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A. Schrijver

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Finding k Disjoint Paths in a Directed Planar Graph

Alexander Schrijver

CWI
Kruislaan 413
1098 SJ Amsterdam
The Netherlands

and

Department of Mathematics
University of Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands

Abstract. We show that for each fixed k, the problem of finding k pairwise vertex-disjoint directed paths between given terminals in a directed planar graph is solvable in polynomial time.

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1. Introduction

In this paper we show that the following problem, the *k disjoint paths problem for directed planar graphs*, is solvable in polynomial time, for any fixed k:

(1) given: a directed planar graph $D = (V, A)$ and k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D;

find: k pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i $(i = 1, \ldots, k)$.

The problem is NP-complete if we do not fix k (even in the undirected case; Lynch [2]). Moreover, it is NP-complete for $k = 2$ if we delete the planarity condition (Fortune, Hopcroft, and Wyllie [1]). This is in contrast to the undirected case (for those believing NP$\neq$P), where Robertson and Seymour [4] showed that, for any fixed k, the k disjoint paths problem is polynomial-time solvable for any graph (not necessarily planar).

In this paper we do not aim at obtaining the best possible running time bound, as we presume that there are much faster (but possibly more complicated) methods for (1) than the one we describe in this paper. In fact, recently Reed, Robertson, Schrijver, and Seymour [3] showed that for undirected planar graphs the k disjoint paths problem can be solved in *linear* time, for any fixed k. This algorithm makes use of methods from Robertson and Seymour's theory of graph minors. A similar algorithm for directed planar graphs might exist, but probably would require extending parts of graph minors theory to the directed case.

Our method is based on cohomology over free (nonabelian) groups. For the k disjoint paths problem we use free groups with k generators. It extends methods given in [5] for undirected graphs on surfaces based on homotopy. Cohomology is in a sense dual to

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P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

homology, and can be defined in any directed graph, also if it is not embedded on a surface. We apply cohomology to an *extension* of the planar graph dual of D—just using homology to D itself seems not powerful enough.

We remark that in our approach free groups and (co)homology are used mainly as a framework to formulate certain ideas smoothly; they give us a convenient tool of recording shifts of curves over the plane. No deep group theory or topology is used. We could avoid free groups and cohomology by adopting a more complex notation and terminology, that however implicitly would mimic free groups and cohomology. The present approach also readily allows application of the algorithm where the embedding of the graph in the plane is given combinatorially, viz. by a list of the cycles that bound the faces of the graph.

2. Free groups

The *free group* G_k generated by the generators $g_1, g_2, \ldots, g_k$ consists of all words $b_1 b_2 \ldots b_t$ where $t \geq 0$ and $b_1, \ldots, b_t \in \{g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$ such that $b_i b_{i+1} \neq g_j g_j^{-1}$ and $b_i b_{i+1} \neq g_j^{-1} g_j$ for $i = 1, \ldots t-1$ and $j = 1, \ldots, k$. The product $x \cdot y$ of two such words is obtained from the concatenation xy by deleting iteratively all occurrences of any $g_j g_j^{-1}$ and $g_j^{-1} g_j$. (So in our notation generally not $x \cdot y = xy$.) This defines a group, with unit element 1 equal to the empty word $\emptyset$. The *size* $|x|$ of a word x is the number of symbols occurring in it, counting multiplicities. We call $g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}$ the *symbols*.

A word y is called a *segment* of word w if $w = xyz$ for certain words x, z. A subset Γ of a free group is called *hereditary* if with each word $y \in \Gamma$, each segment of y belongs to Γ. If $w = yz$ for some word z, y is called a *beginning segment* of w, denoted by $y \leq w$. This partial order gives a lattice if we extend G_k with an element ∞ at infinity. We denote the meet and join by $\wedge$ and $\vee$.

Note that for any symbol b and words x, z one has:

$$\text{(2)} \qquad \text{if } x \leq b \cdot z \text{ and } z \leq b^{-1} \cdot x \text{ then } x^{-1} \cdot b \cdot z = 1 \text{ or } x = z = 1.$$

To see this, let $y := x^{-1} \cdot b \cdot z$. As $x \leq b \cdot z$, $b \cdot z = xy$; as $z \leq b^{-1} \cdot x$, $b^{-1} \cdot x = zy^{-1}$, i.e., $x^{-1} \cdot b = yz^{-1}$. If $y \neq 1$ then $xyz^{-1} = x \cdot y \cdot z^{-1} = b$, implying $x = z = 1$ (as b is a symbol).

3. The cohomology feasibility problem for free groups

Let $D = (V, A)$ be a weakly connected directed graph, let $r \in V$, and let $(G, \cdot)$ be a group. Two functions $\phi, \psi : A \longrightarrow G$ are called *r-cohomologous* if there exists a function $f : V \longrightarrow G$ such that

(3) (i) $f(r) = 1$;

(ii) $\psi(a) = f(u)^{-1} \cdot \phi(a) \cdot f(w)$ for each arc $a = (u, w)$.

One easily checks that this gives an equivalence relation.

Consider the following *cohomology feasibility problem for free groups*:

(4) given: a weakly connected directed graph $D = (V, A)$, a vertex r, a function $\phi : A \longrightarrow G_k$, and for each $a \in A$ a nonempty hereditary subset $\Gamma(a)$ of G_k;

find: a function $\psi : A \longrightarrow G_k$ such that ψ is r-cohomologous to ϕ and such that $\psi(a) \in \Gamma(a)$ for each arc a.

We give a polynomial-time algorithm for this problem. The running time of the algorithm is bounded by a polynomial in $|A| + \sigma + k$, where σ is the maximum size of the words $\phi(a)$ and the words in the $\Gamma(a)$. (In fact we can drop k and assume that G_k is the free group generated by the generators occurring in the $\phi(a)$ and the words in the $\Gamma(a)$.)

Note that, by the definition of r-cohomologous, equivalent to finding a ψ as in (4), is finding a function $f : V \longrightarrow G_k$ satisfying:

(5) (i) $f(r) = 1$;

(ii) $f(u)^{-1} \cdot \phi(a) \cdot f(w) \in \Gamma(a)$ for each arc $a = (u, w)$.

We call such a function f *feasible.*

In solving the cohomology feasibility problem for free groups we may assume

(6) $|\phi(a)| \leq 1$ for each arc a; with each arc $a = (u, w)$ also $a^{-1} = (w, u)$ is an arc, with $\phi(a^{-1}) = \phi(a)^{-1}$ and $\Gamma(a^{-1}) = \Gamma(a)^{-1}$.

Here $\Gamma(a)^{-1} := \{x^{-1} | x \in \Gamma(a)\}$. The first condition can be attained by replacing any arc $a = (u, w)$ with $\phi(a) = b_1 \ldots b_t, t \geq 2$ by a $u - w$ path $a_1 \ldots a_t$ with $\phi(a_i) := b_i$ $(i = 1, \ldots, t)$ and $\Gamma(a_1) := \Gamma(a)$ and $\Gamma(a_i) := \{1\}$ $(i = 2, \ldots, t)$.

4. Pre-feasible functions

Let input $D = (V, A), r, \phi, \Gamma$ for the cohomology feasibility problem for free groups (4) be given, assuming (6). We call a function $f : V \longrightarrow G_k$ *pre-feasible* if $f(r) = 1$ and for each arc $a = (u, w)$ with $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ one has $f(u) = f(w) = 1$.

Pre-feasibility behaves nicely with respect to the partial order $\leq$ on the set G_k^V of all functions $f : V \longrightarrow G_k$ induced by the partial order $\leq$ on G_k as: $f \leq g \Leftrightarrow f(v) \leq g(v)$ for each $v \in V$. It is easy to see that G_k^V forms a lattice if we add an element ∞ at infinity. Let $\wedge$ and $\vee$ denote the meet and join. Then:

Proposition 1. *If f_1 and f_2 are pre-feasible then so is $f := f_1 \wedge f_2$.*

Proof. Clearly $f(r) = 1$. Suppose $y := f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ while not $f(u) = f(w) = 1$. By (2) and by symmetry we may assume $f(u) \not\leq \phi(a) \cdot f(w)$. Since $f(w) = f_1(w) \wedge f_2(w)$, there is an $i \in \{1, 2\}$ such that $y \leq f(u)^{-1} \cdot \phi(a) \cdot f_i(w)$. As $f(u) \not\leq \phi(a) \cdot f(w)$, the first symbols of $f(u)^{-1}$ and y are equal, and hence $f_i(u)^{-1} \cdot \phi(a) \cdot f_i(w) \notin \Gamma(a)$, as it contains y as segment. So $f_i(u) = f_i(w) = 1$ and therefore $f(u) = f(w) = 1$. ∎

So for any function $f : V \longrightarrow G_k$ there exists a smallest pre-feasible function $\bar{f} \geq f$, provided there exists at least one pre-feasible function $g \geq f$. If no such g exists we set $\bar{f} := \infty$. We observe:

Proposition 2. *Let $\bar{f}$ be finite. Then:*

(7) (i) $f(r) = 1$ *and* $|f(v)| < 2\sigma|V|$ *for each vertex* v*;*

(ii) *for each arc* $a = (u, w)$*: if* $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ *then* $f(u) \leq \phi(a) \cdot f(w)$ *or* $f(w) \leq \phi(a^{-1}) \cdot f(u)$.

Proof. Clearly, $f(r) \leq \bar{f}(r) = 1$. Moreover, by induction on the minimum number t of arcs in any $r - v$ path one shows $|\bar{f}(v)| \leq 2\sigma t$ (if $a = (u, v)$ is the last arc in the path then $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(v)$ belongs to $\Gamma(a)$ or is equal to $\phi(a)$, and hence has size at most σ; so $|\bar{f}(v)| \leq |\bar{f}(u)| + |\phi(a)| + \sigma \leq 2\sigma(t-1) + 2\sigma = 2\sigma t$). So $|f(v)| \leq |\bar{f}(v)| < 2\sigma|V|$.

To see (ii), suppose $f(u) \not\leq \phi(a) \cdot f(w)$ and $f(w) \not\leq \phi(a^{-1}) \cdot f(u)$. The first implies that the first symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the first symbol of $f(u)^{-1}$. The second implies that the last symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the last symbol of $f(w)$. Since $f \leq \bar{f}$, it follows that $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is a segment of $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)$. So $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w) \notin \Gamma(a)$. So $\bar{f}(u) = \bar{f}(w) = 1$, and hence $f(u) = f(w) = 1$, a contradiction. ∎

5. A subroutine finding $\bar{f}$

Let input $D = (V, A), r, \phi, \Gamma$ for the cohomology feasibility problem for free groups (4) be given, again assuming (6). We describe a polynomial-time subroutine that outputs $\bar{f}$ for any given $f : V \longrightarrow G_k$.

If f is pre-feasible output $\bar{f} := f$. If f violates (7) output $\bar{f} := \infty$. Otherwise choose an arc $a = (u, w)$ satisfying $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ and $f(w) \not\leq \phi(a^{-1}) \cdot f(u)$ (as f is not pre-feasible and satisfies (7), such an arc exists by (2)). Perform the following

Iteration: Write $\phi(a) \cdot f(w) = xy$, with $y \in \Gamma(a)$ and $|y|$ as large as possible, and reset $f(u) := x$,

and iterate.

Proposition 3. *In the iteration, resetting* f *increases* $|f(u)|$ *and does not change* $\bar{f}$.

Proof. Consider the iteration. Denote by f' the reset f. As (7)(ii) holds, $f(u) \leq \phi(a) \cdot f(w)$. Since $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$, $f(u)$ should be a proper segment of x. So $|f'(u)| > |f(u)|$.

To see $\bar{f}' = \bar{f}$, we must show $f' \leq \bar{f}$, that is, $f'(u) \leq \bar{f}(u)$ if $\bar{f}$ is finite. Since $f(w) \not\leq \phi(a^{-1}) \cdot f(u)$, the last symbol of $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ is equal to the last symbol of $f(w)$. Hence $f(u)^{-1} \cdot \phi(a) \cdot f(w) \leq f(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)$. Since $\bar{f}(u)^{-1} \cdot \phi(a) \cdot \bar{f}(w)$ belongs to $\Gamma(a)$, it follows that $f'(u) = x \leq \bar{f}(u)$. ∎

Since at each iteration $|f(u)|$ increases for some vertex u, after $2\sigma|V|^2$ iterations (7) is violated. Thus the subroutine is polynomial-time.

6. Algorithm for the cohomology feasibility problem for free groups

Let input $D = (V, A), r, \phi, \Gamma$ for the cohomology feasibility problem for free groups (4) be given, again assuming (6). We find a feasible function f as follows.

For every $a = (u, w) \in A$ let f_a be the function defined by: $f_a(u) := \phi(a)$ and $f_a(v) := \emptyset$ for each $v \neq u$. Let E be the set of pairs $\{a, a'\}$ from A for which $\bar{f}_a \vee \bar{f}_{a'}$ is finite and pre-feasible. Let E' be the set of pairs $\{a, a^{-1}\}$ with $a \in A$ and $\phi(a) \notin \Gamma(a)$.

We search for a subset X of A such that each pair in X belongs to E and such that X intersects each pair in E'. This is a special case of the 2-satisfiability problem, and hence can be solved in polynomial time.

Proposition 4. *If X exists then the function $f := \bigvee_{a \in X} \bar{f}_a$ is feasible. If X does not exist then there is no feasible function.*

Proof. Since $\bar{f}_a \vee \bar{f}_{a'}$ is finite and pre-feasible for each two a, a' in X, f is finite and $f(r) = 1$. Moreover, suppose $f(u)^{-1} \cdot \phi(a) \cdot f(w) \notin \Gamma(a)$ for some arc $a = (u, w)$. Let $f(u) = \bar{f}_{a'}(u)$ and $f(w) = \bar{f}_{a''}(w)$ for $a', a'' \in X$. As $\bar{f}_{a'} \vee \bar{f}_{a''}$ is pre-feasible, $\bar{f}_{a'}(u) = \bar{f}_{a''}(w) = 1$. So $\phi(a) \notin \Gamma(a)$, and hence a or a^{-1} belongs to X. By symmetry we may assume $a \in X$. Then $\phi(a) = f_a(u) \leq \bar{f}_a(u) \leq \bar{f}_{a'}(u) = 1$, a contradiction.

Assume conversely that there exists a feasible function f. Let X be the set of arcs $a = (u, w)$ with the property that $\phi(a) \leq f(u)$. Then X intersects each pair in E'. For suppose that for some arc $a = (u, w)$ with $\phi(a) \notin \Gamma(a)$, one has $a \notin X$ and $a' \notin X$, that is, $\phi(a) \not\leq f(u)$ and $\phi(a^{-1}) \not\leq f(w)$. Hence $f(u)^{-1} \cdot \phi(a) \cdot f(w)$ contains $\phi(a)$ as segment, contradicting $f(u)^{-1} \cdot \phi(a) \cdot f(w) \in \Gamma(a)$.

Moreover, each pair in X belongs to E. For let $\{a', a''\}$ be a pair in X. We show that $\{a', a''\} \in E$, that is, $f' := \bar{f}_{a'} \vee \bar{f}_{a''}$ is pre-feasible. As $\bar{f}_{a'} \leq f$ and $\bar{f}_{a''} \leq f$, f' is finite and $f'(r) = 1$. Consider an arc $a = (u, w)$ with $y := f'(u)^{-1} \cdot \phi(a) \cdot f'(w) \notin \Gamma(a)$. We may assume $f'(u) = \bar{f}_{a'}(u)$ and $f'(w) = \bar{f}_{a''}(w)$ (since $f_{a'}$ and $f_{a''}$ are pre-feasible). To show $f'(u) = f'(w) = 1$, by (2) we may assume $f'(w) \not\leq \phi(a^{-1}) \cdot f'(u)$.

Suppose $f'(u) \not\leq \phi(a) \cdot f'(w)$. Then the first and the last symbol of y are equal to the first symbol of $f'(u)^{-1}$ and the last symbol of $f'(w)$, respectively. Since $f' \leq f$ this implies that y is a segment of $f(u)^{-1} \cdot \phi(a) \cdot f(w) \in \Gamma(a)$. This contradicts the hereditarity of $\Gamma(a)$ as $y \notin \Gamma(a)$.

So $f'(u) \leq \phi(a) \cdot f'(w)$. As $f_{a''}(u) \leq f'(u)$ and $y \notin \Gamma(a)$ it follows that $f_{a''}(u)^{-1} \cdot \phi(a) \cdot f_{a''}(w) \notin \Gamma(a)$. As $f_{a''}$ is pre-feasible, $f_{a''}(u) = f_{a''}(w) = 1$; so $f'(w) = 1$. Hence $f'(u) \leq \phi(a)$ and therefore, since $y \notin \Gamma(a)$, $f'(u) = 1$. ∎

Thus we have:

Theorem 1. *The cohomology feasibility problem for free groups is solvable in time bounded by a polynomial in $|A| + \sigma + k$.* ∎

7. Directed planar graphs and homologous functions

Let $D = (V, A)$ be a directed planar graph and let $(G, \cdot)$ be a group. Let R be one of the faces of D. We call two functions $\phi, \psi : A \longrightarrow G$ *R-homologous* if there exists a function $f : \mathcal{F} \longrightarrow G$ such that

(8) (i) $f(R) = 1$;

(ii) $f(F)^{-1} \cdot \phi(a) \cdot f(F') = \psi(a)$ for each arc a, where F and F' are the faces at the left-hand side and right-hand side of a, respectively.

The relation to cohomologous is direct by duality. The *dual* graph $D^* = (\mathcal{F}, A^*)$ of D has as vertex set the collection $\mathcal{F}$ of faces of D, while for any arc a of D there is an arc a^* of D^* from the face of D at the left-hand side of a to the face at the right-hand side. Define for any function ϕ on A the function ϕ^* on A^* by $\phi^*(a^*) := \phi(a)$ for each $a \in A$. Then ϕ and ψ are R-homologous (in D) if and only if ϕ^* and ψ^* are R-cohomologous (in D^*).

8. Enumerating homology types

Let input $D = (V, A), r_1, s_1, \ldots, r_k, s_k \in V$ for the k disjoint paths problem for directed planar graphs (1) be given. We may assume that D is weakly connected and that $r_1, s_1, \ldots, r_k, s_k$ are distinct and each of them is incident with exactly one arc. For any solution $\Pi = (P_1, \ldots, P_k)$ of (1) let $\phi_\Pi : A \longrightarrow G_k$ be defined by: $\phi_\Pi(a) := g_i$ if path P_i traverses a $(i = 1, \ldots, k)$, and $\phi_\Pi(a) := 1$ if a is not traversed by any of the P_i. Let R be the unbounded face. Now:

Proposition 5. *For each fixed k, we can find in polynomial time $\phi_1, \ldots, \phi_N : A \longrightarrow G_k$ such that for each solution Π of* (1)*, ϕ_Π is R-homologous to at least one of $\phi_1, \ldots, \phi_N$.*

Proof. We may assume that each vertex has degree at most 3. (We can 'decontract' vertices.) Let A' be an inclusion-wise minimal set of arcs that contains (undirected) paths between any pair of points among $r_1, s_1, \ldots, r_k, s_k$. So A' forms a tree, with exactly $2k$ vertices of degree 1 and with each vertex having degree at most 3. Hence the arcs of D' fall apart in $4k - 3$ series classes (paths).

Clearly, for each solution Π of (1) there is a unique function ψ_Π that is R-homologous to ϕ_Π such that $\psi_\Pi(a) = 1$ for each $a \notin A'$. Now ψ_Π can be determined from the numbers $|\psi_\Pi(a)|$ for $a \in A'$. Indeed, replace any arc $a \in A'$ by $|\psi_\Pi(a)|$ parallel arcs, resulting in graph (V, A''). Then there is a unique partition of A'' into pairwise noncrossing and nonself-crossing paths $Q_1, \ldots, Q_k$ such that each Q_i is an $r_i - s_i$ path not traversing two arcs in the same parallel class consecutively, except at any r_j or s_j where Q_i turns round the start or end of Q_j. (The uniqueness of the partition follows from the fact that each vertex has degree at most 3. At any vertex v the arcs (except for one if $v \in \{r_1, s_1, \ldots, r_k, s_k\}$) incident with v can be paired up in at most one way so as to possibly form paths as required.) It is easy to find this partition from the numbers $|\psi_\Pi(a)|$. We obtain ψ_Π by 'assigning' symbol g_i to Q_i.

Since $|\psi_\Pi(a)| \leq |A|$ for each $a \in A'$ and since $|\psi_\Pi(a)| = |\psi_\Pi(a')|$ if a and a' are in the same series class, there are at most $(|A| + 1)^{4k-3}$ choices for the $|\psi_\Pi(a)|$. So for fixed k this gives a polynomial-time procedure. ∎

9. The disjoint paths problem

Theorem 2. *For each fixed k, the k disjoint paths problem for directed planar graphs* (1) *is solvable in polynomial time.*

Proof. By Proposition 5 we can find in polynomial time (fixing k) a list of functions $\phi_1, \ldots, \phi_N : A \longrightarrow G$ such that for each solution Π of (1), ϕ_Π is R-homologous to at least one of the ϕ_j.

Consider the dual graph $D^* = (\mathcal{F}, A^*)$ of D. We construct the 'extended' dual graph $D^+ = (\mathcal{F}, A^+)$ by adding in each face of D^* all chords. (So D^+ need not be planar.) More precisely, for any F, F' and any (undirected) $F-F'$ path π on the boundary of any face of D^*, extend the graph with an arc a_π from F to F'. For any $\phi : A \longrightarrow G$ define $\phi^+ : A^+ \longrightarrow G$ by $\phi^+(a^*) := \phi(a)$ and $\phi^+(a_\pi) := \phi(a_1)^{\varepsilon_1} \cdot \ldots \cdot \phi(a_t)^{\varepsilon_t}$ for any path $\pi = (a_1^*)^{\varepsilon_1} \ldots (a_t^*)^{\varepsilon_t}$ with $\varepsilon_1, \ldots, \varepsilon_t \in \{+1, -1\}$. (Here $(a_i^*)^{-1}$ means that π traverses a_i^* in backward direction.) Moreover, let $\Gamma(a^*) := \{1, g_1, \ldots, g_k\}$ and $\Gamma(a_\pi) := \{1, g_1, g_1^{-1}, \ldots, g_k, g_k^{-1}\}$.

By Theorem 1 we can find, for each $j = 1, \ldots, N$ in polynomial time a function ψ that is R-cohomologous to ϕ_j^+ in D^+, with $\psi(b) \in \Gamma(b)$ for each arc b of D^+, provided that such a ψ exists. If we find one, for $i = 1, \ldots, k$ let P_i be any directed $r_i - s_i$ path traversing only arcs a satisfying $\psi(a^*) = g_i$. If such paths exist, they form a solution to the disjoint paths problem. (They are automatically disjoint, as if a and b are arcs of D both incident with a vertex v and $\psi(a^*) = g_i^{\pm 1}$ and $\psi(b^*) = g_j^{\pm 1}$ then $i = j$, since $|\psi(a_\pi)| \leq 1$ for each a_π.)

If for none of $j = 1, \ldots, N$ we find such paths we may conclude that problem (1) has no solution. For suppose $\Pi := (P_1, \ldots, P_k)$ is a solution. Then there exists a $j \in \{1, \ldots, N\}$ such that ϕ_Π and ϕ_j are R-homologous. However, for this j there exists a ψ as above, viz. $\psi := (\phi_\Pi)^+$. Moreover, for any ψ' R-cohomologous to $(\phi_\Pi)^+$ there exists for each $i = 1, \ldots, k$ a directed $r_i - s_i$ path P_i' traversing only arcs a satisfying $\psi'(a^*) = g_i$. This contradicts our assumption. ∎

Quite directly one can extend the method to the following problem:

(9) given: a directed planar graph $D = (V, A)$, k pairs $(r_1, s_1), \ldots, (r_k, s_k)$ of vertices of D, and subsets $A_1, \ldots, A_k$ of A;

find: k pairwise vertex-disjoint directed paths $P_1, \ldots, P_k$ in D, where P_i runs from r_i to s_i and uses only arcs in A_i $(i = 1, \ldots, k)$.

The polynomial-time solvability of this problem (for fixed k) follows by restricting in the proof of Theorem 2 each $\Gamma(a^*)$ to those g_i for which A_i contains a.

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