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M. de Graaf, A. Schrijver

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Grid Minors of Graphs on the Torus

Maurits de Graaf

Department of Mathematics,
University of Amsterdam,
Plantage Muidergracht 24,
1018 TV Amsterdam, The Netherlands

Alexander Schrijver

CWI, Kruislaan 413,
1098 SJ Amsterdam, The Netherlands
and
Department of Mathematics,
University of Amsterdam,
Plantage Muidergracht 24,
1018 TV Amsterdam, The Netherlands

Abstract. We show that any graph G embedded on the torus with face-width $r \geq 5$ contains the toroidal $\lfloor \frac{2}{3}r \rfloor$-grid as a minor. (The *face-width* of G is the minimum value of $|C \cap G|$ where C ranges over all homotopically nontrivial closed curves on the torus. The *toroidal k-grid* is the product $C_k \times C_k$ of two copies of a k-circuit C_k.) For each fixed $r \geq 5$, the value $\lfloor \frac{2}{3}r \rfloor$ is largest possible.
This applies to a theorem of Robertson and Seymour showing, for each graph H embedded on any compact surface S, the existence of a number ρ_H such that every graph G embedded on S with face-width at least ρ_H contains H as a minor. Our result implies that for $H = C_k \times C_k$ embedded on the torus, $\rho_H := \lceil \frac{3}{2}k \rceil$ is the smallest possible value.
Our proof is based on deriving a result in the geometry of numbers. It implies that for any symmetric convex body K in $\mathbb{R}^2$ one has $\lambda_2(K) \cdot \lambda_1(K^*) \leq \frac{3}{2}$, and that this bound is smallest possible. (Here $\lambda_i(K)$ denotes the minimum value of λ such that $\lambda \cdot K$ contains i linearly independent integer vectors. K^* is the polar convex body.)

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1. Introduction

For any graph G embedded on a surface S, the *face-width* (or *representativity*) $r(G)$ of G is the minimum of $|C \cap G|$, where C ranges over all homotopically nontrivial closed curves on S. Robertson and Seymour [1] showed:

(1) for each graph H embedded on a compact surface S there exists an integer ρ_H so that each graph G embedded on S with $r(G) \geq \rho_H$ contains H as a minor.

In this paper we determine the smallest value of ρ_H for a certain class of graphs H embedded on the torus, viz. the toroidal grids. For each $k \geq 3$, the *toroidal k-grid* is the product $C_k \times C_k$ of two k-circuits C_k. (By definition, $C_k \times C_k$ has vertices (i, j) for $0 \leq i, j \leq k-1$, where (i, j) and (i', j') are adjacent if either $i = i'$ and $j = j \pm 1 \pmod{k}$ or $j = j'$ and $i = i' \pm 1 \pmod{k}$.)

Clearly, each toroidal k-grid can be embedded on the torus. In fact, there is a unique embedding, up to homeomorphisms (of the torus and of the grid). (If $k \geq 5$, this follows easily from the fact that each face of the embedded graph should be a quadrangle. For $k = 3$ and 4 this takes some elaboration.)

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We show:

Theorem 1. *For the toroidal k-grid $H = C_k \times C_k$ embedded on the torus, $\rho_H := \lceil \frac{3}{2}k \rceil$ is the smallest integer value one can take for ρ_H in* (1).

We derive this from:

Theorem 2. *Any graph G embedded on the torus contains the toroidal $\lfloor \frac{2}{3}r(G) \rfloor$-grid as a minor (if $r(G) \geq 5$). For each integer $r \geq 3$ there exists a graph G embedded on the torus with $r(G) = r$ and not containing the toroidal $\lfloor \frac{2}{3}r \rfloor + 1$-grid as a minor.*

Proof of the implication Theorem 2 ⇒ Theorem 1. Choose $k \geq 3$. Let G be a graph with $r(G) \geq \lceil \frac{3}{2}k \rceil$. Since $k = \lfloor \frac{2}{3} \lceil \frac{3}{2}k \rceil \rfloor \leq \lfloor \frac{2}{3}r(G) \rfloor$, Theorem 2 implies that G contains the toroidal k-grid as a minor.

Let $r := \lceil \frac{3}{2}k \rceil - 1$. By Theorem 2 there exists a graph G on the torus with $r(G) = r$ and not containing the toroidal $\lfloor \frac{2}{3}r \rfloor + 1$-grid as a minor. Since $k = \lfloor \frac{2}{3}r \rfloor + 1$, Theorem 1 follows. ∎

To prove Theorem 2, we use some results from [2] and [3]. Represent the torus as the product $S^1 \times S^1$ of two copies of the unit circle S^1 in the complex plane. For $(m,n) \in \mathbb{Z}^2$, let $C_{m,n} : S^1 \longrightarrow S^1 \times S^1$ be the closed curve on the torus given by:

$$C_{m,n}(z) := (z^m, z^n) \tag{2}$$

for $z \in S^1$.

Let G be a graph embedded on the torus. Define $\varphi_G : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ by:

$$\varphi_G(m,n) := \min_{C \sim C_{m,n}} \mathrm{cr}(C,G), \tag{3}$$

where $C \sim C'$ means that C is a closed curve freely homotopic to closed curve C' and where $\mathrm{cr}(C,G)$ denotes the number of intersections of C and G, counting multiplicities. So $r(G)$ is equal to the minimum value of $\varphi_G(m,n)$ over all vectors $(m,n) \neq (0,0)$ in $\mathbb{Z}^2$.

Let $P(G)$ be the following set in $\mathbb{R}^2$:

$$P(G) := \{(x,y) \in \mathbb{R}^2 | mx + ny \leq \varphi_G(m,n) \text{ for all } (m,n) \in \mathbb{Z}^2\}. \tag{4}$$

Then $P(G)$ is a symmetric integer polygon (i.e., $P(G) = -P(G)$ and it is a polygon with all vertices having integer coordinates only). Define the *height* $\mathrm{height}(K)$ of a polygon K by:

$$\mathrm{height}(K) := \min_{(m,n)\in\mathbb{Z}^2,(m,n)\neq(0,0)} \max\{mx + ny | (x,y) \in K\}. \tag{5}$$

As $\varphi_G(m,n) = \max\{mx + ny | (x,y) \in P(G)\}$ (cf. [2]), we have:

$$r(G) = \mathrm{height}(P(G)). \tag{6}$$

The following was shown in [3]:

(7) let $k \geq 3$; a graph G embedded on the torus contains a toroidal k-grid as a minor, if and only if $\frac{1}{k}P(G)$ contains two linearly independent integer vectors.

Assertions (6) and (7) imply that to prove Theorem 2, it suffices to show:

Theorem 3. *Let $r \geq 3$. Then for each symmetric integer polygon K of height r, the polygon $\lfloor \frac{2}{3}r \rfloor^{-1}K$ contains two linearly independent integer vectors. Here $\lfloor \frac{2}{3}r \rfloor$ cannot be replaced by any larger integer.*

We show Theorem 3 in Section 2. We first note that it implies the following bound in the geometry of numbers. Let K be a symmetric convex body in $\mathbb{R}^2$ (i.e., K is a compact full-dimensional convex set with $K = -K$). Let $\lambda_1(K)$ denote the minimum value of λ so that $\lambda \cdot K$ contains a nonzero integer vector. Let $\lambda_2(K)$ denote the minimum value of λ so that $\lambda \cdot K$ contains two linearly independent integer vectors. Let K^* denote the *polar* convex body:

(8) $$K^* := \{y \in \mathbb{R}^2 | x^T y \leq 1 \text{ for all } x \in K\}.$$

Then:

Corollary 3a. *For each symmetric convex body K in $\mathbb{R}^2$ one has $\lambda_2(K) \cdot \lambda_1(K^*) \leq \frac{3}{2}$. The bound $\frac{3}{2}$ is smallest possible.*

Proof. It suffices to show the corollary for symmetric integer polygons K with $r :=$ height(K) being a multiple of 3. Now $r := \lambda_1(K^*)$, while by Theorem 3, $\lambda_2(K) \leq (\frac{2}{3}r)^{-1}$. So $\lambda_2(K) \cdot \lambda_1(K^*) \leq \frac{3}{2}$.

Similarly, any better value in the corollary would imply a better factor in Theorem 3. ∎

2. Proof of Theorem 3

Call a symmetric integer polygon K *r-minimal*, if height$(K) \geq r$ while height$(K') < r$ for each symmetric integer polygon $K' \neq K$ contained in K. So Theorem 3 follows from:

(9) let $r \geq 2$; then for each r-minimal symmetric integer polygon K, the polygon $\frac{3}{2r} \cdot K$ contains two linearly independent integer vectors; moreover, there exists an r-minimal symmetric integer polygon K so that $(\lfloor \frac{2r}{3} \rfloor + 1)^{-1} \cdot K$ does not contain two linearly independent integer vectors.

In order to prove (9), we use the classification of r-minimal symmetric integer polygons given in [3]. Each of these polygons is a quadrangle or a hexagon. The quadrangles arise as follows. Choose integer values $0 \leq \alpha < r$ and $0 \leq \beta < r$. Let $Q_{\alpha,\beta}$ be the convex hull of the points $\pm(r, \alpha), \pm(-\beta, r)$. Then $Q_{\alpha,\beta}$ is r-minimal, and all symmetric r-minimal integer polygons that are quadrangles arise in this way, up to unimodular transformations (= linear transformations of $\mathbb{R}^2$ fixing $\mathbb{Z}^2$).

The hexagons arise as follows. Choose integer values $0 < \alpha < r$, $0 < \beta < r$ and $0 < \gamma < r$. Let $H_{\alpha,\beta,\gamma}$ be the convex hull of the points $\pm(r, \alpha), \pm(r - \beta, r), \pm(-\gamma, r - \gamma)$. Again, $H_{\alpha,\beta,\gamma}$ is r-minimal, and all symmetric r-minimal integer polygons that are hexagons arise in this way, up to unimodular transformations.

So it suffices to show the following two lemmas.

Lemma 1. *For each choice of integers $0 \le \alpha < r$ and $0 \le \beta < r$, $\lambda_2(Q_{\alpha,\beta}) \le \frac{3}{2r}$. For fixed r, we cannot replace $\frac{3}{2r}$ by k^{-1} for any integer $k > \frac{2r}{3}$.*

Proof. One easily finds that $Q_{\alpha,\beta}$ is determined by the following inequalities:

$$(10)\qquad \begin{aligned} \left|\frac{r-\alpha}{r^2+\alpha\beta}x + \frac{r+\beta}{r^2+\alpha\beta}y\right| &\le 1, \\ \left|\frac{r+\alpha}{r^2+\alpha\beta}x + \frac{\beta-r}{r^2+\alpha\beta}y\right| &\le 1 \end{aligned}$$

For each vector (x, y), let the norm $\|(x, y)\|$ be the minimum λ for which (x, y) belongs to $\lambda \cdot Q_{\alpha,\beta}$. Note that (x, y) can be easily calculated from (10):

$$(11)\qquad \|(x,y)\| = \frac{\max\{|(r-\alpha)x + (r+\beta)y|, |(r+\alpha)x + (\beta-r)y|\}}{r^2+\alpha\beta}.$$

To show the first statement in the lemma, we have to find two linearly independent integer vectors each with norm at most $\lambda(r)$. We may assume $\alpha \le \beta$.

Then

$$(12)\qquad \|(1,0)\| = \frac{r+\alpha}{r^2+\alpha\beta} \le \frac{r+\alpha}{r^2+\alpha^2} \le \frac{3}{2r}.$$

(The first inequality follows from $\alpha \le \beta$. The second inequality follows from the fact that $(1+x) \le \frac{3}{2}(1+x^2)$ for all $x \in \mathbb{R}$.)

If $\beta < r/3$, then:

$$(13)\qquad \|(0,1)\| = \frac{r+\beta}{r^2+\alpha\beta} < \frac{r+r/3}{r^2} < \frac{3}{2r}.$$

If $\beta \ge r/3$, then:

$$(14)\qquad \|(0,1)\| + \|(1,-1)\| = \frac{r+\beta}{r^2+\alpha\beta} + \frac{2r+\alpha-\beta}{r^2+\alpha\beta} = \frac{3r+\alpha}{r^2+\alpha\beta} \le \frac{3r+3\alpha\beta/r}{r^2+\alpha\beta} = \frac{3}{r},$$

implying that at least one of $(0,1), (1,-1)$ has norm at most $3/2r$. This shows the first statement of the lemma.

To show the second statement, choose $r \ge 3$. Let $k := \lfloor\frac{2r}{3}\rfloor + 1$. Let $\alpha := 0$ and $\beta := \lfloor r/2 \rfloor$. We define a norm as in (11). Let (x, y) be any integer vector with norm at most $1/k$. We show that $y = 0$, implying that there do not exist two linearly independent integer vectors each with norm at most $1/k$. We may assume $x \ge 0$.

First let r be even. Then $\|(x,y)\| = \max\{|x + \frac{3}{2}y|, |x - \frac{1}{2}y|\}/r \le 1/k$. If $x = 0$ then $|\frac{3}{2}y| \le r/k < \frac{3}{2}$, and hence $y = 0$. If $x \ge 1, y \ge 1$, then $r/k \ge |x + \frac{3}{2}y| \ge \frac{5}{2} > r/k$. If $x \ge 1, y \le -1$, then $r/k \ge |x - \frac{1}{2}y| \ge \frac{3}{2} > r/k$.

Next let r be odd. Then $\|(x,y)\| = \max\{|x + (\frac{3}{2} - \frac{1}{2r})y|, |x - (\frac{1}{2} + \frac{1}{2r})y|\}/r \le 1/k$. Note that $k \ge \frac{2}{3}r + \frac{1}{3}$, implying $k(\frac{3}{2} - \frac{1}{2r}) \ge (\frac{2r}{3} + \frac{1}{3})((\frac{3}{2} - \frac{1}{2r}) = r + \frac{1}{6} - \frac{1}{6r} > r$. If $x = 0$ then $|(\frac{3}{2} - \frac{1}{2r})y| \le r/k < (\frac{3}{2} - \frac{1}{2r})$, yielding $y = 0$. If $x \ge 1, y \ge 1$, then $r/k \ge |x + (\frac{3}{2} - \frac{1}{2r})y| \ge \frac{5}{2} - \frac{1}{2r} > \frac{3}{2} > r/k$. If $x \ge 1, y \le -1$ then $r/k \ge |x - (\frac{1}{2} + \frac{1}{2r})y| > \frac{3}{2} > r/k$. So also if $x \ge 1$ then $y = 0$. ∎

Lemma 2. *For each choice of integers* $0 < \alpha < r, 0 < \beta < r$ *and* $0 < \gamma < r$, $\lambda_2(H_{\alpha,\beta,\gamma}) < \frac{3}{2r}$.

Proof. One easily finds that $H_{\alpha,\beta,\gamma}$ is determined by the following inequalities:

$$
\begin{aligned}
|\frac{r-\alpha}{r^2+\alpha\beta-\alpha r}x + \frac{\beta}{r^2+\alpha\beta-\alpha r}y| &\leq 1, \\
|\frac{\gamma}{r^2+\beta\gamma-\beta r}x + \frac{\beta-r-\gamma}{r^2+\beta\gamma-\beta r}y| &\leq 1, \\
|\frac{\gamma-r-\alpha}{r^2+\gamma\alpha-\gamma r}x + \frac{r-\gamma}{r^2+\gamma\alpha-\gamma r}y| &\leq 1.
\end{aligned}
\tag{15}
$$

For each vector (x, y), let the norm $||(x, y)||$ be the minimum λ for which (x, y) belongs to $\lambda \cdot H_{\alpha,\beta,\gamma}$. Again, (x, y) can be easily calculated from (15). It follows that:

$$
\begin{aligned}
||(1,0)|| &= \frac{r+\alpha-\gamma}{r^2+\gamma\alpha-\gamma r}, \\
||(0,1)|| &= \frac{r+\gamma-\beta}{r^2+\beta\gamma-\beta r}, \\
||(1,1)|| &= \frac{r+\beta-\alpha}{r^2+\alpha\beta-\alpha r}.
\end{aligned}
\tag{16}
$$

We show that at least two of these norms are less than $\frac{3}{2r}$. Suppose not. By symmetry we may assume that $||(1,0)|| \geq \frac{3}{2r}$ and $||(0,1)|| \geq \frac{3}{2r}$. As $0 < \gamma < r$, the first norm in (16) is monotonically increasing in α, while the second norm is monotonically decreasing in β. So:

$$
\frac{r+\alpha-\gamma}{r^2+\gamma\alpha-\gamma r} < \frac{2r-\gamma}{r^2} \text{ and } \frac{r+\gamma-\beta}{r^2+\beta\gamma-\beta r} < \frac{r+\gamma}{r^2}.
\tag{17}
$$

Since $2r-\gamma \leq \frac{3}{2}r$ or $r+\gamma \leq \frac{3}{2}r$ (as $(2r-\gamma)+(r+\gamma) = 3r$), this contradicts our assumption. ∎

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