1992

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Department of Operations Research, Statistics, and System Theory Report BS-R9203 April

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Classification of Minimal Graphs of Given Face-width on the Torus

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Abstract. For any graph G embedded on the torus, the *face-width* $r(G)$ of G is the minimum number of intersections of G and C, where C ranges over all nonnullhomotopic closed curves on the torus. We call G *r-minimal* if $r(G) \geq r$ and $r(G') < r$ for each proper minor G' of G.
We classify the r-minimal graphs by means of certain symmetric integer polygons in the plane $\mathbb{R}^2$. Up to a certain natural equivalence, the number of r-minimal graphs on the torus is equal to $\frac{1}{6}r^3 + \frac{5}{6}r$ if r is odd and to $\frac{1}{6}r^3 + \frac{4}{3}r$ if r is even.

1991 Mathematics Subject Classification: 05C10, 11H06.
Key words and phrases: Graph, face-width, torus, integer, polygon.

1. Introduction

Let S be the torus. A closed curve on S is called *nontrivial* if it is not nullhomotopic. For any graph G embedded on S, the *face-width* (or *representativity*) $r(G)$ of G is the minimum of $|C \cap G|$ where C ranges over all nontrivial closed curves on S. (We identify a graph G embedded on S with its image on S.)

We call a graph G embedded on S *r-minimal* if $r(G) \geq r$, G has no isolated vertices, and $r(G') < r$ for each graph G' obtained from G by deleting or contracting any edge of G. (Clearly, contracting an edge of G, induces an embedding of the contracted graph on S.)

It is easy to see that, for any fixed r, r-minimality is maintained under the following operations:

(1)
(i) replacing G by $\phi(G)$, where $\phi : S \longrightarrow S$ is a homeomorphism;
(ii) replacing G by its surface dual;
(iii) ΔY-exchange.

Here ΔY-*exchange* means replacing a triangular face by a vertex connected to the three vertices of the triangle, and conversely.

The operations (1) imply an equivalence relation for r-minimal graphs (which we denote by $\sim$). In this paper we classify the equivalence classes. The classification is based on considering symmetric integer polygons related to graphs on the torus.

Report BS-R9203
ISSN 0924-0659
CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

A *polygon* in $\mathbb{R}^2$ is the convex hull of a finite nonempty set of points in $\mathbb{R}^2$. (We do not require full dimensionality.) A polygon P in $\mathbb{R}^2$ is *integer* if all its vertices have integer coordinates only. Call P *symmetric* (*about the origin*) if $P = -P$. The *height* height(P) of a polygon P is the minimum value of $\max\{c^T x | x \in P\}$, where the minimum ranges over all nonzero integer vectors $c \in \mathbb{Z}^2$. An integer polygon P is *r-minimal* if height$(P) \geq r$, while height$(P') < r$ for each integer polygon $P' \neq P$ contained in P. Two polygons P, P' are called *equivalent* (denoted by $P \sim P'$) if there exists a unimodular transformation $U : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $UP = P'$. (A *unimodular transformation* is a linear transformation U satisfying $U\mathbb{Z}^2 = \mathbb{Z}^2$. Equivalently, it is a linear transformation $x \to Ax$ where A is an integer matrix with determinant ± 1.)

We give, for each $r \geq 1$, a one-to-one relation between equivalence classes of r-minimal graphs on the torus and equivalence classes of symmetric r-minimal integer polygons in $\mathbb{R}^2$. For each fixed r there exist only finitely many such classes. We also give a description of the classes, yielding a formula for the number of equivalence classes of r-minimal graphs.

Remark 1. Scott Randby showed that for the projective plane, for each r, there is exactly one equivalence class of r-minimal graphs. We do not know any extension to nonorientable compact surfaces of higher genus. ∎

Remark 2. Integer polygons. A polygon P in $\mathbb{R}^2$ is fully determined by the function $f : \mathbb{Z}^2 \longrightarrow \mathbb{R}$ defined by:

$$f(c) := \max\{c^T x | x \in P\} \text{ for } c \in \mathbb{Z}^2. \tag{2}$$

It can be shown quite easily that P is integer if and only if f takes only integer values (this is a special case of a more general theorem of Hoffman [4]).

In fact it is quite standard to see that for any function $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$, there exists an integer polygon P satisfying (2), if and only if f satisfies the following ('norm-type') conditions:

$$\begin{aligned} &\text{(i) } f(c + c') \leq f(c) + f(c') \text{ for all } c, c' \in \mathbb{Z}^2; \\ &\text{(ii) } f(k \cdot c) = |k| \cdot f(c) \text{ for all } k \in \mathbb{Z}, c \in \mathbb{Z}^2 \end{aligned} \tag{3}$$

(cf. [8]). If f satisfies (3), the corresponding polygon is:

$$P := \{x \in \mathbb{R}^2 | c^T x \leq f(c) \text{ for each } c \in \mathbb{Z}^2\}. \tag{4}$$

∎

2. Integer polygons obtained from graphs on the torus

Since the torus S can be obtained from the plane $\mathbb{R}^2$ by identifying any two vectors y and y' whenever $y - y'$ is an integer vector, it is not surprising that the plane is of help in studying the torus. In particular, graphs on the torus can be studied with the help of polygons in $\mathbb{R}^2$ (cf. [8]).

We represent the torus S as the product $S^1 \times S^1$ of two copies of the unit circle S^1 in the complex plane $\mathbb{C}$. For $m, n \in \mathbb{Z}$, let $C_{m,n} : S^1 \longrightarrow S^1 \times S^1$ be the closed curve on S defined by:

(5) $$C_{m,n}(z) := (z^m, z^n)$$

for $z \in S^1$. As is well-known (cf. Stillwell [10]), the curves $C_{m,n}$ form a system of representatives for the homotopy classes of curves on the torus.

For each graph G on the torus, let $f_G : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ be defined by:

(6) $$f_G(m,n) := \min\{\mathrm{cr}(G,C) | C \sim C_{m,n}\}$$

for $(m,n) \in \mathbb{Z}^2$. Here $\mathrm{cr}(G,C)$ denotes the number of intersections of G and C, counting multiplicities. By $\sim$ we denote free homotopy of closed curves on the torus. (That is, $C \sim C'$ if there exists a homotopic shift of C over the torus, bringing C to C'. We do not fix a 'base point'.)

It is not difficult to show that the function f_G satisfies (3). (The inequality in (i) follows from the fact that if $C \sim C_{m,n}$ and $C' \sim C_{m',n'}$ and (m,n) and (m',n') are linearly independent, then C and C' have a crossing. We can concatenate C and C' at this crossing so as to obtain a closed curve $C'' \sim C_{m+m',n+n'}$ with $\mathrm{cr}(G,C'') = \mathrm{cr}(G,C) + \mathrm{cr}(G,C')$.)

Hence, the set $P(G)$ defined by:

(7) $$P(G) := \{x \in \mathbb{R}^2 | c^T x \leq f_G(c) \text{ for each } c \in \mathbb{Z}^2\}$$

is an integer polygon.

Note that $P(G)$ is full-dimensional (i.e., not a line segment and not a point) if and only if G is cellularly embedded. (A graph G is *cellularly embedded* if each face is an open disk; equivalently, if $r(G) > 0$.)

The operation $G \to P(G)$ maintains equivalence:

Theorem 1. *If graphs G and G' (embedded on the torus) are equivalent, then $P(G)$ and $P(G')$ are equivalent.*

Proof. The polygon $P(G)$ is trivially maintained under the operations (1)(ii) and (iii) (as f_G is trivially maintained under these operations). Consider now a homeomorphism $\phi : S \longrightarrow S$. Let $m, n, m', n' \in \mathbb{Z}$ be so that $\phi \circ C_{1,0} \sim C_{m,n}$ and $\phi \circ C_{0,1} \sim C_{m',n'}$. Then $(1,0) \longmapsto (m,n), (0,1) \longmapsto (m',n')$ defines a unimodular transformation $U : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, so that for each integer vector $c \in \mathbb{Z}^2$ one has: $\phi \circ C_c \sim C_{Uc}$. Then for each $c \in \mathbb{Z}^2$: $f_G(c) = f_{\phi(G)}(Uc)$. So $P(\phi(G))$ arises by a unimodular transformation from $P(G)$, i.e. $P(G)$ and $P(\phi(G))$ are equivalent. ∎

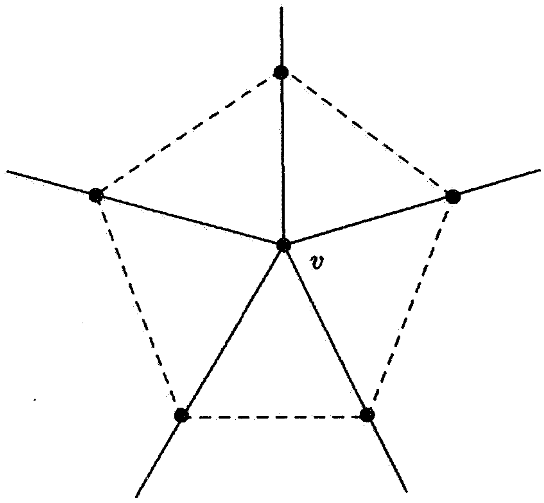
3. Kernels

In studying r-minimal graphs, the concept of 'kernel' introduced in [7] is helpful. A graph G embedded on the torus S is called a *kernel*, if $f_{G'} \neq f_G$ for each graph G' obtained from G by deleting an edge or contracting an edge that is not a loop.

It was shown in [7] that

(8) a cellularly embedded graph G on the torus is a kernel if and only if the medial graph $H(G)$ of G is the union of a minimally crossing system of simple nontrivial closed curves $D_1, \ldots, D_k$.

Here we use the following terminology. For any graph G embedded on the torus, 'the' *medial graph* $H(G)$ of G is 'the' 4-regular graph obtained by putting a vertex on each edge of G, and by joining, for each vertex v of G, the vertices on the edges incident with v, by edges so as to form a circuit, like the interrupted lines in:



If H is the medial graph of G, then G is called a *radial graph* of H. Each 4-regular graph H cellularly embedded on the torus S so that the faces can be bicolored, has a radial graph. Note that each two radial graphs of a cellularly embedded graph can be obtained from each other by homotopic shifts and taking surface duals.

Closed curves $D_1, \ldots, D_k$ form a *minimally crossing* system of closed curves if for all $i \neq i'$, D_i and $D_{i'}$ have a minimal number of intersections among all closed curves D and D' freely homotopic to D_i and $D_{i'}$ respectively. (So each intersection of D_i and $D_{i'}$ is a crossing (and not a touching).)

It was also shown in [7] that if G is a kernel, and $D_1, \ldots, D_k$ are as in (8), then

(9) $$f_G(m,n) = \frac{1}{2}\sum_{i=1}^{k} \text{mincr}(C_{m,n}, D_i)$$

for each $(m,n)^T \in \mathbb{Z}^2$. Here mincr(C, D) denotes the minimum number of crossings (counting multiplicities) of C' and D', where C' and D' range over all closed curves freely homotopic to C and D respectively.

Clearly, f_G and $P(G)$ are maintained under the following operations on graphs embedded on the torus:

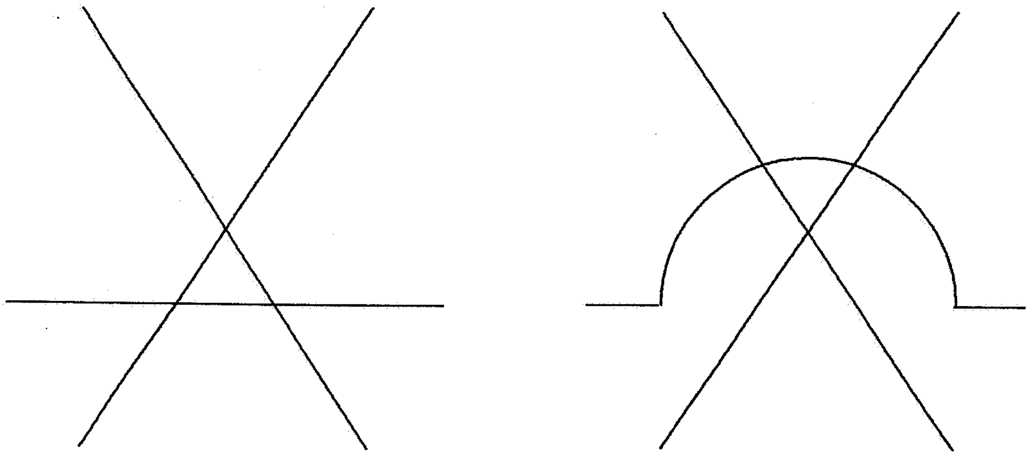
(10)
(i) homotopic shifts of the graph over the torus;
(ii) taking the surface dual;
(iii) ΔY-exchange.

(We take the surface dual only if the graph is cellularly embedded.)

It was shown in [7] that

(11) if G and G' are kernels with $P(G) = P(G')$, then G' can be obtained from G by the operations (10).

(The reason is that the closed curve systems making up the medial graphs $H(G)$ and $H(G')$ can be moved to each other, using only '$\Delta\nabla$-exchange':



This induces ΔY-exchanges bringing G to G' (up to duality).)

This is a special case of a more general result for compact orientable surfaces. On the other hand, for the torus, a stronger statement can be proved. Let G and G' be graphs embedded on the torus. We call a graph G' a ΔY-*minor* of G if G' arises from some minor of G by the operations (10) (maintaining the embedding throughout).

Theorem 2. *Let G and G' be graphs embedded on the torus, where G is a kernel. Then G is a ΔY-minor of G' if and only if $P(G) \subseteq P(G')$.*

Proof. Necessity of the condition is easy, since $P(G)$ is maintained under the operations (10), while $P(G) \subseteq P(G')$ if G is a minor of G'.

To see sufficiency, assume $P(G) \subseteq P(G')$. If G is not cellularly embedded, then $P(G)$ is not full-dimensional. In that case, G consists of just a number k of pairwise disjoint simple closed curves, each freely homotopic to one simple closed curve, C say. Then G is a ΔY-minor of G', if and only if G' contains k pairwise disjoint simple closed curves each freely homotopic to C. In [5] (cf. [9], [1]) it was shown that this last holds if and only if for each closed curve D one has $\mathrm{cr}(G', D) \geq k \cdot \mathrm{mincr}(C, D)$. But this last is equivalent to $f_{G'} \geq f_G$, i.e., to $P(G') \supseteq P(G)$.

So we may assume that G' is cellularly embedded (in particular, connected). Let $H(G)$ and $H(G')$ be the medial graphs of G and G' respectively. Since G is a kernel, by (8) $H(G)$ is the union of a system of simple closed curves $D_1, \ldots, D_k$, where each two D_i and D_j are minimally crossing.

Now $P(G) \subseteq P(G')$ implies that $f_G(c) \leq f_{G'}(c)$ for each integer vector c, and hence

(12) $$\mathrm{mincr}(H(G), C) \leq \mathrm{mincr}(H(G'), C)$$

for each closed curve C. Here $\text{mincr}(H, C)$ denotes the minimum number of crossings of H and C', where C' ranges over all closed curves freely homotopic to C so that C' does not traverse vertices of H.

Combining (9) and (12) gives

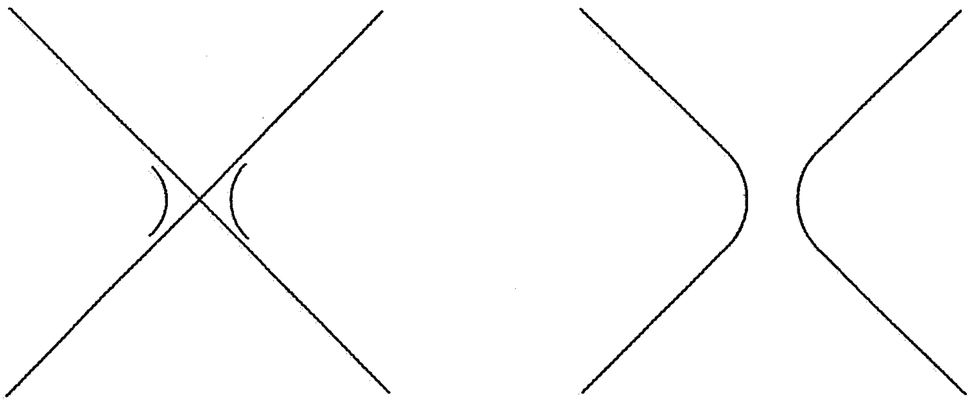
$$\text{(13)} \qquad \text{mincr}(H(G'), C) \geq \sum_{i=1}^{k} \text{mincr}(C, D_i),$$

for each closed curve C on S.

It is shown in [2] that (13) is equivalent to the fact that $H(G')$ contains closed curves $D'_1, \ldots, D'_k$, so that no edge of G is traversed more than once, and so that $D'_i \sim D_i$ for $i = 1, \ldots, k$. We may assume (cf. [6]) that $D'_1, \ldots, D'_k$ form a minimally crossing system of simple nontrivial closed curves.

In fact, we can assume that the system $D'_1, \ldots, D'_k$ traverses each edge of $H(G')$ exactly once. This can be seen as follows. We can decompose the edges of $H(G')$ not used by $D'_1, \ldots, D'_k$ into pairwise noncrossing, simple closed curves $D'_{k+1}, \ldots, D'_l$. Any trivial closed curve among $D'_{k+1}, \ldots, D'_l$ can be inserted in one of the other curves without increasing the total number of crossings. (This can be done since $H(G')$ is connected.) So we may assume that each of $D'_{k+1}, \ldots, D'_l$ is nontrivial. Since they are simple and pairwise noncrossing, they must be pairwise freely homotopic. Since $H(G')$ is a medial graph, each closed curve not traversing vertices of $H(G')$ has an even number of crossings with $H(G')$. Also, since $H(G)$ is a medial graph, each closed curve has an even number of crossings with $D_1, \ldots, D_k$, and hence with $D'_1, \ldots, D'_k$. So each closed curve has an even number of crossings with $D'_{k+1}, \ldots, D'_l$. So $l - k$ is even. Therefore, if $l > k$ we can insert D'_{k+1} and D'_{k+2} into one of the curves among $D'_1, \ldots, D'_k$, without changing its homotopy. Repeating this, we find $D'_1, \ldots, D'_k$ as required.

Now at any 'touching' of two D'_i and D'_j (possibly $i = j$), we can 'open' the graph as in:



Doing this at each touching we have transformed $H(G')$ to a graph H'' that is the union of a minimally crossing system of simple closed curves $D''_1, \ldots, D''_k$, with $D''_i \sim D'_i$ for $i = 1, \ldots, k$. Since openings of $H(G')$ correspond to deleting and contracting edges of G', H'' is the medial graph of some minor G'' of G'.

By (8), G'' is a kernel, and by (9), $f_{G''} = f_G$. So by (11), G'' arises by operations (10) from G. So G is ΔY-minor of G'. ∎

This theorem states that for each function $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ satisfying (3) there exists a unique minor-minimal graph among all graphs G with $f_G \geq f$ — unique up to the operations (10). This is more general than (11), which states that there exists a unique minor-minimal graph among all graphs G with $f_G = f$.

We give a corollary on 'toroidal grids'. Let $k \geq 3$. The product $C_k \times C_k$ of two copies of the k-circuit C_k is called the *toroidal k-grid.* Clearly, the toroidal k-grid can be embedded on the torus, in fact in a unique way, up to homeomorphisms of the torus and of the grid. Let H be the embedding of $C_k \times C_k$ on the torus, consisting of k disjoint circuits freely homotopic to $C_{1,0}$ crossed by k disjoint circuits freely homotopic to $C_{0,1}$.

By (11), H is a kernel. Since it is self-dual and does not allow ΔY-exchange (as all vertices have degree 4 and each face is bounded by 4 edges), Theorem 2 implies:

Corollary 2a. *Let G be a graph embedded on the torus. Then G contains H as an embedded minor, if and only if $P(G)$ contains $(k,0)$ and $(0,k)$.*

Proof. This follows directly from Theorem 2, since $P(H)$ is the convex hull of $\pm(k,0)$ and $\pm(0,k)$. ∎

This directly gives:

Corollary 2b. *Let G be a graph embedded on the torus, and let $k \geq 3$. Then G contains a toroidal k-grid as a minor, if and only if $\frac{1}{k}P(G)$ contains two linearly independent integer vectors.*

Proof. Directly from Corollary 2a. ∎

In [3] we derive from this result that every graph G embedded on the torus contains a toroidal $\lfloor \frac{2}{3} r(G) \rfloor$-grid minor.

4. Kernels obtained from symmetric integer polygons

Above we saw that each graph G on the torus gives a symmetric integer polygon $P(G)$ in $\mathbb{R}^2$. We now show conversely that for each symmetric integer polygon P in $\mathbb{R}^2$ there exists a graph G such that $P(G) = P$. So there exists a kernel G with $P(G) = P$, which should be unique by (11). We give a construction.

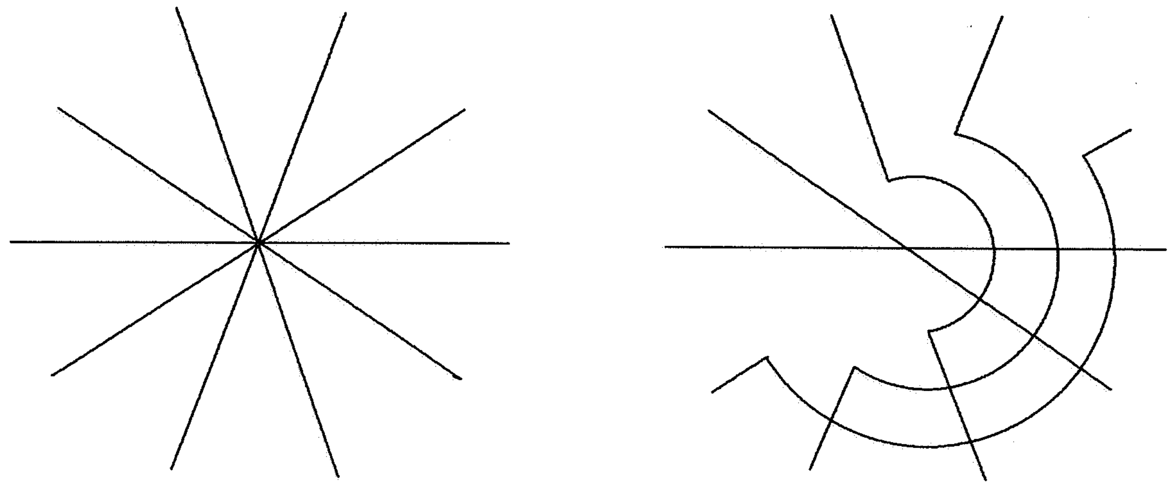
Let P be a symmetric integer polygon in $\mathbb{R}^2$. We first construct a graph Γ_P embedded in $\mathbb{R}^2$ as follows.

Let $v_1, \ldots, v_{2k}$ be the vertices of P, in cyclic order. (So $v_{j+k} = -v_j$ for $j = 1, \ldots, k$.) If P just consists of the origin, we take $k = 0$. Let Γ_P be the set of all points x in $\mathbb{R}^2$ for which $(v_{i+1} - v_i)^T x$ is an integer, for at least one $i \in \{1, \ldots, k\}$. Then Γ_P is a graph in $\mathbb{R}^2$, generally with an infinite set of vertices.

We can obtain the torus S from $\mathbb{R}^2$ by identifying any two vectors v, v' whenever $v - v'$ is an integer vector. Since Γ_P is invariant under translations by an integer vector, this identification makes Γ_P to a graph, denoted by H_P, embedded in the torus S.

The faces of H_P can be colored black and white so that adjacent faces have different colors. This follows from the fact that we can color the faces of Γ_P black and white so that adjacent faces have different colors and so that the coloring is invariant under translations by an integer vector. (Color $x \in \mathbb{R}^2 \setminus \Gamma_P$ black if $\sum_{i=1}^{k} \lfloor (v_{i+1} - v_i)^T x \rfloor$ is even, and white if this sum is odd. Here $\lfloor \ \rfloor$ denotes lower integer part.)

Let H'_P arise from H_P by 'rerouting' slightly the 'curves' traversing any vertex of H_P of degree larger than four, in such a way that each point of S is traversed by not more than two of the curves, not introducing any new crossings. E.g., a vertex of degree 10 can be changed as follows:



If P is full-dimensional, then H'_P is 4-regular and cellularly embedded. Then G_P is by definition some radial graph of H'_P.

If P is not full-dimensional, H_P consists of a number $2t$ of pairwise disjoint nontrivial closed curves on S, each freely homotopic to some curve C, say. In this case G_P will be a graph consisting of t pairwise disjoint nontrivial closed curves each freely homotopic to C. In fact, if P has vertices v_1 and v_2 with $v_2 = -v_1$ then we can take $C = C_{m,n}$, where (m,n) is any integer vector orthogonal to v_1 with m and n relative prime. If P only consists of the origin, then H_P and G_P are empty.

It can be derived from (8) that G_P indeed is a kernel (as H'_P consists of a system of closed curves that are minimally crossing). In fact:

Theorem 3. *G_P is a kernel with $P(G_P) = P$.*

Proof. We must show $P(G_P) = P$, or equivalently,

$$f_{G_P}(c) = \max\{c^T x \mid x \in P\} \qquad (14)$$

for all $c \in \mathbb{Z}^2$.

Choose $c \in \mathbb{Z}^2$. By symmetry we may assume that $\max\{c^T x | x \in P\}$ is attained at vertex v_1. So $c^T v_1 \geq c^T v_2 \geq \ldots \geq c^T v_{k+1}$.

Let B be any curve in $\mathbb{R}^2$ connecting vectors y and y' with $y' - y = c$, in such a way that B does not traverse any vertex of Γ_P, and has end points not in Γ_P. Then by the construction of Γ_P, B should cross at least

$$(15) \qquad \sum_{i=1}^{k} |c^T(v_{i+1} - v_i)| = \sum_{i=1}^{k} c^T(v_i - v_{i+1}) = c^T v_1 - c_{k+1}^T = 2c^T v_1$$

edges of Γ_P.

So the projection of B onto the torus S (under the quotient map) should cross at least $2c^T v_1$ edges of H_P, and hence it intersects G_P at least $c^T v_1$ times. As this minimum can be attained by taking for B a straight line segment, we know that $f_G(c) = c^T v_1$. ∎

5. Equivalence of polygons and graphs

The equivalence relation of graphs on the torus is strongly related to the equivalence relation of symmetric integer polygons:

Theorem 4. *Two symmetric integer polygons P and P' are equivalent, if and only if the graphs G_P and $G_{P'}$ are equivalent.*

Proof. Let P and P' be two equivalent symmetric integer polygons. Let U be a unimodular transformation bringing P to P'. Then it is not difficult to check that there exists a homeomorphism $\phi : S \longrightarrow S$ bringing G_P to $G_{P'}$.

Conversely, if G_P and $G_{P'}$ are equivalent, then by Theorem 1 $P(G_P)$ and $P(G_{P'})$ are equivalent. Since $P = P(G_P)$ and $P' = P(G_{P'})$ it follows that P and P' are equivalent. ∎

Theorem 4 implies that there exists a one-to-one relation between equivalence classes of kernels on the torus and equivalence classes of symmetric integer polygons in $\mathbb{R}^2$, given by:

(16) (i) $\langle G \rangle \longmapsto \langle P(G) \rangle$, where G is a kernel;

(ii) $\langle P \rangle \longmapsto \langle G_P \rangle$, where P is a symmetric integer polygon.

Here $\langle .. \rangle$ denotes the equivalence class of .. .

We finally come to the classification of equivalence classes of r-minimal graphs. Let $\mathcal{P}_r$ denote the collection of all symmetric r-minimal integer polygons.

Theorem 5. *For each $P \in \mathcal{P}_r$ the graph G_P is r-minimal. Each r-minimal graph is equivalent to G_P for some $P \in \mathcal{P}_r$.*

Proof. Let $P \in \mathcal{P}_r$. Then $r(G_P) = \text{height}(P) = r$. For each proper minor G' of G_P one has that $P(G') \neq P$, implying that $r(G') = \text{height}(P(G')) < \text{height}(P) = r$. So G_P is r-minimal.

Let G be an r-minimal graph. Then $P(G)$ is r-minimal. For suppose not. Then $P(G)$ contains a symmetric integer polygon $P' \neq P(G)$ with $\text{height}(P') = r$. By Theorem 2, G contains a minor G' that arises by the operations (10) from $G_{P'}$. Since $P(G') = P(G_{P'}) =$

$P' \neq P(G)$, G' is a proper minor of G. However, $r(G') = \text{height}(P(G')) = \text{height}(P') = r$, contradicting the r-minimality of G. ∎

6. r-minimal integer polygons

Fix $r \geq 1$. We give a construction of symmetric r-minimal integer polygons. Each of them is either a quadrangle or a hexagon. For any choice of integers $0 \leq \alpha < r$ and $0 \leq \beta < r$, let $Q_{\alpha,\beta}$ be the convex hull of the four points $\pm(r,\alpha), \pm(-\beta,r)$. For any choice of integers $0 < \alpha < r$, $0 < \beta < r$ and $0 < \gamma < r$, let $H_{\alpha,\beta,\gamma}$ be the convex hull of the six points $\pm(r,\alpha), \pm(r-\beta,r), \pm(-\gamma,r-\gamma)$.

Theorem 6. *Each $Q_{\alpha,\beta}$ belongs to $\mathcal{P}_r$.*

Proof. To show that $\text{height}(Q_{\alpha,\beta}) \geq r$, let (c,d) be a nonzero integer vector. We show $\max\{(c,d)\cdot(x,y)|(x,y) \in Q_{\alpha,\beta}\} \geq r$. (Here $\cdot$ denotes inner product.) We may assume that the last nonzero in (c,d) is positive. If $c \geq 1$ then $d \geq 0$, implying $(c,d)\cdot(r,\alpha) \geq cr \geq r$. If $c \leq 0$ then $d \geq 1$, implying $(c,d)\cdot(-\beta,r) \geq dr \geq r$.

Since for $(c,d) := (1,0)$ and $(c,d) := (0,1)$, the maximum is uniquely attained at (r,α) and $(-\beta,r)$ respectively, $Q_{\alpha,\beta}$ is r-minimal. ∎

Theorem 7. *Each $H_{\alpha,\beta,\gamma}$ belongs to $\mathcal{P}_r$.*

Proof. To show that $\text{height}(H_{\alpha,\beta,\gamma}) \geq r$, let (c,d) be a nonzero integer vector. We show $\max\{(c,d)\cdot(x,y)|(x,y) \in H_{\alpha,\beta,\gamma}\} \geq r$. We may assume that the last nonzero in $(c,d,c+d)$ is positive. If $c+d \geq 1$ and $c \geq 1$ then $(c,d)\cdot(r,\alpha) \geq (c,1-c)\cdot(r,\alpha) = c(r-\alpha)+\alpha \geq r$. If $c+d \geq 1$ and $d \geq 1$ then $(c,d)\cdot(r-\beta,r) \geq (1-d,d)\cdot(r-\beta,r) = d\beta + r - \beta \geq r$. If $c+d = 0$ then $c \leq -1, d \geq 1$, implying $(c,d)\cdot(-\gamma,r-\gamma) = dr \geq r$.

Since for $(c,d) := (1,0)$, $(c,d) := (0,1)$, and $(c,d) := (-1,1)$ the maximum is uniquely attained at (r,α), $(r-\beta,r)$, and $(-\gamma,r-\gamma)$, respectively, $H_{\alpha,\beta,\gamma}$ is r-minimal. ∎

Theorem 8. *Each polygon in $\mathcal{P}_r$ is equivalent to at least one of the $Q_{\alpha,\beta}, H_{\alpha,\beta,\gamma}$.*

Proof. I. Let $P \in \mathcal{P}_r$. We first show that for each vertex v of P

(17) there exists a nonzero integer vector c such that $c^T v = r$ and $c^T x \leq r-1$ for each integer vector $x \neq v$ in P.

Indeed, by the r-minimality of P there exists a nonzero integer vector d such that $d^T v = r' \geq r$ and $d^T x \leq r-1$ for each integer vector $x \neq v$ in P. If $r' = r$ we are done, so suppose $r' > r$. We may assume that the components of d are relatively prime (otherwise we could divide d by the g.c.d. of the components), and therefore we may assume that $d = (1,0)^T$. So $v = (r',\lambda)^T$ for some λ.

Now there exists an $i \in \{1,\ldots,r'-r\}$ such that $i\lambda - \lfloor\frac{i\lambda}{r'}\rfloor r' \leq r$. Otherwise there would exist $i < j$ in $\{1,\ldots,r'-r\}$ such that $i\lambda - \lfloor\frac{i\lambda}{r'}\rfloor r' = j\lambda - \lfloor\frac{j\lambda}{r'}\rfloor r'$ (since each $i\lambda - \lfloor\frac{i\lambda}{r'}\rfloor r'$ would be in $\{r+1,\ldots,r'-1\}$). Then

$$(18) \qquad x := \begin{pmatrix} r'-j+i \\ \lambda - \lfloor\frac{j\lambda}{r'}\rfloor + \lfloor\frac{i\lambda}{r'}\rfloor \end{pmatrix} = \frac{r'-j+i}{r'}\begin{pmatrix} r' \\ \lambda \end{pmatrix} = \frac{r'-j+i}{r'}v$$

would be an integer vector $x \neq v$ in P with $d^T x = r' - j + i \geq r$, contradicting our assumption.

Let

$$(19) \qquad c := \begin{pmatrix} -\lfloor \frac{i\lambda}{r'} \rfloor \\ i \end{pmatrix}.$$

So $c^T v \leq r$. We show that $c^T x \leq r - 1$ for each integer vector $x \neq v$ in P, thus proving (17).

Suppose $x \neq v$ is an integer vector in P with $c^T x \geq r$. Let x' be the point on the line segment connecting x and the origin such that $c^T x' = c^T v$. Now consider the point

$$(20) \qquad u := v - \begin{pmatrix} i \\ -\lfloor \frac{i\lambda}{r'} \rfloor \end{pmatrix}.$$

Then $c^T u = c^T v$. So u, v and x' are on a line. Since $d^T v = r' > r$, $r \leq d^T u = r' - i < r'$, and $d^T x' \leq d^T x \leq r - 1$, u is on the line segment connecting v and x'. This implies that u belongs to P, contradicting the fact that u is an integer vector with $u \neq v$ and $d^T u \geq r$.

II. We next show the theorem. Let $v_1, \ldots, v_{2k}$ be the vertices of P, in counterclockwise order. (So $v_{j+k} = -v_j$ for $j = 1, \ldots, k$.) Write $v_j = (v_j', v_j'')^T$ for $j = 1, \ldots, 2k$.

By (17), for each $j = 1, \ldots, 2k$, there exists an integer vector c_j satisfying $c_j^T v_j = r$ and $c_j^T x < r$ for all $x \neq v_j$ in P. We may assume that $c_{j+k} = -c_j$.

Then for each two distinct j, j' from $\{1, \ldots, k\}$ one has that c_j and $c_{j'}$ form a basis for $\mathbb{Z}^2$. Otherwise, the triangle with vertices $c_j, c_{j'}, 0$ would contain a nonzero integer vector d with $d \neq c_j$ and $d \neq c_{j'}$. Then we would have $d^T x < r$ for each vector x in P. This contradicts the fact that height$(P) \geq r$.

We may assume that $c_1 = (1, 0)^T$ and $c_2 = (0, 1)^T$. Moreover, $k = 2$ or $k = 3$, since each c_j with $3 \leq j \leq k$ should be equal to $(\pm 1, \pm 1)^T$.

If $k = 2$, then $v_1' = r, |v_1''| < r$ and $v_2'' = r, |v_2'| < r$. Moreover, $v_2' v_1'' \leq 0$. For suppose $v_2' v_1'' > 0$. If $v_2' < 0$ and $v_1'' < 0$ then $\max\{x' + x'' | x \in P\} < r$, as this maximum is attained at v_1 or at v_2, while $v_1' = v_2'' = r$. If $v_2' > 0$ and $v_1'' > 0$ then $-v_2' < 0$ and $-v_1'' < 0$, and hence $\max\{-x' + x'' | x \in P\} < r$, as this maximum is attained at v_2 or at v_3, while $v_2'' = -v_3' = r$.

This shows $v_2' v_1'' \leq 0$. By symmetry we may assume $v_1'' \geq 0$ and $v_2' \leq 0$. So $P = Q_{\alpha,\beta}$ for $\alpha := v_1''$ and $\beta := -v_2'$.

If $k = 3$, then we may assume that $c_3 = (-1, 1)^T$. Then $v_1' = r, |v_1''| < r, |v_1'' - v_1'| < r$, $v_2'' = r, |v_2'| < r, |v_2'' - v_2'| < r$, and $v_3'' - v_3' = r, |v_3'| < r, |v_3''| < r$. So $P = H_{\alpha,\beta,\gamma}$ for $\alpha := v_1'', \beta := r - v_2', \gamma := -v_3'$. ∎

Remark 3. One can show that for any polygon P in $\mathcal{P}_r$ the number of edges of the kernel G_P is equal to the area of P. This follows from the fact that the number of vertices of the graph H_P' is equal to the number of crossings of the curves making up H_P (counting multiplicities). Let $v_1, \ldots, v_{2k}$ be the vertices of P in cyclic order. Let L_i be the set of points x in $\mathbb{R}^2$ for which $(v_{i+1} - v_i)^T x$ is an integer. Then L_i is the union of a collection of parallel lines. Let R be a unit square so that no two of the L_i cross each other on the boundary of R.

Then the number of crossings of L_i and L_{i+1} on R is equal to $|\det[(v_{i+2}-v_{i+1})\ (v_{i+1}-v_i)]|$, as one easily checks.

Now since $k \leq 3$, one has that the area of P is equal to $\sum_{i=1}^{k} |\det[(v_{i+2}-v_{i+1})\ (v_{i+1}-v_i)]|$. This gives the required equality.

Note that the area of P is equal to the number of integer vectors contained in the interior in P, plus half of the number of integer vectors on the boundary of P, minus 1. ∎

7. Counting equivalence classes

As above, two polygons Q, Q' are called *equivalent* (denoted by $Q \sim Q'$) if there exists a unimodular transformation $U : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $UQ = Q'$. It follows directly from Theorem 8 that:

Theorem 9. *For each r, the number of equivalence classes in $\mathcal{P}_r$ is finite.*

Proof. Directly from Theorem 8, since the number of $Q_{\alpha,\beta}$ and $H_{\alpha,\beta,\gamma}$ is finite. ∎

In fact, an explicit formula for the number of equivalence classes can be given. First we count those containing quadrangles. To this end we first note:

Theorem 10. *For any $0 \leq \alpha, \alpha' < r$ and $0 \leq \beta, \beta' < r$, $Q_{\alpha,\beta}$ is equivalent to $Q_{\alpha',\beta'}$ if and only if*

(i) $\{\alpha, \beta\} = \{\alpha', \beta'\}$, *or*

(ii) $\{\alpha, \beta\} = \{0, \gamma\}, \{\alpha', \beta'\} = \{0, r-\gamma\}$ *for some γ.*

Proof. Sufficiency is easy. (Note that $Q_{0,\gamma}$ goes to $Q_{0,r-\gamma}$ by the unimodular transformation $(x, y) \rightarrow (x+y, -y)$.)

To see necessity, let $Q_{\alpha,\beta}$ be equivalent to $Q_{\alpha',\beta'}$. Let U be a unimodular matrix bringing $Q_{\alpha,\beta}$ to $Q_{\alpha',\beta'}$. Since the unimodular transformation $(x, y) \rightarrow (-y, x)$ brings $Q_{\alpha,\beta}$ to $Q_{\beta,\alpha}$, and since the conditions (i) and (ii) are independent of the order of α, β, we may assume that U brings $(r, \alpha)^T$ to $(r, \alpha')^T$. Then U brings $(-\beta, r)$ either to $(-\beta', r)$ or to $(\beta', -r)$.

If U brings $(-\beta, r)$ to $(-\beta', r)$, then the matrix corresponding to U is:

$$(21)\qquad \begin{pmatrix} r & -\beta' \\ \alpha' & r \end{pmatrix} \begin{pmatrix} r & -\beta \\ \alpha & r \end{pmatrix}^{-1} = \frac{1}{r^2+\alpha\beta} \begin{pmatrix} r^2+\alpha\beta' & r(\beta-\beta') \\ r(\alpha'-\alpha) & r^2+\alpha'\beta \end{pmatrix}.$$

Since this is an integer matrix, $r^2 + \alpha\beta$ should divide both $r(\alpha' - \alpha)$ and $r(\beta - \beta')$. So $\alpha = \alpha'$ and $\beta = \beta'$.

If U brings $(-\beta, r)$ to $(\beta', -r)$, then the matrix corresponding to U is:

$$(22)\qquad \begin{pmatrix} r & \beta' \\ \alpha' & -r \end{pmatrix} \begin{pmatrix} r & -\beta \\ \alpha & r \end{pmatrix}^{-1} = \frac{1}{r^2+\alpha\beta} \begin{pmatrix} r^2-\alpha\beta' & r(\beta+\beta') \\ r(\alpha'+\alpha) & -r^2+\alpha'\beta \end{pmatrix}.$$

Again since this is an integer matrix, $r^2 + \alpha\beta$ should divide both $r(\alpha + \alpha')$ and $r(\beta + \beta')$. So $\alpha\beta = 0$, and both $\alpha + \alpha'$ and $\beta + \beta'$ belong to $\{0, r\}$. This yields (ii) or (if $\alpha = \beta = \alpha' =$

$\beta' = 0$) (i). ∎

This implies:

Theorem 11. *For each fixed r, the number of equivalence classes in $\mathcal{P}_r$ consisting of quadrangles is equal to $\frac{1}{2}r^2 + \frac{1}{2}$ if r is odd and to $\frac{1}{2}r^2 + 1$ if r is even.*

Proof. From Theorem 10 it follows that the number of classes is equal to the number of sets $\{\alpha, \beta\}$ with $\alpha, \beta \in \{1, \ldots, r-1\}$ (possibly $\alpha = \beta$), plus the number of sets $\{0, \beta\}$ with $\beta \in \{0, \ldots, \lfloor r/2 \rfloor\}$. This number is equal to $r - 1 + \binom{r-1}{2} + 1 + \lfloor r/2 \rfloor$, which equals the values given in the theorem. ∎

Next we count classes containing hexagons. We first show:

Theorem 12. *For any $0 < \alpha, \alpha' < r$, $0 < \beta, \beta' < r$ and $0 < \gamma, \gamma' < r$, $H_{\alpha,\beta,\gamma}$ is equivalent to $H_{\alpha',\beta',\gamma'}$ if and only if $(\alpha', \beta', \gamma')$ is a cyclic permutation of (α, β, γ) or of $(r-\gamma, r-\beta, r-\alpha)$.*

Proof. Sufficiency can be seen easily. To see necessity, first observe that $c := (1,0)$ is the only integer vector for which $\max\{c^T x | x \in H_{\alpha,\beta,\gamma}\}$ equals r and is uniquely attained at (r, α). Similarly for $c := (0,1)$ with respect to $(r-\beta, r)$ and for $c := (-1,1)$ with respect to $(-\gamma, r-\gamma)$. So any unimodular transformation U that brings $H_{\alpha,\beta,\gamma}$ to $H_{\alpha',\beta',\gamma'}$ would bring the set $\{\pm(1,0), \pm(0,1), \pm(-1,1)\}$ to itself. It implies the condition given in the theorem. ∎

This implies:

Theorem 13. *For each fixed r, the number of classes in $\mathcal{P}_r$ consisting of hexagons is equal to $\frac{1}{6}r^3 - \frac{1}{2}r^2 + \frac{5}{6}r - \frac{1}{2}$ if r is odd and to $\frac{1}{6}r^3 - \frac{1}{2}r^2 + \frac{4}{3}r - 1$ if r is even.*

Proof. We use Theorem 12. If α, β, γ are distinct and $\{\alpha, \beta, \gamma\} \neq \{r-\alpha, r-\beta, r-\gamma\}$ then there exist six triples $(\alpha', \beta', \gamma')$ such that $Q_{\alpha',\beta',\gamma'} \sim Q_{\alpha,\beta,\gamma}$.

If α, β, γ are distinct and $\{\alpha, \beta, \gamma\} = \{r-\alpha, r-\beta, r-\gamma\}$ then there exist three triples $(\alpha', \beta', \gamma')$ such that $Q_{\alpha',\beta',\gamma'} \sim Q_{\alpha,\beta,\gamma}$. The number of such triples (α, β, γ) is equal to 0 if r is odd and to $3(r-2)$ if r is even.

If $|\{\alpha, \beta, \gamma\}| = 2$, then there exist six triples $(\alpha', \beta', \gamma')$ such that $Q_{\alpha',\beta',\gamma'} \sim Q_{\alpha,\beta,\gamma}$. The number of such (α, β, γ) is equal to $3(r-1)(r-2)$.

If $|\{\alpha, \beta, \gamma\}| = 1$ and $\alpha \neq r - \alpha$, then there exist two triples $(\alpha', \beta', \gamma')$ such that $Q_{\alpha',\beta',\gamma'} \sim Q_{\alpha,\beta,\gamma}$. The number of such (α, β, γ) is equal to $r-1$ if r is odd and to $r-2$ if r is even.

If $|\{\alpha, \beta, \gamma\}| = 1$ and $\alpha = r - \alpha$, then there exists one triple $(\alpha', \beta', \gamma')$ such that $Q_{\alpha',\beta',\gamma'} \sim Q_{\alpha,\beta,\gamma}$. The number of such (α, β, γ) is equal to 0 if r is odd, and to 1 if r is even.

This all gives that if r is odd, the number of equivalence classes is equal to

(23) $$\tfrac{1}{6}(r-1)(r-2)(r-3)+0+\tfrac{1}{6}(3(r-1)(r-2))+\tfrac{1}{2}(r-1)+0$$
$$=\tfrac{1}{6}r^3-\tfrac{1}{2}r^2+\tfrac{5}{6}r-\tfrac{1}{2}.$$

If r is even, it is equal to

(24) $$\tfrac{1}{6}((r-1)(r-2)(r-3)-3(r-2))+\tfrac{1}{3}(3(r-2))+\tfrac{1}{6}(3(r-1)(r-2))+\tfrac{1}{2}(r-2)+1$$
$$=\tfrac{1}{6}r^3-\tfrac{1}{2}r^2+\tfrac{4}{3}r-1.$$ ∎

Combining Theorems 11 and 13 gives:

Theorem 14. *The number of equivalence classes of $\mathcal{P}_r$, and hence of equivalence classes of r-minimal graphs on the torus, is equal to $\frac{1}{6}r^3+\frac{5}{6}r$ if r is odd and to $\frac{1}{6}r^3+\frac{4}{3}r$ if r is even.*

Proof. Directly from Theorems 11 and 13. ∎

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