamma_1 = (\Re a - 1 + N_0)/\rho$, where $[y]$ denotes the integer part of y. This z-dependence of N_0 is typical in hyperasymptotic expansions, and with this N_0 the remainder is minimal. Laplace's method (see OLVER [6, p. 80]) gives the estimate

$$|R_{N_0}(z)| = \mathcal{O}(C_1 \sqrt{\frac{2\pi}{\rho}} e^{-\rho\gamma_1} \gamma_1^{\Re a - \frac{1}{2}}), \qquad \text{as } |z| \to \infty. \tag{2.8}$$

Now we describe the method for obtaining the $(n+1)$th-level of the hyperasymptotic expansion from the remainder of the nth-level. More details are given in [5], and the next section is more detailed. The remainder at the nth-level is of the form

$$R_{N_n}(z) = \frac{e^{-ia\theta}}{\Gamma(a)} \int_0^\infty e^{-\rho\tau} \tau^{a-1+N_0} (\tau - \gamma_1)^{N_1} \cdots (\tau - \gamma_n)^{N_n} f_{n+1}(\tau) d\tau, \tag{2.9}$$

where

$$f_{n+1}(\tau) = \frac{1}{2\pi i} \int_{\Omega_n(\gamma_n, \tau)} \frac{f_n(w) dw}{(w - \gamma_n)^{N_n} (w - \tau)}, \tag{2.10}$$

and where $\Omega_n(\gamma_n, \tau)$ is a contour that encircles γ_n and τ. These contours look like those in Figure 2.1. The estimations of the R_{N_n} are of the form

$$|R_{N_n}(z)| \le C_n \int_0^\infty e^{-\rho\tau} \frac{\tau^{\Re a - 1 + N_0}}{(1-\varepsilon)^{N_0}} \left| \frac{\tau - \gamma_1}{d(\gamma_1, \theta) - \rho_1 \varepsilon} \right|^{N_1} \cdots \left| \frac{\tau - \gamma_n}{d(\gamma_n, \theta) - \rho_n \varepsilon} \right|^{N_n} d\tau, \tag{2.11}$$

where $d(\gamma_m, \theta) = \sqrt{\gamma_m^2 + 2\gamma_m \cos\theta + 1}$ and $\rho_m = 2 - 2^{-m}$.
For obtaining N_n, such that the right-hand side of (2.11) is minimal, we analyse the integrand of (2.11). This integrand is of the form shown in Figure 2.2.
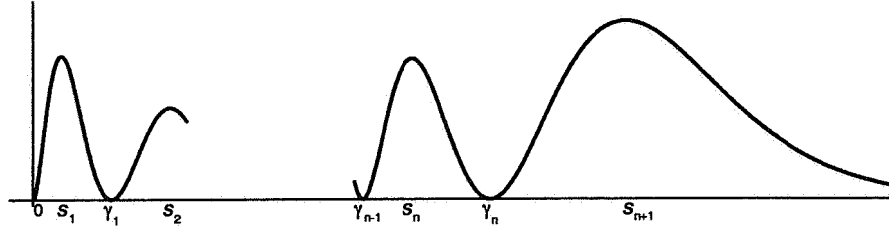

FIGURE 2.2. The graph of the integrand of (2.11)

The local maxima are at $\tau = s_j$, $j = 1, 2, \cdots, n, n+1$, and these s_j fulfil

$$\rho = \frac{\Re a - 1 + N_0}{s_j} + \frac{N_1}{s_j - \gamma_1} + \cdots + \frac{N_n}{s_j - \gamma_n}, \tag{2.12}$$

From (2.12) if follows that $s_1, ..., s_n$ are decreasing functions of N_n, and that s_{n+1} is an increasing function of N_n. It is not difficult to prove that the local maxima at $s_1, ..., s_n$ are decreasing functions of N_n, and that the local maximum at s_{n+1} is decreasing on the interval $s_{n+1} \in (\gamma_n, \gamma_n + d(\gamma_n, \theta) - \rho_n \varepsilon)$ and increasing on the interval $s_{n+1} \in (\gamma_n + d(\gamma_n, \theta) - \rho_n \varepsilon, \infty)$. Thus for some $N_n \in \mathbb{N}$ the global maximum is minimal, and it is attained at s_{n+1}. Now we expand $f_{n+1}(\tau)$ at $\gamma_{n+1} = s_{n+1}$:

$$f_{n+1}(\tau) = b_0 + b_1(\tau - \gamma_{n+1}) + \cdots + b_{N_{n+1}-1}(\tau - \gamma_{n+1})^{N_{n+1}-1} + (\tau - \gamma_{n+1})^{N_{n+1}} f_{n+2}(\tau). \tag{2.13}$$

We substitute this expansion in (2.9) and we obtain the $(n+1)$th-level

$$R_{N_n}(z) = \frac{e^{-ia\theta}}{\Gamma(a)} \sum_{m=0}^{N_{n+1}-1} b_m \int_0^\infty e^{-\rho\tau} \tau^{a-1+N_0} \cdots (\tau - \gamma_n)^{N_n} (\tau - \gamma_{n+1})^m d\tau + R_{N_{n+1}}(z). \tag{2.14}$$

In [5] it is explained how the integrals in (2.14) can be computed.

At the first and the second level of the hyperasymptotic process the minimal global maximum is attained only at the most right maximum (i.e. at s_{n+1}). For the higher levels this is not obvious, and in the next section it is shown what happens if the global maximum is attained at more than 1 of the s_j.

At each level we obtain $|R_{N_n}(z)| = \mathcal{O}(\rho^{-\frac{1}{2}}e^{-\lambda_n\rho})$ as $|z| \to \infty$, where $\lambda_0 \sim 1 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$. We do not have a proof whether or not $\{\lambda_n\}_{n\in\mathbb{N}}$ is an unbounded sequence. For $\theta = 0$ we have $\lambda_1 \sim 1.90$ and $\lambda_2 \sim 3.59$. Thus at the second level our λ_2 exceeds already the limiting value of 2.39 of a different method, which is described at the end of the next section.

3. Hyperasymptotic expansions of $U(a, c, z)$, near the Stokes line $\arg(z) = \pi$

In this section we shall obtain a hyperasymptotic expansion for $U(a, c, \rho e^{i\theta})$ of the form (1.8), which holds for $\delta \leq \theta \leq 2\pi - \delta$. The expansion will be in terms of the function $F_{a+N_0}(z)$, and functions of the form $U(1 - b, c, \lambda\rho e^{i(\theta-\pi)})$, where λ is a positive constant. Notice that for confluent hypergeometric functions with the present z-domain, we already derived a hyperasymptotic expansion in §2 and in [5].

We start with the expansion (2.5), and we use [8] where an integral representation for $R_{N_0}(z)$ is given in the form

$$R_{N_0}(z) = (-1)^{N_0}\frac{e^{-i\theta(a+N_0-1)}}{\Gamma(a)\Gamma(b)}\int_0^\infty \frac{U(1-b,c,\rho\tau)e^{-\rho\tau}\tau^{a+N_0-1}}{\tau + e^{i\theta}}d\tau. \tag{3.1}$$

This integral is defined for $-\pi < \theta < \pi$ and $N_0 > \max(\Re(-a), \Re(-b))$. Now we substitute $U(1-b, c, \rho\tau) = \{U(1 - b, c, \rho\tau) - U(1 - b, c, \rho e^{i(\theta-\pi)})\} + U(1-b, c, \rho e^{i(\theta-\pi)})$ in (3.1) and we obtain

$$\begin{aligned}R_{N_0}(z) =&(-1)^{N_0}\frac{e^{-i\theta(a+N_0-1)}}{\Gamma(a)\Gamma(b)}\int_0^\infty g_1(\tau, z)e^{-\rho\tau}\tau^{a+N_0-1}d\tau\\&+\frac{2\pi(-1)^{N_0}}{\Gamma(a)\Gamma(b)}e^z U(1-b, c, \rho e^{i(\theta-\pi)})F_{a+N_0}(z),\end{aligned} \tag{3.2}$$

where we used the integral representation (1.6) for the second term, and where

$$g_1(t, z) = \frac{U(1-b, c, \rho t) - U(1-b, c, \rho e^{i(\theta-\pi)})}{t + e^{i\theta}}. \tag{3.3}$$

This function is regular in $t = e^{i(\theta-\pi)}$, thus the right-hand side of (3.2) is an analytic continuation of $R_{N_0}(z)$ for $-\pi < \theta < 3\pi$.

In the remaining part of this section we shall give an hyperasymptotic expansion for the first term of the right-hand side of (3.2), that is

$$S_{N_0}(z) = (-1)^{N_0}\frac{e^{-i\theta(a+N_0-1)}}{\Gamma(a)\Gamma(b)}\int_0^\infty g_1(\tau, z)e^{-\rho\tau}\tau^{a+N_0-1}d\tau. \tag{3.4}$$

In the appendix we shall prove that we can bound $g_1(t, z)$ by

$$|g_1(t, z)| \leq C_1(t, \varepsilon)\rho^{\Re b-1}, \qquad \Re(t) > 0, \tag{3.5}$$

where $C_1(t, \varepsilon) = C_1\max(|t|^{-|\Re a|-|\Re b|}, \varepsilon)$, with ε a small positive constant. With (3.5) we estimate

$$|S_{N_0}(z)| \leq C_1\rho^{\Re b-1}\int_0^\infty e^{-\rho\tau}\tau^{\Re a+N_0-1}d\tau, \tag{3.6}$$

where C_1 depends on a, b and ε, and for small ε we can obtain the estimate $|S_{N_0}(z)| = \mathcal{O}(\rho^{\Re b-\frac{3}{2}}e^{-\rho})$, as $\rho \to \infty$.

For the Taylor series of $g_1(t, z)$ at $t = \lambda$, where λ is a positive number, we can use the next expression

$$\frac{d^n}{dt^n} g_1(t, z) = \frac{1}{t + e^{i\theta}} \left[(-\rho)^n (1 - b)_n U(1 - b + n, c + n, \rho t) - n \frac{d^{n-1}}{dt^{n-1}} g_1(t, z) \right], \qquad (3.7)$$

which is proved in the appendix. With the recursion relation

$$U(\alpha + 2, \gamma + 2, z) = \frac{\gamma - z}{z(\alpha + 1)} U(\alpha + 1, \gamma + 1, z) + \frac{1}{z(\alpha + 1)} U(\alpha, \gamma, z), \qquad (3.8)$$

which follows from Kummer's equation, it follows that the coefficients of the Taylor series of $g_1(t, z)$, at $t = \lambda$, can be computed easily from the 'known' $U(1 - b, c, \rho\lambda)$, $U(2 - b, c + 1, \rho\lambda)$ and $U(1 - b, c, \rho e^{i(\theta - \pi)})$.

Now we expand $g_1(\tau, z)$ in a Taylor series at $\tau = \gamma_1$, where γ_1 is the maximum of the integrand of (3.6), (see §2):

$$g_1(\tau, z) = b_{0,1} + b_{1,1}(\tau - \gamma_1) + \cdots + b_{N_1-1,1}(\tau - \gamma_1)^{N_1-1} + (\tau - \gamma_1)^{N_1} g_2(\tau, z), \qquad (3.9)$$

where

$$g_2(t, z) = \frac{1}{2\pi i} \int_{\Omega_1(\gamma_1, t)} \frac{g_1(w, z)}{(w - \gamma_1)^{N_1}(w - t)} dw, \qquad (3.10)$$

and where $\Omega_1(\gamma_1, t)$ is a contour that encircles γ_1 and τ. For obtaining sharp estimates for $g_2(t, z)$, we choose here and below for $n = 1, 2, 3, \ldots$

$$\Omega_n(\gamma_n, t) = \begin{cases} \{ w \in \mathbb{C} \mid |w - \gamma_n| = \gamma_n - \rho_n\varepsilon \text{ or } |w - t| = 2^{-n}\varepsilon \} & \text{for} \quad \Re t > \rho_n\varepsilon, \\ \{ w \in \mathbb{C} \mid |w - \gamma_n| = \gamma_n - \rho_n\varepsilon \text{ or } |w - t| = \frac{1}{2}\Re t \} & \text{for} \quad 0 < \Re t \le \rho_n\varepsilon, \end{cases} \qquad (3.11)$$

where $\rho_n = 2 - 2^{1-n}$. Thus $\Omega_n(\gamma_n, t)$ is the union of two circles. In the case that these discs intersect, we take $\Omega_n(\gamma_n, t)$ the outer part of these two circles. (See Figure 2.1). With this $\Omega_1(\gamma_1, t)$ it is not difficult to prove that

$$\left| g_2(t, z) \right| \le C_2(t, \varepsilon) \rho^{\Re b-1} (\gamma_1 - \rho_1\varepsilon)^{-N_1}, \qquad \Re(t) > 0, \qquad (3.12)$$

where $C_2(t, \varepsilon)$ is of the same form as $C_1(t, \varepsilon)$, used in (3.5).

We substitute (3.9) in (3.4) and we obtain

$$S_{N_0}(z) = (-1)^{N_0} \frac{e^{-i\theta(a+N_0-1)}}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{N_1-1} b_{n,1} \int_0^\infty e^{-\rho\tau} \tau^{a+N_0-1}(\tau - \gamma_1)^n d\tau + S_{N_1}(z), \qquad (3.13)$$

where

$$S_{N_1}(z) = (-1)^{N_0} \frac{e^{-i\theta(a+N_0-1)}}{\Gamma(a)\Gamma(b)} \int_0^\infty g_2(\tau, z) e^{-\rho\tau} \tau^{a+N_0-1}(\tau - \gamma_1)^{N_1} d\tau. \qquad (3.14)$$

Again, in [5] it is explained how to compute the integrals in (3.13). With (3.12) we estimate

$$|S_{N_1}(z)| \le C_2 \rho^{\Re b-1} \int_0^\infty e^{-\rho\tau} \tau^{\Re a+N_0-1} \left| \frac{\tau - \gamma_1}{\gamma_1 - \rho_1\varepsilon} \right|^{N_1} d\tau, \qquad (3.15)$$

where C_2 only depends on a, b and ε. The graph of the integrand of (3.15) looks like Figure 2.2; it has 2 local maxima, and it is not difficult to prove that the most right maximum is the global maximum. This maximum is minimal for $N_1 = \rho(\gamma_1 - \rho_1\varepsilon)^2(2\gamma_1 - \rho_1\varepsilon)^{-1}$. So we choose $N_1 = [\rho(\gamma_1 - \rho_1\varepsilon)^2(2\gamma_1 - \rho_1\varepsilon)^{-1}]$, and we call γ_2 the place where the integrand has its maximum. We have $\gamma_2 = 2\gamma_1 - \rho_1\varepsilon + \mathcal{O}(\rho^{-1})$, as $\rho \to \infty$, and for small ε we obtain $|S_{N_1}(z)| = \mathcal{O}(\rho^{\Re b - \frac{3}{2}} e^{-(1.30..)\rho})$, as $\rho \to \infty$.

Next we want to expand $g_2(\tau, z)$ in a Taylor series at $\tau = \gamma_2$. By writing $g_2(\tau, z) = \{g_1(\tau, z) - \sum_{n=0}^{N_1-1} b_{n,1}(\tau - \gamma_1)\}(\tau - \gamma_1)^{-N_1}$, it follows that the Taylor series of $g_2(\tau, z)$ at $\tau = \gamma_2$, can be computed with the Taylor series of $g_1(\tau, z)$ at $\tau = \gamma_2$. We write

$$g_2(\tau, z) = b_{0,2} + b_{1,2}(\tau - \gamma_2) + \cdots + b_{N_2-1,2}(\tau - \gamma_2)^{N_2-1} + (\tau - \gamma_2)^{N_2} g_3(\tau, z), \qquad (3.16)$$

and we obtain

$$S_{N_1}(z) = (-1)^{N_0} \frac{e^{-i\theta(a+N_0-1)}}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{N_2-1} b_{n,2} \int_0^\infty e^{-\rho\tau} \tau^{a+N_0-1}(\tau - \gamma_1)^{N_1}(\tau - \gamma_2)^n d\tau + S_{N_2}(z), \qquad (3.17)$$

where

$$S_{N_2}(z) = (-1)^{N_0} \frac{e^{-i\theta(a+N_0-1)}}{\Gamma(a)\Gamma(b)} \int_0^\infty g_3(\tau, z) e^{-\rho\tau} \tau^{a+N_0-1}(\tau - \gamma_1)^{N_1}(\tau - \gamma_2)^{N_2} d\tau. \qquad (3.18)$$

With estimates for $g_3(t, z)$, which look like (3.12), we estimate

$$|S_{N_2}(z)| \le C_3 \rho^{\Re b-1} \int_0^\infty e^{-\rho\tau} \tau^{\Re a+N_0-1} \left|\frac{\tau - \gamma_1}{\gamma_1 - \rho_1 \varepsilon}\right|^{N_1} \left|\frac{\tau - \gamma_2}{\gamma_2 - \rho_2 \varepsilon}\right|^{N_2} d\tau. \qquad (3.19)$$

Now the integrand has 3 local maxima, at $\tau = s_1, s_2, s_3$, with $s_1 < s_2 < s_3$. The local maxima at s_1 and s_2 are decreasing functions of N_2, and the local maximum at s_3 is decreasing on the interval $s_3 \in (\gamma_2, 2\gamma_2 - \rho_2 \varepsilon)$, and increasing on the interval $s_3 \in (2\gamma_2 - \rho_2 \varepsilon, \infty)$. For small and medium N_2, the global maximum is at s_1, but for a certain large N_2 ($N_2 \sim (1.85..)\rho$ with ε small) the global maximum is minimal and it is attained at s_1 and s_3. With the add of the computer we were able to show that $|S_{N_2}(z)| = \mathcal{O}(\rho^{\Re b-\frac{3}{2}} e^{-(1.94..)\rho})$, as $\rho \to \infty$.

An expansion of $g_3(\tau, z)$ at $\tau = s_1$, will give an estimate of $|S_{N_3}(z)|$, which is of the form (3.19), with an extra factor $\left|(\tau - s_1)(s_1 - \rho_3 \varepsilon)^{-1}\right|^{N_3}$ in the integrand. When N_3/ρ is bounded away from 0, for large ρ, the extra factor is too large in a neighbourhood of s_3, and the estimate of $|S_{N_3}(z)|$ explodes. Thus, for obtaining the next level of the hyperasymptotic expansion, we expand $g_3(\tau, z)$ at $\tau = \gamma_3 = s_3$. The method for obtaining this level is the same as the method for obtaining the previous levels, but much more terms are needed for a large exponentially improvement. This follows from the estimate

$$|S_{N_3}(z)| \le C_4 \rho^{\Re b-1} \int_0^\infty e^{-\rho\tau} \tau^{\Re a+N_0-1} \left|\frac{\tau - \gamma_1}{\gamma_1 - \rho_1 \varepsilon}\right|^{N_1} \left|\frac{\tau - \gamma_2}{\gamma_2 - \rho_2 \varepsilon}\right|^{N_2} \left|\frac{\tau - \gamma_3}{\gamma_3 - \rho_3 \varepsilon}\right|^{N_3} d\tau, \qquad (3.20)$$

and now the global maximum of this integrand is greater than the previous maximum times $\left[(\gamma_3 - s_1)(\gamma_3 - \rho_3 \varepsilon)^{-1}\right]^{N_3}$. But the method still works and with this method we can obtain nth-level hyperasymptotic expansions.

NOTE. Olver's expansion (1.5) is obtained by substituting

$$U(1 - b, c, \rho\tau) = (\rho\tau)^{b-1} \sum_{m=0}^{M-1} (-1)^m \frac{(1-a)_m(1-b)_m}{m!(\rho\tau)^m} + R'_M(\rho\tau), \qquad (3.21)$$

in (3.1). The remainder $R_M(z)$ in (1.5) is

$$R_M(z) = \frac{(-1)^{N+M} e^{-i\theta(a+N-1)}}{\Gamma(a)\Gamma(b)\Gamma(1-a)\Gamma(1-b)} \int_0^\infty \frac{e^{-\rho\tau} \tau^{a+N-1}}{\tau + e^{i\theta}} \int_0^\infty \frac{U(a, c, \rho\tau s) e^{-\rho\tau s} s^{M-b}}{s + 1} ds \, d\tau, \qquad (3.22)$$

where $M < N$. This substituting of the asymptotic expansion of the U-function in the remainder can be repeated, and a hyperasymptotic expansion is obtained. This method is described in more detail in BERRY & HOWLS [2]. And in that paper it is proven that at each level the optimal number of terms

(the N_n) is smaller than the optimal number of terms at the previous level. The final error in [2] is $R_{N_\infty}(z) = \mathcal{O}(e^{-(2.39..)|z|})$, as $|z| \to \infty$. In this method the universal hyperterminants are 'horrible' multiple integrands, while the coefficients are very simple. In our method the universal components are polynomials (see [5]), easily calculated by recursion, but the coefficients, the b_{nm}, are more difficult to obtain.

At each level we obtain $|S_{N_n}(z)| = \mathcal{O}(|z|^{\Re b - \frac{3}{2}} e^{-\lambda_n |z|})$ as $|z| \to \infty$, where $\lambda_0 < \lambda_1 < \lambda_2 < ...,$ and $\delta \leq \arg(z) \leq 2\pi - \delta$. For the first 5 of the λ_n we have $\lambda_0 \sim 1$, $\lambda_1 \sim 1.30$, $\lambda_2 \sim 1.94$, $\lambda_3 \sim 2.38$ and $\lambda_4 \sim 2.46$, as $|z| \to \infty$. Thus in neighbourhoods of the Stokes lines our λ_4 exceeds already Berry and Howls' limiting value of 2.39.

4. Discussion

In §2 and in [5], it has been shown how a hyperasymptotic expansion can be obtained for $U(a, c, z)$. This expansion is valid far from the Stokes lines. This method can be generalized to any function $F(z)$ expressible as a Laplace transform:

$$F(z) = \int_0^\infty t^{a-1} e^{-zt} f(t) dt, \tag{4.1}$$

where $\Re a > 0$, and $f(t)$ is an analytic function in a sector containing $[0, \infty)$, with $|f(t)| \leq Ce^{\sigma t}$, $t \in [0, \infty)$, C, $\sigma > 0$.

In §3 it is shown how a hyperasymptotic expansion can be obtained for $U(a, c, z)$, which is valid near the Stokes line. This method can be generalized to any function $G(z)$ expressible as a Stieltjes transform of the form:

$$G(z) = \int_0^\infty t^{a-1} e^{-zt} \frac{f(t)}{t+1} dt, \tag{4.2}$$

with the same conditions on f and a.

Presently, there are two main methods in the literature, for obtaining hyperasymptotic expansions of integrals. In the first method, by Berry and Howls, the 'difficult part' of the integral representing the remainder (the function $\tau \mapsto U(1 - b, c, \rho\tau)$ in (3.1)), is expanded in a simple asymptotic expansion. The next level of the hyperasymptotic expansion is obtained by substituting the simple asymptotic expansion into the integral representing the remainder. This method is used in [2], JONES [4], [8], and in PARIS [9] & [10]. In [9] Paris starts with a Mellin-Barnes integral representation of $U(a, c, z)$, and he obtains expansion (1.5). The advantage of the first method is that it remains valid across the Stokes lines.

The second method was anticipated in BOYD [3]. Boyd indicates how hyperasymptotic expansions of Stieltjes transforms can be obtained. This method is worked out in more detail in [5]. In this paper we extended this method to neighbourhoods of Stokes lines. We start by expanding the 'large parameter independent part' of the integral representing the remainder (the function $f_{n+1}(\tau)$ in (2.9)), in a Taylor series at the main saddle point. The next level of the hyperasymptotic expansion is obtained by substituting the Taylor series into the integral representing the remainder. With this method we can achieve greater accuracy.

5. Numerical illustrations

We illustrate the method of the previous sections with the K-Bessel function. We choose $a = b = \frac{1}{2}$ and have

$$U(\tfrac{1}{2}, 1, z) = \pi^{-\frac{1}{2}} e^{\frac{1}{2}z} K_0(\tfrac{1}{2}z). \tag{6.1}$$

In the following illustrations we calculated the UF-contribution of (3.2), that is the second term on the right-hand side of (3.2), 'exact'. The zeroth-level is (2.5). The UF-level is the zeroth-level plus the UF-contribution of (3.2). The first level is the previous plus (3.13). And so on.

Table 5.1. *Hyperasymptotic approximations to $U(\frac{1}{2}, 1, z)$ for $z = 10e^{\pi i}$ and $\varepsilon = \frac{1}{100}$*

Level	approximation	exact−approximation	
zeroth	$-0.32531887427684i$	$1.4\times10^{-5}+1.2\times10^{-6}i$	$N_0 = 11$
UF	$0.00001403163469 - 0.32531593185480i$	$-1.7\times10^{-6}i$	
first	$0.00001403163469 - 0.32531763374864i$	$1.4\times10^{-8}i$	$N_1 = 5$
second	$0.00001403163469 - 0.32531762008956i$	$-1.8\times10^{-11}i$	$N_2 = 19$
third	$0.00001403163469 - 0.32531762010763i$	$-9.2\times10^{-14}i$	$N_3 = 67$
exact	$0.00001403163469 - 0.32531762010773i$	0	

Notice that on the Stokes line $\arg(z) = \pi$ the UF-contribution of (3.2) takes into account the real part of the U-function. This UF-contribution is of the same order as $R_{N_0}(10e^{\pi i})$, and it is due to this UF-contribution that the hyperasymptotic expansion in §2 is limited to $|\arg(z)| < \pi$.

Table 5.2. *Hyperasymptotic approximations to $U(\frac{1}{2}, 1, z)$ for $z = 10e^{\frac{1}{2}\pi i}$ and $\varepsilon = \frac{1}{100}$*

Level	approximation	exact−approximation	
zeroth	$0.22848404036066 - 0.21753754623525i$	$6.9\times10^{-9}\ -2.4\times10^{-6}\ i$	$N_0 = 11$
UF	$0.22848580197590 - 0.21753930930013i$	$-1.7\times10^{-6}\ -6.4\times10^{-7}\ i$	
first	$0.22848403527347 - 0.21753996260505i$	$1.2\times10^{-8}\ +8.9\times10^{-9}\ i$	$N_1 = 5$
second	$0.22848404727888 - 0.21753995365582i$	$-1.4\times10^{-11}-1.3\times10^{-11}i$	$N_2 = 19$
third	$0.22848404726479 - 0.21753995366845i$	$-6.9\times10^{-14}-6.5\times10^{-14}i$	$N_3 = 67$
exact	$0.22848404726472 - 0.21753995366852i$	0	

In these numerical illustrations the third level approximation is better than the final approximation that can be obtained with the method described at the end of §3.

Appendix

For obtaining estimates for $g_1(t, z)$ on $\Re t > 0$ we write $z = \rho e^{i\theta}$ and $t = se^{i\nu}$, where $0 < \theta < 2\pi$ and $-\frac{1}{2}\pi < \nu < \frac{1}{2}\pi$. We use (1.1) in (3.3) and obtain

$$g_1(t, z) = \frac{\rho^{b-1}}{\Gamma(1-b)} \int_0^\infty e^{-y} y^{-b} \left[\frac{(1 + \frac{y}{\rho s e^{i\nu}})^{a-1}(se^{i\nu})^{b-1} - (1 + \frac{y}{\rho e^{i(\theta-\pi)}})^{a-1}(e^{i(\theta-\pi)})^{b-1}}{se^{i\nu} - e^{i(\theta-\pi)}} \right] dy, \quad (A.1)$$

where the function between brackets is bounded for $s > \varepsilon$. Thus

$$|g_1(t, z)| \leq C_1^+ \rho^{\Re b - 1}, \qquad \Re t > 0 \text{ and } |t| > \varepsilon. \quad (A.2)$$

For $|t| < \varepsilon$ we first estimate $U(1 - b, c, \rho t)$ with

$$|U(1 - b, c, \rho t)| \leq \frac{1}{|\Gamma(1-b)|} \int_0^\infty e^{-\rho s x} x^{-\Re b}(1 + x)^{\Re a - 1} dx. \quad (A.3)$$

In the case that $\Re a \leq 1$ we can estimate the right-hand side of (A.3) with

$$\frac{1}{|\Gamma(1-b)|} \int_0^\infty e^{-\rho s x} x^{-\Re b}(1 + x)^{\Re a - 1} dx \leq \frac{1}{|\Gamma(1-b)|} \int_0^\infty e^{-\rho s x} x^{-\Re b} dx$$
$$\leq \frac{\Gamma(1 - \Re b)}{|\Gamma(1-b)|}(\rho s)^{\Re b - 1}. \quad (A.4)$$

And in the case that $\Re a > 1$ we can estimate the right-hand side of $(A.3)$ with

$$\frac{1}{|\Gamma(1-b)|}\int_0^\infty e^{-\rho s x}x^{-\Re b}(1+x)^{\Re a-1}dx = \frac{\rho^{\Re b-1}}{|\Gamma(1-b)|}\int_0^\infty e^{-sy}y^{-\Re b}(1+\frac{y}{\rho})^{\Re a-1}dy$$

$$\leq \frac{\rho^{\Re b-1}}{|\Gamma(1-b)|}\int_0^\infty e^{-sy}y^{-\Re b}(1+y)^{\Re a-1}dy$$

$$= \frac{\Gamma(1-\Re b)}{|\Gamma(1-b)|}\rho^{\Re b-1}U(1-\Re b,\Re c,s) \qquad (A.5)$$

$$\leq_* C\rho^{\Re b-1}s^{-|\Re c|}.$$

In $*$ we used that $U(\alpha,\gamma,z) = o(z^{-|\Re\gamma|})$, as $z \to 0$. This rough estimate follows from ABRAMOWITZ & STEGUN [1,13.5.5...13.5.12]. Thus we obtain the estimate

$$|U(1-b,c,\rho t)| \leq C|t|^{\Re b-1}s^{-|\Re a|-|\Re b|}, \qquad (A.6)$$

and finally

$$|g_1(t,z)| \leq C_1^-\rho^{\Re b-1}|t|^{-|\Re a|-|\Re b|}, \qquad \Re t > 0 \text{ and } |t| \leq \varepsilon. \qquad (A.7)$$

Estimate (3.5) is the union of estimates $(A.2)$ and $(A.7)$.

For the proof of (3.7) we start with

$$g_1(t,z) = \frac{1}{\Gamma(1-b)}\int_0^\infty \frac{e^{-\rho tx}-e^{\rho e^{i\theta}x}}{t+e^{i\theta}}x^{-b}(1+x)^{a-1}dx$$

$$= -\frac{\rho}{\Gamma(1-b)}\int_0^\infty\int_0^1 e^{\rho xy(t+e^{i\theta})}dy\, e^{-\rho tx}x^{1-b}(1+x)^{a-1}dx \qquad (A.8)$$

$$= -\rho\frac{\Gamma(2-b)}{\Gamma(1-b)}\int_0^1 U\big(2-b,c+1,\rho[t(1-y)-ye^{i\theta}]\big)dy.$$

In the following derivation we use $\frac{d^n}{dy^n}U(\alpha,\gamma,y) = (-y)^n(\alpha)_n U(\alpha+n,\gamma+n,y)$

$$\frac{d^n}{dt^n}g_1(t,z) = -\rho(1-b)\int_0^1 \frac{d^n}{dt^n}U\big(2-b,c+1,\rho[t(1-y)-ye^{i\theta}]\big)dy$$

$$=(-\rho)^{n+1}(1-b)_{n+1}\int_0^1 (1-y)^n U\big(2-b+n,c+n+1,\rho[t(1-y)-ye^{i\theta}]\big)dy$$

$$=\frac{(-\rho)^n(1-b)_n}{t+e^{i\theta}}\bigg[U(1-b+n,c+n,\rho t) \qquad (A.9)$$

$$-n\int_0^1 (1-y)^{n-1}U\big(1-b+n,c+n,\rho[t(1-y)-ye^{i\theta}]\big)dy\bigg]$$

$$=\frac{1}{t+e^{i\theta}}\bigg[(-\rho)^n(1-b)_n U(1-b+n,c+n,\rho t)-n\frac{d^{n-1}}{dt^{n-1}}g_1(t,z)\bigg].$$

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