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Graphs on the Torus and Geometry of Numbers

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Abstract. We show that if G is a graph embedded on the torus S and each nonnullhomotopic closed curve on S intersects G at least r times, then G contains at least $\lfloor\frac{3}{4}r\rfloor$ pairwise disjoint nonnullhomotopic circuits. The factor $\frac{3}{4}$ is best possible.
We prove this by showing the equivalence of this bound to a bound in the 2-dimensional geometry of numbers. To show the equivalence, we study *integer* norms, i.e., norms $||.||$ such that $||x||$ is an integer for each integer vector x. In particular, we show that each integer norm in two dimensions has associated with it a graph embedded on the torus, and conversely.

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1. Description of results

Call a closed curve on the torus *nontrivial* if it is not nullhomotopic. For any graph G embedded on the torus S, the *representativity* (or *face width*) $r(G)$ of G is the minimum of $|C \cap G|$, where C ranges over all nontrivial closed curves on S.

We will show the following theorem.

Theorem 1. (i) *Any graph G embedded on the torus contains at least $\lfloor\frac{3}{4}r(G)\rfloor$ pairwise disjoint nontrivial circuits.*

(ii) *The factor $\frac{3}{4}$ is best possible.*

Here $\lfloor x\rfloor$ denotes the lower integer part of x. A *circuit* is a simple closed curve contained in G.

Remark 1. The representativity of a graph embedded on a surface is recently a focus of attention in the study of minimal genus embeddings of graphs and of graph minors and disjoint paths (see [1], [8], [13], [14], [15], [23]).

In particular, Robertson and Seymour [13] showed:

(1) for any compact surface S and any graph H embedded on S there exists a number r so that any graph G embedded on S with representativity at least r contains H as a minor.

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In fact, Robertson and Seymour showed that for any graph H embedded on a compact surface S such that each vertex of H has degree at most three, there exists a number r with the following property: for each graph G embedded on S with representativity at least r there exists a homeomorphism $\phi : S \longrightarrow S$ such that $\phi[H] \subseteq G$. (This implies (1).)

One of the simplest special cases is that for each natural number k there exists a number $r(k)$ such that any graph G embedded on the torus S with representativity at least $r(k)$ contains k pairwise disjoint nontrivial circuits.

Theorem 1 asserts that we can take $r(k) = \lceil \frac{4}{3}k \rceil$, where $\frac{4}{3}$ is the best possible factor. ∎

We will show that Theorem 1 is equivalent to the following result in the geometry of numbers. For any symmetric convex body K (i.e., full-dimensional compact convex set K with $K = -K$) in $\mathbb{R}^n$, let

$$K^* := \{y \in \mathbb{R}^n \mid y^T x \leq 1 \text{ for each } x \in K\}. \quad (2)$$

As is well-known, K^* is again a symmetric convex body, and $(K^*)^* = K$.

Now Theorem 1 is equivalent to:

Theorem 2. (i) *For any symmetric convex body K in $\mathbb{R}^2$, there exists a nonzero integer vector in K or there exists a nonzero integer vector in $\frac{4}{3} \cdot K^*$.*

(ii) *The factor $\frac{4}{3}$ is best possible.*

Although we assume that this result belongs to the folklore of the geometry of numbers, we were not able to locate a proof in the literature. (The best result in this direction we found in the literature was by Mahler [11] who proved a factor $\sqrt{2}$ replacing $\frac{4}{3}$ in Theorem 2(i).) Therefore, for completeness we describe a proof of Theorem 2 in Section 3 of this paper.

Remark 2. As is well-known (cf. Cassels [3], Lekkerkerker [10]), there are several equivalent forms for Theorem 2(i). First, for any symmetric convex body K in $\mathbb{R}^2$ not containing a nonzero integer vector, there exists a nonzero integer vector c such that $c^T x \leq \frac{4}{3}$ for each vector x in K.

Second, define for each symmetric convex body K, $\lambda(K)$ to be the smallest value of λ for which $\lambda \cdot K$ contains a nonzero integer vector. Then for any symmetric convex body K in $\mathbb{R}^2$, $\lambda(K) \cdot \lambda(K^*) \leq \frac{4}{3}$.

Third, for any norm $||.||$ in $\mathbb{R}^n$, the *dual norm* $||.||_*$ is defined by

$$||y||_* := \sup_x \frac{y^T x}{||x||}, \quad (3)$$

where the supremum ranges over all nonzero vectors x in $\mathbb{R}^n$. Then, for any norm $||.||$ in $\mathbb{R}^2$, there exist nonzero integer vectors x and y such that $||x|| \cdot ||y||_* \leq \frac{4}{3}$. ∎

The equivalence of Theorems 1 and 2 is proved with the help of the following theorem given in [18] ([20] and [5] gave more direct proofs, and extensions to the directed case). Call two closed curves on the torus *freely homotopic* if one can be shifted continuously to the other over the torus. (So there is no fixed "base point".)

For any two closed curves C and D, let $\text{mincr}(C, D)$ denote the minimum number of crossings of C' and D' (counting multiplicities), where C' and D' range over all closed curves freely homotopic to C and D, respectively.

Theorem 3. *Let G be a graph embedded on the torus S, and let C be a simple closed curve on S. Then G contains k pairwise disjoint circuits each freely homotopic to C, if and only if each closed curve D on S has at least $k \cdot \text{mincr}(C, D)$ intersections with G (counting multiplicities).*

To describe the equivalence of Theorems 1 and 2, represent the torus as the product $S^1 \times S^1$, where S^1 is the unit circle in the complex plane. For each $(m, n) \in \mathbb{Z}^2$, let $C_{m,n} : S^1 \longrightarrow S^1 \times S^1$ be the closed curve on the torus defined by

$$C_{m,n}(z) := (z^m, z^n), \tag{4}$$

for $z \in S^1$.

Now, as is well-known (cf [22: Section 6.2.2]), the $C_{m,n}$ form a system of representatives for the free homotopy classes of closed curves on the torus. Moreover,

$$\text{mincr}(C_{m,n}, C_{m',n'}) = |mn' - m'n| \tag{5}$$

for all $m, n, m', n' \in \mathbb{Z}$.

Let G be a graph on the torus, such that each face of G is an open disk, i.e., such that $r(G) > 0$. (This clearly will be no restriction in Theorem 1.) Define for each $(m, n) \in \mathbb{Z}^2$, $f_G(m, n)$ as the minimum number of intersections of C' and G (counting multiplicities), where C' ranges over all closed curves homotopic to $C_{m,n}$.

It is not difficult to see that

$$\begin{aligned} &\text{(i) } f_G(m + m', n + n') \leq f_G(m, n) + f_G(m', n') \text{ and} \\ &\text{(ii) } f_G(km, kn) = |k| \cdot f_G(m, n) \end{aligned} \tag{6}$$

hold for all $(m, n), (m', n') \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$. (The inequality in (i) follows from the fact that if C is freely homotopic to $C_{m,n}$ and C' is freely homotopic to $C_{m',n'}$ and (m, n) and (m', n') are linearly independent, then C has a crossing with C'. We can concatenate C and C' at this crossing so as to obtain a closed curve C'' freely homotopic to $C_{m+m',n+n'}$ with $\text{cr}(G, C'') = \text{cr}(G, C) + \text{cr}(G, C')$, where cr denotes the number of crossings. The equality in (ii) is easy.)

Hence there exists a unique norm $||.||$ in $\mathbb{R}^2$ with the property that $||(m, n)|| = f_G(m, n)$ for each $(m, n) \in \mathbb{Z}^2$.

Having this we give one half of the proof of the equivalence of Theorems 1 and 2:

Proof of the implications Theorem 2(i) $\Longrightarrow$ Theorem 1(i) and Theorem 1(ii) $\Longrightarrow$ Theorem 2(ii). The representativity $r(G)$ of G is equal to the minimum of $||(m, n)||$ over all nonzero integer vectors (m, n). Hence, by Theorem 2(i) (third variant in Remark 2), there exists a nonzero integer vector (m', n') such that $||(m', n')||_* \leq \frac{4}{3} r(G)^{-1}$.

By definition (3) of $||.||_*$,

$$\frac{(m', n')^T (m, n)}{||(m, n)||} \leq \frac{4}{3r(G)} \tag{7}$$

for each nonzero vector (m,n) in $\mathbb{R}^2$. This implies

$$(8) \qquad \tfrac{3}{4}r(G)|m'm+n'n| \leq f_G(m,n)$$

for each integer vector (m,n). Therefore, as $|m'm+n'n| = \text{mincr}(C_{m,n}, C_{n',-m'})$, by Theorem 3 G contains $\lfloor \tfrac{3}{4}r(G) \rfloor$ pairwise disjoint circuits each freely homotopic to $C_{n',-m'}$. This shows Theorem 1(i).

This construction also shows that Theorem 1(ii) implies Theorem 2(ii), since any better factor in 2(i) would imply a better factor in 1(i). ∎

To see the other implications, we consider integer norms. We call a norm $\|.\|$ in $\mathbb{R}^n$ an *integer norm* if $\|x\|$ is an integer for each x in $\mathbb{Z}^n$.

Above we saw that each graph G embedded on the torus gives an integer norm $\|.\|$ in $\mathbb{R}^2$ such that $f_G(m,n) = \|(m,n)\|$ for each integer vector (m,n). In fact *each* integer norm in $\mathbb{R}^2$ can be constructed in this way:

Theorem 4. *For each integer norm $\|.\|$ in $\mathbb{R}^2$ there exists a graph G embedded on the torus such that $f_G(m,n) = \|(m,n)\|$ for each integer vector (m,n).*

We will give a proof of this theorem in Section 2 below.

Proof of the implications Theorem 1(i) $\Longrightarrow$ Theorem 2(i) and Theorem 2(ii) $\Longrightarrow$ Theorem 1(ii). We first show the first implication. Let K be a symmetric convex body in $\mathbb{R}^2$ not containing any nonzero integer vector. We show that $\tfrac{4}{3} \cdot K^*$ contains a nonzero integer vector.

We may assume that K is a polygon in $\mathbb{R}^2$ with rational vertices (since we can make K slightly larger). Then also K^* is a polygon with rational vertices.

Define the norm $\|.\|$ in $\mathbb{R}^2$ by

$$(9) \qquad \|x\| := \min\{\lambda \mid x \in \lambda \cdot K\} = \max\{x^T y \mid y \in K^*\}$$

for $x \in \mathbb{R}^2$. Let t be a common multiple of the denominators of the components of the vertices of K^*, with the further property that t is also a multiple of four.

Then $t \cdot \|.\|$ is an integer norm, as the maximum in (9) is attained at a vertex of K^*. Hence, by Theorem 4, there exists a graph G embedded on the torus such that $f_G(m,n) = t \cdot \|(m,n)\|$ for each integer vector (m,n).

As K contains no nonzero integer vector, we know that $\|(m,n)\| > 1$ for each nonzero integer vector (m,n), and hence $f_G(m,n) > t$ for each nonzero integer vector (m,n). So the representativity $r(G)$ of G is larger than t.

By Theorem 1(i), G contains $\tfrac{3}{4}t$ pairwise disjoint nontrivial circuits. They all are mutually freely homotopic, say they are all freely homotopic to $C_{m,n}$. So, by the necessity of the condition in Theorem 2 and by (5), for each integer vector (m',n'):

$$(10) \qquad \tfrac{3}{4}t \cdot |mn' - m'n| = \tfrac{3}{4}t \cdot \text{mincr}(C_{m,n}, C_{m',n'}) \leq f_G(m',n') = t \cdot \|(m',n')\|.$$

Hence $\|(n,-m)\|_* \leq \tfrac{4}{3}$, and therefore, $(n,-m)$ belongs to $\tfrac{4}{3} \cdot K^*$. This shows Theorem 2(i).

Again, any better factor in Theorem 1(i) would imply a better factor in Theorem 2(i). This gives the implication Theorem 2(ii) $\Longrightarrow$ Theorem 1(ii). ∎

In Section 2 we will prove Theorem 4, and develop some further results on integer norms in relation to graphs on the torus, and in Sections 3 we give a proof of Theorems 2.

2. Integer norms and graphs on the torus

In this section we give a proof of Theorem 4 above. To this end, we derive some further results.

The following theorem follows directly from the 'cutting plane theorem' of Chvátal [4]. It is a slight extension of a result of Hoffman [9] for polytopes (extended by Edmonds and Giles [7] to polyhedra, forming the basis for the theory of *total dual integrality* – cf. [17:Chapter 23]).

A *polytope* is the convex hull of a finite set of vectors. A polytope P is called *integer* if each vertex of P is an integer vector.

Theorem 5. *Let C be a nonempty compact convex set in $\mathbb{R}^n$. Then C is an integer polytope, if and only if $\max\{c^Tx|x \in C\}$ is an integer for each integer vector $c \in \mathbb{R}^n$.*

This implies:

Theorem 6. *For any integer norm $||.||$ in $\mathbb{R}^n$ there exist integer vectors $y_1, \ldots, y_t$ in $\mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$:*

(11) $$||x|| = \max\{y_1^T x, \ldots, y_t^T x\}.$$

Proof. Let $K := \{x \in \mathbb{R}^n | \; ||x|| \leq 1\}$. Hence for each $x \in \mathbb{R}^n$:

(12) $$||x|| = \max\{x^T y \mid y \in K^*\}.$$

As $||.||$ is an integer norm, this maximum is an integer for each integer vector x. Hence, by Theorem 5, K^* is an integer polytope. So we can take for $y_1, \ldots, y_t$ the vertices of K^*. ∎

Remark 3. One similarly shows the following related result. Any function $\varphi : \mathbb{Z}^n \longrightarrow \mathbb{Z}_+$ satisfies

(13) (i) $\varphi(x + x') \leq \varphi(x) + \varphi(x')$ for all $x, x' \in \mathbb{Z}^n$, and

(ii) $\varphi(k \cdot x) = |k| \cdot \varphi(x)$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{Z}^n$,

if and only if there exist integer vectors $y_1, \ldots, y_t$ such that

(14) $$\varphi(x) = \max\{|y_1^T x|, \ldots, |y_t^T x|\}$$

for each $x \in \mathbb{Z}^n$.

Equivalently, in terms of groups: Let G be an abelian group. Then any function $\varphi : G \longrightarrow \mathbb{Z}_+$ satisfies

(15) (i) $\varphi(x + x') \leq \varphi(x) + \varphi(x')$ for all $x, x' \in G$, and

(ii) $\varphi(k \cdot x) = |k| \cdot \varphi(x)$ for all $k \in \mathbb{Z}$ and $x \in G$,

if and only if there exist homomorphisms $\varphi_1, \ldots, \varphi_t : G \longrightarrow \mathbb{Z}$ such that

(16) $$\varphi(x) = \max\{|\varphi_1(x)|, \ldots, |\varphi_t(x)|\}$$

for each $x \in G$. ∎

For integer norms in $\mathbb{R}^2$ we derive from Theorem 6 a further characterization.

Theorem 7. *A norm $\|.\|$ in $\mathbb{R}^2$ is integer, if and only if there exist integer vectors $z_1, \ldots, z_k$ in $\mathbb{R}^2$ such that*

(17) $$\|x\| = \frac{1}{2}\sum_{i=1}^{k} |z_i^T x|$$

for each $x \in \mathbb{R}^2$ and such that both components of the vector $z_1 + \ldots + z_k$ are even.

Proof. Sufficiency of the condition follows from the fact that, for any integer vector x,

(18) $$\frac{1}{2}\sum_{i=1}^{k} z_i^T x = (\frac{1}{2}(z_1 + \ldots + z_k))^T x$$

is an integer, that differs by an integer value, viz.

(19) $$\sum_{i=1}^{k} \frac{1}{2}(|z_i^T x| - z_i^T x),$$

from $\|x\|$ (by (17)).

To see necessity, let $K := \{x \in \mathbb{R}^2 | \ \|x\| \leq 1\}$. By Theorem 6, K^* is a polygon with integer vertices, $y_1, \ldots y_{2k}$, say, in cyclic order. So $y_{i+k} = -y_i$ for $i = 1, \ldots, k$. Define

(20) $$z_i := y_{i+1} - y_i$$

for $i = 1, \ldots, k$.

So $z_1 + \ldots + z_k = y_{k+1} - y_1 = 2y_{k+1}$ is an even vector. We show that (17) holds for each $x \in \mathbb{R}^2$.

Since $\|x\| = \max\{y^T x | y \in K^*\}$, we know

(21) $$\|x\| = \max\{y_1^T x, \ldots, y_{2k}^T x\}.$$

Let the maximum be attained by $y_j^T x$. Without loss of generality, $1 \leq j \leq k$. It follows that $z_1^T x, \ldots, z_{j-1}^T x \geq 0$ and $z_j^T x, \ldots, z_k^T \leq 0$. Hence

(22) $$\sum_{i=1}^{k} |z_i^T x| = \sum_{i=1}^{j-1} z_i^T x \ - \ \sum_{i=j}^{k} z_i^T x.$$

Now $y_1 = -y_{k+1} = -\frac{1}{2}(z_1 + \ldots + z_k)$, implying $y_j = y_1 + z_1 + \ldots + z_{j-1} = \frac{1}{2}(z_1 + \ldots + z_{j-1} - z_j - \ldots - z_k)$. Hence the right hand side of (22) is equal to $2y_j^T x = 2\|x\|$. ∎

We are now able to prove:

Theorem 4. *For each integer norm $\|.\|$ in $\mathbb{R}^2$ there exists a graph G embedded on the torus such that $f_G(m,n) = \|(m,n)\|$ for each integer vector (m,n).*

Proof. Let $\|.\|$ be an integer norm in $\mathbb{R}^2$. By Theorem 7, there exist integer vectors $z_1, \ldots, z_k$ in $\mathbb{R}^2$ such that

$$\|x\| = \frac{1}{2}\sum_{i=1}^{k} |z_i^T x| \tag{23}$$

holds for each $x \in \mathbb{R}^2$ and such that $z_1 + \ldots + z_k$ is an even vector.

We may assume that, for each $i = 1, \ldots, k$, the two components of z_i are relatively prime (as $z_1, \ldots, z_k$ need not all be different). Write $z_i = (z_i', z_i'')^T$ for $i = 1, \ldots, k$.

Again, let $S = S^1 \times S^1$ be the torus. Let $\Pi : \mathbb{R}^2 \longrightarrow S$ be the usual projection of $\mathbb{R}^2$ to the torus (i.e., $\Pi(x) := (e^{2\pi i x'}, e^{2\pi i x''})$ for each $x = (x', x'')^T$ in $\mathbb{R}^2$). Call a simple closed curve D on S *geodesic* if each component of $\Pi^{-1}[D]$ is a straight line in $\mathbb{R}^2$.

For each $i = 1, \ldots, k$, let D_i be a geodesic simple closed curve on S freely homotopic to $C_{z_i'', -z_i'}$, in such a way that each two of the D_i are different and no point of S is in more than two of the D_i. So $\text{mincr}(C_{m,n}, D_i) = |mz_i' + nz_i''|$ for all m, n.

Let H be the 4-regular graph on the torus formed by the union of $D_1, \ldots, D_k$. Then one easily checks that for each $(m,n) \in \mathbb{Z}^2$:

(24) each closed curve C freely homotopic to $C_{m,n}$, not traversing vertices of H, has at least

$$\sum_{i=1}^{k} \text{mincr}(C_{m,n}, D_i) = \sum_{i=1}^{k} |mz_i' + nz_i''| = 2\|(m,n)\|$$

crossings with H; moreover, at least one such curve has exactly this number of crossings with H.

(Indeed, C has at least $\text{mincr}(C_{m,n}, D_i)$ crossings with part D_i of H. This gives the lower bound. Equality can be attained by $C_{m,n}$ itself or a slight shift of it.)

Since $z_1 + \ldots + z_k$ is even, we know that each closed curve on S not traversing vertices of H, has an even number of crossings with H. So we can color each face of H black or white in such a way that adjacent faces have different colors.

Hence we can construct a 'radial' graph G as follows. In each black face F, put a vertex, and connect it by (pairwise disjoint) lines through F, to each of the vertices of H incident with F. Doing this for each black face of H, we obtain a graph G, called a *radial* graph G.

Now each closed curve on S intersecting H r times and not intersecting vertices of H, can be shifted slightly so that it intersects G $\frac{1}{2}r$ times (in vertices of G). So from (24) we have that $f_G(m,n) = \|(m,n)\|$ for each integer vector (m,n). ∎

Remark 4. The graph G in Theorem 7 need not be unique, but (as was shown in [19]) the minimal such graphs are unique, in the following sense.

Let G be a graph embedded on the torus S. A *minor* of G is any graph obtained from G by a series of deletions and contractions of edges (contracting loops only if they enclose a face). Any minor of G has a natural embedding on S derived from the embedding of G. It is a *proper* minor if at least one edge is deleted or contracted. Call a graph G

embedded on the torus S a *kernel* if for each proper minor G' of G one has $f_{G'} \neq f_G$ (i.e., $f_{G'}(m,n) < f_G(m,n)$ for at least one integer vector (m,n)).

So for each integer norm $||.||$ in $\mathbb{R}^2$ there exists at least one kernel G on S with $f_G(m,n) = ||(m,n)||$ for all integer vectors (m,n). Now by the results in [19], for any fixed integer norm $||.||$, each two such kernels can be obtained from each other by a series of the following operations:

(25)
(i) shifting the graph over the torus;
(ii) taking the surface dual of the graph;
(ii) ΔY-exchange.

Here ΔY-*exchange* means replacing a triangular face F by a vertex in the face connected to the three vertices incident with F, or conversely. (This operation was introduced by Steinitz [21], who called it the θ-process.) ∎

3. Proof of Theorem 2

Although Theorem 2 is nothing but a simple exercise in plane geometry, for completeness we give a proof here. As a preparation, we first give another simple fact.

Theorem 8. *For any nonsingular* 2×2 *matrix* A *there exist nonzero integer vectors* x *and* y *in* $\mathbb{R}^2$ *such that*

$$(26) \qquad ||Ax||_\infty \cdot ||y^T A^{-1}||_1 \leq \tfrac{1}{2}(\sqrt{2}+1).$$

Proof. We may assume that $\det A = 1$. Let Λ and Λ^* be the pair of dual lattices

$$(27) \qquad \Lambda := \{Ax \mid x \in \mathbb{Z}^2\},\ \Lambda^* := \{y^T A^{-1} \mid y \in \mathbb{Z}^2\}.$$

We may assume that Λ has a basis $b = (b_1, b_2)^T, c = (c_1, c_2)^T$ satisfying

$$(28) \qquad b_1 \geq b_2 \geq 0 \text{ and } c_2 \geq -c_1 \geq 0.$$

Indeed, let b be a nonzero vector in Λ minimizing $||b||_\infty$. Without loss of generality, $||b||_\infty = b_1 \geq b_2 \geq 0$. Let c be a nonzero vector in Λ minimizing $|c_2|$ over all nonzero vectors $c \in \Lambda$ with $|c_1| < ||b||_\infty$. We may assume that $c_2 \geq 0$, and that the triangle Δ with vertices $0, b$, and c has minimal area. If b and c do not form a basis, Δ would contain another vector c' with the required properties, contradicting the minimality of Δ. Moreover, $||c||_\infty = c_2 \geq ||b||_\infty > |c_1|$. If $c_1 > 0$, we can replace c by $c - b$. Thus we obtain b and c satisfying (28).

The arithmetic-geometric inequality $(\alpha\beta \leq (\frac{1}{2}\alpha + \frac{1}{2}\beta)^2$ applied to $\alpha = (\sqrt{2}-1)b_1c_2, \beta = -(\sqrt{2}+1)b_2c_1$ and the fact that $b_1c_2 - b_2c_1 = \det A = 1$ give

$$(29) \qquad -b_1b_2c_1c_2 \leq (\tfrac{1}{2}(\sqrt{2}-1)b_1c_2 - \tfrac{1}{2}(\sqrt{2}+1)b_2c_1)^2 = (\tfrac{1}{2}(\sqrt{2}+1) - b_1c_2)^2.$$

Now $\frac{1}{2}(\sqrt{2}+1) - b_1c_2 \geq 1 - b_1c_2 = -b_2c_1 \geq 0$. Hence at least one of b_2c_2 and $-b_1c_1$ is at most $\frac{1}{2}(\sqrt{2}+1) - b_1c_2$. That is, at least one of $(b_1+b_2)c_2$ and $b_1(-c_1+c_2)$ is at most $\frac{1}{2}(\sqrt{2}+1)$. Since b and c belong to Λ and since $(b_2, -b_1)$ and $(c_2, -c_1)$ belong to Λ^*, we have the required vectors. ∎

In fact, bound $\frac{1}{2}(\sqrt{2}+1)$ in Theorem 8 is best possible, as is shown by the matrix

$$A = \begin{pmatrix} 1 & 1-\sqrt{2} \\ \sqrt{2}-1 & 1 \end{pmatrix}. \tag{30}$$

Theorem 2.(i) *For any symmetric convex body K in $\mathbb{R}^2$, there exists a nonzero integer vector in K or there exists a nonzero integer vector in $\frac{4}{3} \cdot K^*$.*

(ii) *The factor $\frac{4}{3}$ is best possible.*

Proof. (i) We may assume that K is a polygon. We show that if K contains no nonzero integer vectors in its interior, then $\frac{4}{3}K^*$ contains a nonzero integer vector.

We may assume that each edge of K contains an integer vector in its relative interior (otherwise, we can shift the edge until it contains an integer vector in its relative interior or until the edge 'disappears').

If K has four edges, the result directly follows from Theorem 8 (applied to the matrix A with rows the coefficients of the inequalities determining the edges of K), since $\frac{1}{2}(\sqrt{2}+1) < \frac{4}{3}$.

If K has at least six edges, let $v_1, \ldots, v_{2k}$ be the vertices of K (in cyclic order), and let z_i be an integer vector in the relative interior of the edge connecting v_{i-1} and v_i ($i = 1, \ldots, 2k$, taking indices mod $2k$).

By Minkowski's theorem [12], the volume of K is at most 4. Hence, there exists an $i = 1, \ldots, 2k$ so that the volume V of the quadrangle $(0, z_i, v_i, z_{i+1})$ is at most $4/2k$. As the triangle $(0, z_i, z_{i+1})$ contains no further integer vectors, z_i and z_{i+1} form a basis for the lattice $\mathbb{Z}^2$. So the vector c satisfying $c^T z_i = c^T z_{i+1} = 1$ is an integer vector. Let V_1 and V_2 be the volumes of the triangles $(0, z_i, z_{i+1})$ and (z_i, z_{i+1}, v_i) respectively. So $V_1 = 1/2$ and $V_2 = V - V_1$. Moreover, $V_2/V_1 = (c^T v_i - c^T z_i)/c^T z_i$. This implies $c^T v_i = 2V$. Hence

$$\max\{c^T x | x \in K\} = c^T v_i = 2V \leq 2\tfrac{4}{2k} \leq \tfrac{4}{3}. \tag{31}$$

(ii) Let K be the convex hull of the vectors $\pm(\frac{2}{3}, \frac{4}{3}), \pm(\frac{4}{3}, \frac{2}{3}), \pm(-\frac{2}{3}, \frac{2}{3})$. Then K^* is the convex hull of the vectors $\pm(-\frac{1}{2}, 1), \pm(1, -\frac{1}{2}), \pm(\frac{1}{2}, \frac{1}{2})$. Since no slight shrinking of K and of $\frac{4}{3} \cdot K^*$ contains any nonzero integer vector, we obtain that $\frac{4}{3}$ is best possible. ∎

Remark 5. In fact, in this proof k cannot exceed 3, as no two of the vectors z_i and $z_{i'}$ for $i, i' = 1, \ldots, k$ are equal mod 2 (otherwise $\frac{1}{2}(z_i + z_{i'})$ would be an integer vector in the interior of K). (This is a special case of a result of Doignon [6] (cf. [2], [16]).) ∎

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