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One-Third-Integrality in the Metric Polytope

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The metric polytope $\mathcal{MP}_n$ is defined by the triangle inequalities $x_{ij} - x_{ik} - x_{jk} \leq 0$ and $x_{ij} + x_{ik} + x_{jk} \leq 2$ for $i, j, k \in V = \{1, \ldots, n\}$. A graph G is called $\frac{1}{d}$-integral if the program $\max(c^T x : x \in \mathcal{MP}_n)$ has a solution whose coordinates belong to $\{\frac{i}{d} : 0 \leq i \leq d\}$ for every objective function supported by G. For $\frac{1}{3}$-integral graphs, the ratio between the optimum solution over $\mathcal{MP}_n$ and the maximum cut in G is bounded by 4/3. We study several operations preserving $\frac{1}{d}$-integrality, in particular, the k-sum operation for $0 \leq k \leq 3$. We present several minimal forbidden minors for $\frac{1}{3}$-integrality. In particular, we characterize the $\frac{1}{3}$-integral graphs on 7 nodes. We prove that series parallel graphs are characterized by the following stronger property. The program $\max(c^T x : x \in \mathcal{MP}_n, \ell \leq x \leq u)$ has a $\frac{1}{3}$-integral optimizing vector for every objective function c supported by G and every $\frac{1}{3}$-integral bounds ℓ, u imposed on the edges of G.

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1 Introduction

We study the polytope $\mathcal{MP}_n \subseteq \Re^{\binom{n}{2}}$ given by the system of inequalities

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \tag{1}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \tag{2}$$

for all triples $\{i, j, k\} \subseteq V = \{1, \ldots, n\}$. The polytope $\mathcal{MP}_n$ is called the *metric polytope*, and the inequalities (1) and (2) are called the *triangle inequalities*. We study several properties of the metric polytope, in particular the fractionality of its vertices.

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Let us mention that a closely related object, the *metric cone* which is defined only by the inequalities (1), has been studied by Avis (1980a,b) and Grishukhin (1989).

There are several reasons that motivate our study of the metric polytope. The main motivation comes from the polyhedral description of the max-cut problem. The metric polytope is a very natural and simple relaxation of the cut polytope, since every triangle inequality is a facet of the cut polytope. We recall that the *cut polytope* $\mathcal{P}(K_n)$ is defined as the convex hull of the characteristic vectors of all cuts of the complete graph K_n.

Hence, the optimum of the linear program

$$\max c^T x,\ x \in \mathcal{MP}_n \tag{3}$$

always provides an upper bound on the optimum of

$$\max c^T x,\ x \in \mathcal{P}(K_n). \tag{4}$$

Since the max-cut problem is NP-hard, it is important to study for which objective functions c the linear program (3) provides a good approximation for (4).

A graph G is called *integral*, if the linear program (3) has an integral solution for every objective function c supported by G. Such a solution is also optimum for (4), because $\mathcal{MP}_n$ does not have any other integral vertices than those of $\mathcal{P}(K_n)$. Barahona and Mahjoub (1986) proved that a graph is integral if and only if it is not contractible to K_5.

We say that a graph G is $\frac{1}{d}$*-integral* if the optimum solution of (3) is achieved at a $\frac{1}{d}$-integral vector for every objective function c supported by G. In particular, we are interested in $\frac{1}{3}$-integral graphs, because then (3) provides a 4/3 approximation of (4) for every nonnegative objective function supported by such a graph. Moreover, $\frac{1}{3}$-integral graphs are the "first" non integral graphs, in the sense that 3 is the smallest possible denominator for a fractional vertex of the metric polytope.

We present several results on $\frac{1}{d}$-integral graphs. We show in Section 3 that this class is preserved by sum operations: the 0-sum and 1-sum of two $\frac{1}{d}$-integral graphs is $\frac{1}{d}$-integral, and the 2-sum and 3-sum, with some restriction in the latter case, of a $\frac{1}{d}$-integral graph and an integral graph is $\frac{1}{d}$-integral. In consequence, the class is closed also under subdivisions of edges, and, with some restriction, under the ΔY-operation.

The class of $\frac{1}{d}$-integral graphs is closed under minors. We present in section 4 four minimal forbidden minors for $\frac{1}{3}$-integrality. In particular, all subgraphs of K_6 are $\frac{1}{3}$-integral and we characterize the $\frac{1}{3}$-integral graphs on 7 nodes. We also include the full description of $\mathcal{MP}_n$ for $n \leq 6$.

In Section 5 we study a more constrained linear program

$$\max c^T x,\ x \in \mathcal{MP}_n,\ \ell_e \leq x_e \leq u_e\ (e \in E) \tag{5}$$

where c is supported by a graph G, and the bounds ℓ_e and u_e, $e \in E$, are $\frac{1}{3}$-integral. We prove that this program has a $\frac{1}{3}$-integral optimum for every c, ℓ, u if and only if G is a series-parallel graph.

Section 2 contains some tools and operations. We recall the description of the projection $\mathcal{S}(G)$ of the metric polytope $\mathcal{MP}_n$. We point out that a graph G is $\frac{1}{d}$-integral if and only if the polytope $\mathcal{S}(G)$ has only $\frac{1}{d}$-integral vertices. We consider some operations on the vertices of $\mathcal{S}(G)$ which are intensively used later, namely the 0- and 1-extension, switching, and the union operation. In general, a projection of a vertex of the metric polytope $\mathcal{MP}_n$ is not a vertex of $\mathcal{MP}_{n-1}$. We give a combinatorial characterization of those vertices x of $\mathcal{MP}_n$ which admit a projection to a vertex y of $\mathcal{MP}_{n-1}$. In consequence, if y has denominator d, then x has denominator either d or $2d$.

Let us briefly mention some other works on the metric polytope. The metric polytope enjoys a lot of interesting geometrical properties which have been investigated by Deza, Laurent and Poljak (1992). The metric polytope 'wraps' the cut polytope very tightly, as the following properties indicate.

- every face of dimension $d = 0, 1, 2$ (i.e. vertices, edges, and 2-faces) of the cut polytope is also a face of the metric polytope, and moreover, this property holds for most faces of dimension d up to $d = log_2 n$.

- The triangle inequalities are the facets of $\mathcal{P}(K_n)$ which are very close to its barycentrum; they are proved to be closest among all facets with coefficients in $0, \pm 1$, and they are conjectured to be the closest among all facets.

Several classes of vertices, mainly arising from graphs, have been constructed and studied by Laurent (1991). It has been confirmed that all the vertices considered in that paper are adjacent to integral vertices (cf. our conjecture in Section 5).

The relation between the linear programs (3) and (4) has been studied also by Poljak (1991) and Poljak and Tuza (1992) in the case when the objective function is given by $c_e = 1$ if $e \in E(G)$, and $c_e = 0$ otherwise where G is a graph. In the latter paper it is shown that the expected value of the ratio between (3) and (4) tends to 4/3 for a random graph with fixed edge probabilities. However, the ratio can be arbitrarily close to 2 on a class of sparse graphs, as observed in Poljak (1991).

Some notation.

Alternatively, we will use $K(V)$ to denote the complete graph on a vertex set V, and $\mathcal{MP}(V)$ the corresponding metric polytope. If $x \in \Re^E$ is a vector indexed by the edges of a graph $G = (V, E)$, we denote its coordinates alternatively by x_e, $x(e)$, x_{ij}, or $x(i, j)$, for an edge $e = (i, j)$ of G.

Let $G_t = (V_t, E_t)$ be a graph, for $t = 1, 2$. When the subgraph induced by $V_1 \cap V_2$ is a clique on k nodes in both G_1 and G_2, we define the *k-sum* of G_1 and G_2 as the graph $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

2 Operations

A vector is said to be *integral* if all its coordinates are integers. Given an integer $d \geq 2$, a vector x is called $\frac{1}{d}$*-integral* if dx is integral; if d is the smallest such integer, we also say that x has *denominator d*. A vector x is called *fully fractional* if none of its coordinates is integral. In particular, the terminology will be used in connection with the vertices of a polytope, i.e. we will speak about $\frac{1}{d}$-fractional vertices, fully fractional vertices, integral vertices etc. We say that a vector $c \in \Re^{\binom{n}{2}}$ is *supported* by a graph $G = (V, E)$ (or, with support in G) if $c_{ij} = 0$ for all $ij \notin E$.

DEFINITION 2.1 *A graph $G = (V, E)$ is called* $\frac{1}{d}$-integral *if, for every objective function c supported by G the program* $\max(c^T x : x \in \mathcal{MP}_n)$ *admits a $\frac{1}{d}$-integral optimizing vector.*

2.1 The cycle relaxation of the cut polytope.

Let $G = (V, E)$ be a graph with node set V and edge set E. Given a subset S of V, $\delta_G(S)$ denotes the cut in G determined by S, i.e. the set $\delta_G(S) = \{ij \in E : i \in S, j \notin S\}$. The *cut polytope* $\mathcal{P}(G) \subset \Re^E$ is defined as the convex hull of the incidence vectors of the cuts of G. The following inequalities are valid for the cut polytope $\mathcal{P}(G)$ (Barahona and Mahjoub 1986):

$$x(F) - x(C - F) \leq |F| - 1 \text{ for } F \subseteq C, |F| \text{ odd }, C \text{ cycle of } G \quad (6)$$

$$0 \leq x_e \leq 1 \text{ for } e \in E \quad (7)$$

The polytope defined by the inequalities (6) and (7) is denoted by $\mathcal{S}(G)$ and called the *cycle relaxation* of the cut polytope $\mathcal{P}(G)$.

It is easy to see that the non redundant inequalities (6) are for C chordless cycle of G and the non redundant inequalities (7) are for the edges e that do not belong to any triangle of G. In particular, the polytope $\mathcal{S}(K_n)$ coincides with the metric polytope $\mathcal{MP}_n$. In fact, in general, the polytope $\mathcal{S}(G)$ is the projection of $\mathcal{MP}_n$ on the space $\Re^E$ (Barahona 1983). More precisely, the following holds.

LEMMA 2.2 *Let $G = (V, E)$ be a graph and let e be an edge of $K(V)$ which does not belong to G. Let $G + e$ denote the graph obtained by adding the edge e to G.*

(i) If $x \in \mathcal{MP}(V)$, then the projection x_E of x on $\Re^E$ belongs to $\mathcal{S}(G)$.

(ii) If $y \in \mathcal{S}(G)$, there exists $x \in \mathcal{S}(G+e)$ whose projection x_E on $\Re^E$ coincides with y. Moreover, if y is a $\frac{1}{d}$-integral vertex of $\mathcal{S}(G)$, then there exists such x which is a $\frac{1}{d}$-integral vertex of $\mathcal{S}(G+e)$.

PROOF. *(i)* can be easily verified. We show that *(ii)* holds. Let $y \in \mathcal{S}(G)$. In order to define an extension x of y to $\mathcal{S}(G+e)$, we set $x_f = y_f$ for all edges f of G and we choose the value x_e on the new edge e satisfying:

(i) $0 \leq x_e \leq 1$

(ii) $x_e \leq |F| - 1 + y(C-F) - y(F-e)$ for all $e \in F \subseteq C$, where C is a cycle in $G+e$ and $|F|$ is odd

(iii) $x_e \geq -|F'| + 1 - y(C' - F' \cup \{e\}) + y(F')$ for all $F' \subseteq C'$, $e \in C' - F'$, where C' is a cycle in $G+e$ and $|F'|$ is odd.

The fact that such x_e exists follows from the fact that $y \in \mathcal{S}(G)$. If y is a vertex of $\mathcal{S}(G)$ and if we choose x_e satisfying one of the above inequalities (i)-(iii) as equality, then x is a vertex of $\mathcal{S}(G+e)$. Moreover, if y is $\frac{1}{d}$-integral, then x_e is clearly $\frac{1}{d}$-integral too. □

COROLLARY 2.3 *A graph G is $\frac{1}{d}$-integral if and only if all the vertices of the polytope $\mathcal{S}(G)$ are $\frac{1}{d}$-integral.*

PROOF. The proof is based on Lemma 2.2 and the following observations:

- $\max(c^T z : z \in \mathcal{MP}_n) = \max(c^T z : z \in \mathcal{S}(G))$,
- If $x \in \mathcal{MP}_n$ is an optimizing vector for $\max(c^T z : z \in \mathcal{MP}_n)$, then its projection x_E on $\Re^E$ optimizes $\max(c^T z : z \in \mathcal{S}(G))$,
- If $x \in \mathcal{S}(G)$ optimizes $\max(c^T z : z \in \mathcal{S}(G))$, then any extension of x to $\mathcal{MP}_n$ optimizes $\max(c^T z : z \in \mathcal{MP}_n)$, where c is supported by G.

□

We present two properties of $\frac{1}{d}$-integral graphs. The proofs of Propositions 2.4 and 2.5 will be given in subsection 2.3 as an application of the extension operation.

PROPOSITION 2.4 *If G is $\frac{1}{d}$-integral, then any minor of G is also $\frac{1}{d}$-integral.*

PROPOSITION 2.5 *Assume G is $\frac{1}{3}$-integral. Then, for every objective $c \in \Re^E_+$,*

$$\max(c^T x : x \in \mathcal{S}(G)) \leq \frac{4}{3} mc(G, c),$$

where $mc(G, c)$ denotes the maximum cut of the graph G with the weights c.

2.2 The switching operation

Given a cut $\delta_G(S)$, we define the *switching reflection* $r_{\delta_G(S)}$ of $\Re^E$ by $y = r_{\delta_G(S)}(x)$, where: $y_{ij} = 1 - x_{ij}$ if $ij \in \delta_G(S)$ and $y_{ij} = x_{ij}$ if $ij \in E - \delta_G(S)$. The switching reflection preserves the cut polytope (Barahona and Mahjoub 1986); indeed, $r_{\delta_G(S)}$ maps the cut $\delta_G(T)$ on the cut $\delta_G(S\Delta T)$. In particular, the switching reflection $r_{\delta_G(S)}$ preserves faces and facets of the cut polytope $\mathcal{P}(G)$. Given $v \in \Re^E$, $v_0 \in \Re$, suppose that the inequality $v^T x \leq v_0$ defines a face of $\mathcal{P}(G)$. Define $v^S \in \Re^{\binom{n}{2}}$ by $v^S_{ij} = -v_{ij}$ if $ij \in \delta(S)$ and $v^S_{ij} = v_{ij}$ otherwise. By applying the switching reflection $r_{\delta_G(S)}$, we obtain the inequality $(v^S)^T x \leq v_0 - \sum_{e \in \delta_G(S)} v_e$ which defines a face of the same rank of $\mathcal{P}(G)$. Clearly, the inequalities (6) are preserved under any switching. Therefore, the switching reflections preserve the polytope $\mathcal{S}(G)$. Thus we have

LEMMA 2.6 *If $x \in \mathcal{S}(G)$, then $y = r_{\delta(S)}(x) \in \mathcal{S}(G)$; moreover, y is a vertex of $\mathcal{S}(G)$ whenever x is a vertex of $\mathcal{S}(G)$.*

□

In the case of the complete graph $G = K_n$, it was proved that the switching reflections together with the permutations of the nodes are the only symmetries of the cut polytope $\mathcal{P}(K_n)$ (Deza, Grishukhin and Laurent 1991) and of the metric polytope (Laurent 1991).

2.3 Extension and projection of vertices in $\mathcal{S}(G)$

If $x \in \mathcal{S}(G)$ and $G' = (V, E')$ is a subgraph of G, i.e. $E' \subseteq E$, then the projection $x_{E'}$ of x on $\Re^{E'}$ belongs to $\mathcal{S}(G')$; we also say that x is an extension of $x_{E'}$.

In general, vertices are not preserved by projection. However, a nice feature of the polytope $\mathcal{S}(G)$ is that, essentially, we may always assume to deal with fully fractional vertices, since a vertex of $\mathcal{S}(G)$ with some coordinate 0 or 1 is the extension of a vertex x' of $\mathcal{S}(G')$, where G' comes from G by contracting the edge corresponding to the integral coordinate of x.

Let $G = (V, E)$ be defined on the n nodes $1, \ldots, n$ and suppose that $e = (1, n)$ is an edge of G. Let $G' = (V', E')$ denote the graph obtained by contracting the edge e in G; so, $V' = V - \{n\}$. Let V_1, V_n denote, respectively, the set of nodes of $V - \{1, n\}$ that are adjacent to the node 1, n. Then, $E' = E - \{(n, i) : i \in V_n\} \cup \{(1, i) : i \in V_n - V_1\}$. Given $x' \in \Re^{E'}$, we define its *0-extension* $x \in \Re^E$ by:

$$x_{ij} = \begin{cases} x'_{1j} & \text{for } i = 1,\ j \in V_1 \\ x'_{1j} & \text{for } i = n,\ j \in V_n \\ 0 & \text{for } i = 1,\ j = n \\ x'_{ij} & \text{elsewhere} \end{cases} \tag{8}$$

Conversely, if $x \in \mathcal{S}(G)$ with $x_{1n} = 0$, then, by the triangle inequalities (1), $x_{1j} = x_{nj}$ holds for all $j \in V_1 \cap V_n$. Hence, defining $x' \in \Re^{E'}$ as the projection of x on E', we have that x is the 0-extension of x' as defined by the above relation (8).

Similarly, we define the *1-extension* y of x' by

$$y_{ij} = \begin{cases} x'_{1j} & \text{for } i = 1,\ j \in V_1 \\ 1 - x'_{1j} & \text{for } i = n,\ j \in V_n \\ 1 & \text{for } i = 1,\ j = n \\ x'_{ij} & \text{elsewhere.} \end{cases} \tag{9}$$

Moreover, if $y \in \mathcal{S}(G)$ with $y_{1n} = 1$, then y is the 1-extension of its projection x' on E'.

PROPOSITION 2.7 *Let $x \in \Re^E$ be the 0-extension of $x' \in \Re^{E'}$, i.e. x, x' satisfy (8). Then, $x \in \mathcal{S}(G)$ if and only if $x' \in \mathcal{S}(G')$; moreover, x is a vertex of $\mathcal{S}(G)$ if and only if x' is a vertex of $\mathcal{S}(G')$. The same holds also for x' and its 1-extension y.*

PROOF. It is easy to check that $x \in \mathcal{S}(G)$ if and only if $x' \in \mathcal{S}(G')$.

Let x' be a vertex of $\mathcal{S}(G')$. Let $\mathcal{B}'$ be a family of $|E'|$ linearly independent inequalities (6) and (7) that are satisfied at equality by x'. The inequalities $x_{1n} \geq 0$ and $x_{1j} - x_{1n} - x_{jn} \leq 0$, $2 \leq j \leq n-1$, are satisfied as equality by x. Together with $\mathcal{B}'$, we obtain a set of $|E|$ equalities for x which are linearly independent. Therefore, x is a vertex of $\mathcal{S}(G)$.

Assume now that x is a vertex of $\mathcal{S}(G)$. Let $\mathcal{B}$ be a family of $|E|$ linearly independent equalities chosen among (6) and (7) satisfied by x. We can suppose that $\mathcal{B}$ contains the equalities $x_{1n} = 0$ and $x_{1j} - x_{1n} - x_{nj} = 0$ for $j \in V_1 \cap V_n$. Then, the remaining equalities of $\mathcal{B}$ do not use the edge $(1, n)$; hence, they yield equalities for x'. Therefore, x' is a vertex of $\mathcal{S}(G')$.

The statement about y follows by applying switching and using Lemma 2.6.

□

As a consequence, for many questions, we may restrict ourselves to fully fractional vertices. An easy application of the above proposition is that $\mathcal{S}(G)$ has no fractional $\frac{1}{2}$-integral vertices.

COROLLARY 2.8 *The metric polytope has no fractional $\frac{1}{2}$-integral vertices.*

PROOF. If $\mathcal{MP}_n$ has a fractional $\frac{1}{2}$-integral vertex, then there would exist a vertex of $\mathcal{MP}_m$, for some $m \leq n$, with all coordinates equal to $\frac{1}{2}$. But such vector satisfies none of the inequalities (6) at equality. □

Two other applications are the statements formulated in Propositions 2.4 and 2.5, that we prove now.

PROOF OF PROPOSITION 2.4. Let G be a $\frac{1}{d}$-integral graph and let $e = (1, n)$ be an edge of G.

It is obvious that the graph $G - e$ obtained by deleting the edge e is $\frac{1}{d}$-integral.

We show that the graph G/e obtained by contracting the edge e is $\frac{1}{d}$-integral. We take the same notation as above for V_1, V_n and $G' = G/e$. Let w' be an objective function with support in G'. Define the objective w with support in G by:

$$w_{ij} = \begin{cases} w'_{1j} & \text{for } i = 1,\ j \in V_1 \\ w'_{1j} & \text{for } i = n,\ j \in V_n \\ -M & \text{for } i = 1,\ j = n \\ w'_{ij} & \text{elsewhere} \end{cases} \tag{10}$$

By assumption, the linear program $\max(w^T x : x \in \mathcal{MP}_n)$ admits a $\frac{1}{d}$-integral optimizing vector x. If we choose the constant M large enough, then $x_{1n} = 0$. Let x' denote the projection of x on $\Re^{E'}$. Hence, x' is $\frac{1}{d}$-integral. It is easy to check that x' is an optimizing vector for the linear program $\max(w'^T z : z \in \mathcal{MP}_{n-1})$. Therefore, the graph G' is $\frac{1}{d}$-integral. □

PROOF OF PROPOSITION 2.5. The proof is by induction on n, the number of nodes of G. The statement holds trivially if $n \leq 2$. Let G be a $\frac{1}{3}$-integral graph on $n \geq 3$ nodes and let c be a non-negative objective function supported by G. Let x be a vertex of $\mathcal{S}(G)$ which optimizes the program $\max(c^T x : x \in \mathcal{S}(G))$.

If x is fully fractional, then $x_e = \frac{2}{3}$ for all edges. Therefore, $c^T x = \frac{2}{3} \sum_{e \in E} c_e$. On the other hand, a trivial lower bound for the maximum cut in G is $mc(G, c) \geq \frac{1}{2} \sum_{e \in E} c_e$. Therefore, Proposition 2.5 holds.

Suppose that $x_e = 0$ for some edge $e = (1, n)$. Let x' denote the projection of x on $\Re^{E'}$, where E' is the edge set of $G' = G/e$. Consider the objective $c' \in \Re^{E'}$ defined by:

$$c'_{ij} = \begin{cases} c_{1j} & \text{for } i = 1,\ j \in V_1 - V_n \\ c_{nj} & \text{for } i = n,\ j \in V_n - V_1 \\ c_{1j} + c_{nj} & \text{for } i = 1,\ j \in V_1 \cap V_n \\ c_{ij} & \text{elsewhere} \end{cases} \tag{11}$$

It is easy to see that x' optimizes the objective function c' over $\mathcal{S}(G')$. By the induction hypothesis, the following inequality holds:

$$\max(c'^T z : z \in \mathcal{S}(G')) \leq \frac{4}{3} mc(G', c').$$

But, $mc(G',c') \le mc(G,c)$ holds. Therefore, Proposition 2.5 holds.

Suppose now that $x_f \neq 0$ for all edges f of G, but $x_e = 1$ for some edge $e = (1,n)$. Let $G' = G - \{1,n\}$ with edge set E'. Let c', x' denote the projection of c, x on $\Re^{E'}$, respectively. Since G' is $\frac{1}{3}$-integral, by the induction hypothesis, we have

$$\max(c'^T z : z \in \mathcal{S}(G')) \le \frac{4}{3} mc(G',c').$$

This implies $c'^T x' \le \frac{4}{3} mc(G',c')$. Let $\delta_{G'}(S)$ be an optimizing cut in G' for the weights c'. We have $mc(G,c) \ge \frac{1}{2}(c^T \chi^{\delta_G(S\cup\{1\})} + \chi^{\delta_G(S\cup\{n\})}) = mc(G',c') + c_{1n} + \frac{1}{2}\sum_{u\neq 1,n}(c_{1u}+c_{nu})$. But, x_{1u}, $x_{nu} \le \frac{2}{3}$ for all nodes $u \neq 1,n$ and $mc(G',c') \ge \frac{3}{4}c'^T x'$. Therefore, $mc(G,c) \ge \frac{3}{4}c'^T x' + c_{1n} + \frac{3}{4}\sum_{u\neq 1,n}(c_{1u}x_{1u} + c_{nu}x_{nu})$. We deduce that $mc(G,c) \ge \frac{3}{4}c^T x$. Therefore, Proposition 2.5 holds. □

Finally we observe how a new vertex of the cycle polytope $\mathcal{S}(G)$ can be constructed by 'gluing' together two given vertices of smaller cycle polytopes.

Let $G_i = (V_i, E_i)$ be a graph for $i = 1,2$ and assume that the subgraph induced by $V_1 \cap V_2$ is a clique on $k = |V_1 \cap V_2|$ nodes in both G_1 and G_2. Let $G = (V,E)$ denote the k-sum of G_1 and G_2. Let $x_i \in \Re^{E_i}$, $i = 1,2$, such that x_1 and x_2 coincide on the edges of the common clique $K(V_1 \cap V_2)$. We can define $x \in \Re^E$ by concatenation of x_1 and x_2, i.e. $x(e) = x_i(e)$ for $e \in E_i$, $i = 1,2$. The vector x is called the *k-union* of x_1 and x_2.

PROPOSITION 2.9 *We have*

(i) $x \in \mathcal{S}(G)$ if and only if $x_i \in \mathcal{S}(G_i)$ for $i = 1,2$.

(ii) If x_i is a vertex of $\mathcal{S}(G_i)$ for $i = 1,2$, then x is a vertex of $\mathcal{S}(G)$.

PROOF. The part (i) is clear. We verify (ii). Let x_i be a vertex of $\mathcal{S}(G_i)$, $i = 1,2$. We show that x is a vertex of $\mathcal{S}(G)$. Assume $x = \alpha y + (1-\alpha)z$ for some $0 < \alpha < 1$ and $y,z \in \mathcal{S}(G)$. Denote by y_i, z_i the projection of y,z on E_i for $i = 1,2$. We obtain that $x_i = \alpha y_i + (1-\alpha)z_i$, implying that $x_i = y_i = z_i$ for $i = 1,2$. Hence $x = y = z$ holds, yielding that x is a vertex. □

In particular, if x_i is a vertex of the metric polytope $\mathcal{MP}(V_i)$, for $i = 1,2$, such that x_1 and x_2 coincide on the edges of $K(V_1 \cap V_2)$, then their k-union x is a vertex of $\mathcal{S}(G)$, G denoting the k-sum of $K(V_1)$ and $K(V_2)$. By Lemma 2.2, x can be extended to a vertex y of the metric polytope $\mathcal{MP}(V_1 \cup V_2)$. Moreover, if x_1 and x_2 are $\frac{1}{d}$-integral, then y can be chosen $\frac{1}{d}$-integral. Such y is a common extension of both x_1 and x_2.

2.4 Extension and projection in the metric polytope

It is convenient to specify the extension operation defined previously for $\mathcal{S}(G)$ to the metric polytope. Actually, it was already considered in (Laurent 1991).

Given $x' \in \Re^{\binom{n}{2}}$, we define its *0-extension* x and *1-extension* y, $x, y \in \Re^{\binom{n+1}{2}}$, as follows.

$$x_{ij} = \begin{cases} x'_{ij} & \text{for } 1 \le i < j \le n \\ x'_{1j} & \text{for } i = 1,\ 2 \le j \le n \\ 0 & \text{for } i = 1,\ j = n+1. \end{cases}$$

$$y_{ij} = \begin{cases} x'_{ij} & \text{for } 1 \le i < j \le n \\ 1 - x'_{1j} & \text{for } i = 1,\ 2 \le j \le n \\ 1 & \text{for } i = 1,\ j = n+1. \end{cases}$$

Clearly, $y = r_{\delta(\{n+1\})}(x)$ and the extension of an integral vertex is again an integral vertex. By Proposition 2.7, x' is a vertex of $\mathcal{MP}_n$ if and only if x and y are vertices of $\mathcal{MP}_{n+1}$.

So, the 0- or 1-extension of any vertex is always a vertex; we shall refer to such an extension as a *trivial* extension. We are interested in studying some conditions for the existence of nontrivial extensions.

In the following we represent every triangle inequality $a^T x \le \alpha$ from the system (1), (2) by the vector a. Given a family $\mathcal{B}$ of triangle inequalities and a node $i \in V$, define $\mathcal{B}(i)$ as the set of triangle inequalities of $\mathcal{B}$ that go through node i. Clearly, if x is a vertex of $\mathcal{MP}_n$ and if $\mathcal{B}$ is a base of triangle equalities for x (i.e. $\mathcal{B}$ consists of $\binom{n}{2}$ linearly independent triangle equalities satisfied by x), then $|\mathcal{B}(i)| \ge n-1$ holds for all $i \in V$.

LEMMA 2.10 *Let x be a vertex of $\mathcal{MP}_n$. Then, x admits a projection that is a vertex of $\mathcal{MP}_{n-1}$ if and only if there exists a base $\mathcal{B}$ of triangle equalities for x and a node $i \in V$ such that $|\mathcal{B}(i)| = n-1$.*

Let us introduce some definitions. Let $\mathcal{F}$ be a family of triangle inequalities on V all going through a given node, say n. We define the graph $G(\mathcal{F})$ with $\mathcal{F}$ as nodeset where two triangle inequalities are adjacent if they share a common non zero coordinate. We also define the matrix $M(\mathcal{F})$ as the $k \times k$ matrix ($k = |\mathcal{F}|$) whose rows are the projections on the set $\{(i,n) : i \in V$ covered by some triangle of $\mathcal{F}\}$ of the triangle inequalities of $\mathcal{F}$. Assume that $G(\mathcal{F})$ is a cycle, say $G(\mathcal{F}) = (T_1, T_2, \ldots, T_k)$, where T_i is based on the triple (v_i, v_{i+1}, n) for some nodes $v_1, v_2, \ldots, v_k \in V - \{n\}$ (setting $v_{k+1} = v_1$). Denote by a_i, b_i the component of T_i on the edge (v_i, n), (v_{i+1}, n), respectively. Then, the determinant of $M(\mathcal{F})$ is equal to $a_1 a_2 \ldots a_k - (-1)^k b_1 b_2 \ldots b_k$ which takes value 0, -2 or 2. We call $\mathcal{F}$ an *odd signed cycle* (resp. *even signed cycle*) if $G(\mathcal{F})$ is a cycle and $det(M(\mathcal{F})) \ne 0$ (resp.

$det(M(\mathcal{F})) = 0)$. Hence, when all triangle inequalities of $\mathcal{F}$ are of type (2), then $\mathcal{F}$ is an odd signed (resp. even signed) cycle if and only if $G(\mathcal{F})$ is an odd (resp. even) cycle.

We give a combinatorial characterization of those vertices of $\mathcal{MP}_n$ admitting a projection which is a vertex of $\mathcal{MP}_{n-1}$.

THEOREM 2.11 *Given $x \in \mathcal{MP}_n$, let $\mathcal{B}$ be a family of triangle equalities satisfied by x and assume that $|\mathcal{B}| = \binom{n}{2}$ and $|\mathcal{B}(n)| = n-1$. The following statements are equivalent.*

(i) $\mathcal{B}$ is linearly independent

(ii) $\mathcal{B} - \mathcal{B}(n)$ is linearly independent and $\mathcal{B}(n) - \mathcal{F}_n$ is the disjoint union of odd signed cycles, where $\mathcal{F}_n$ is constructed by the following iterative procedure.

Set $F_n = \{(i,n) : 1 \le i \le n-1\}$ and $X_n = \mathcal{F}_n = \emptyset$ initially. If there exists an edge $e \in F_n - X_n$ at which a unique triangle equality of $\mathcal{B}(n) - \mathcal{F}_n$ has non zero coordinate, then replace X_n by $X_n \cup \{e\}$ and $\mathcal{F}_n$ by $\mathcal{F}_n \cup \{T\}$. Stop when no such edge exists.

PROOF. Set $E_n = \{(i,j) : 1 \le i < j \le n\}$ and $E_{n-1} = E_n - F_n$. Let M denote the incidence matrix of $\mathcal{B}$, let M_2 denote the $\binom{n-1}{2} \times \binom{n-1}{2}$ matrix whose rows are the projections on E_{n-1} of the triangle equalities from $\mathcal{B} - \mathcal{B}(n)$ and let M_1 denote the $n-1 \times n-1$ matrix whose rows are the projections on F_n of the triangle equalities from $\mathcal{B}(n)$. Then, M is nonsingular if and only if M_1, M_2 are nonsingular and M_2 is nonsingular if and only if $\mathcal{B} - \mathcal{B}(n)$ is linearly independent.

Assume first that (ii) holds. Then, $\mathcal{B}(n) = \mathcal{F}_n \cup \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_p$ where each $\mathcal{B}_i$ is an odd signed cycle. Then, M_1 has a block diagonal form with one block for each of $\mathcal{F}_n$,$\mathcal{B}_i$. By construction of $\mathcal{F}_n$, the corresponding block is triangular with all ones on its diagonal and so is nonsingular. Each block corresponding to $\mathcal{B}_i$ is nonsingular since $\mathcal{B}_i$ is an odd signed cycle. Hence, M_1 is nonsingular.

Assume that (i) holds. By construction of $\mathcal{F}_n$, for every edge of $F_n - X_n$, at least two triangle equalities from $\mathcal{B}(n) - \mathcal{F}_n$ have a nonzero coordinate indexed by this edge. A simple counting argument shows that, at each edge of $F_n - X_n$, exactly two triangle equalities of $\mathcal{B}(n) - \mathcal{F}_n$ have a nonzero coordinate. Hence, the graph $G(\mathcal{B}(n) - \mathcal{F}_n)$ is a disjoint union of cycles and each cycle must be odd signed, since M_1 is nonsingular. □

As a consequense of the proof of Theorem (2.11), we obtain that:

PROPOSITION 2.12 *Let x be a vertex of $\mathcal{MP}_n$ and suppose that x is fractional with denominator d. Then, any extension of x which is a vertex is fractional with denominator d or $2d$.*

EXAMPLE 2.13 *We give an example of a vertex x that is $(\frac{1}{3}, \frac{2}{3})$-valued and admits an extension that is fractional with denominator 6. Take $x \in \mathcal{MP}_8$ with $x_{ij} = \frac{2}{3}$ except $x_{ij} = \frac{1}{3}$ for the following pairs (1,2),(1,4),(1,6), (2,3), (4,5), (6,7), (i,8) for $1 \leq i \leq 7$. Take $y \in \mathcal{MP}_9$ the extension of x defined by $y_{i9} = \frac{1}{6}$ for $i = 2,3$, $y_{i9} = \frac{3}{6}$ for $i = 1,4,5,8$ and $y_{i9} = \frac{5}{6}$ for $i = 6,7$. It can be checked that both x, y are vertices.*

EXAMPLE 2.14 *We give an example of a vertex of $\mathcal{MP}_n$ whose projections are all not vertices of $\mathcal{MP}_{n-1}$. Take $x \in \mathcal{MP}_7$ defined by $x_{ij} = \frac{1}{3}$ except $x_{ij} = \frac{2}{3}$ for the pairs (2,3),(2,4),(3,4),(3,6),(5,6),(5,7),(6,7). It can be checked that x is a vertex of $\mathcal{MP}_7$ while any projection of x is not a vertex of $\mathcal{MP}_6$.*

On the other hand, for any fractional vertex x with denominator d, we can construct a nontrivial extension of x that is a vertex with the same denominator d. Assume x is a vertex of $\mathcal{MP}_n$; define the extension $y \in \mathcal{MP}_{n+1}$ of x by $y_{i,n+1} = x_{1i}$ for $2 \leq i \leq n$ and $y_{1,n+1} = \min_{2 \leq i \leq n}(2x_{1i}, 2 - 2x_{1i})$. Then, y is a vertex of $\mathcal{MP}_{n+1}$ with same denominator as x. (y is obtained from x by duplication of a node, see subsection 2.5.)

However, not every vertex with denominator d admits an extension with denominator $2d$. (Every vertex of $\mathcal{MP}_5$ is such a counterexample, because $\mathcal{MP}_6$ has no vertex with denominator 6.)

Let $x \in \mathcal{MP}_n$ such that some projection of x is a vertex of $\mathcal{MP}_{n-1}$. In general, x is not a vertex, but we have the following result.

LEMMA 2.15 *Take $x \in \mathcal{MP}_n$. If there exist three distinct nodes $i, j, k \in V$ such that the projections x^i, x^j, x^k of x on the sets $V - \{i\}, V - \{j\}, V - \{k\}$, respectively, are vertices of $\mathcal{MP}_{n-1}$, then x is a vertex of $\mathcal{MP}_n$.*

PROOF. Let $0 \leq \alpha \leq 1$ and $y, z \in \mathcal{MP}_n$ such that $x = \alpha y + (1 - \alpha)z$. We show that $y = z = x$ holds. Let $h \in \{i, j, k\}$. Denote by y^h, z^h, respectively, the projection of y, z on $V - \{h\}$. Then, $x^h = \alpha y^h + (1 - \alpha)z^h$ holds, implying that $y^h = z^h = x^h$. We deduce that $y = z = x$ holds. □

3 Sums with integral graphs

In this section, we study $\frac{1}{d}$-integrality with respect to the k-sum operation for graphs; d is an integer, $d \geq 3$. We prove the following results.

- $\frac{1}{d}$-integrality is preserved by 0- and 1-sums.

- The 2-sum of a $\frac{1}{d}$-integral graph and an integral graph is $\frac{1}{d}$-integral.
- The 3-sum of an integral graph and a rich $\frac{1}{d}$-integral graph (for the definition of a rich graph, see Definition 3.5 below) is $\frac{1}{d}$-integral.

THEOREM 3.1 *The 0- and 1-sum operations preserve $\frac{1}{d}$-integrality.*

PROOF. Let $G_i = (V_i, E_i)$ be a $\frac{1}{d}$-integral graph, for $i = 1, 2$. We suppose first that G_1 and G_2 have no common node and let $G = (V, E)$ denote their 0-sum. Let x be a vertex of $\mathcal{S}(G)$ and let x_{E_i} denote the projection of x on $\Re^{E_i}$ for $i = 1, 2$. Let $\mathcal{B}$ be a system of $|E|$ linearly independent inequalities from the system (6), (7) that are satisfied at equality by x. Let $\mathcal{B}_i$ denote the susbset of $\mathcal{B}$ consisting of the equations supported by G_i, for $i = 1, 2$. Then, $|B| = |E| = |\mathcal{B}_1| + |\mathcal{B}_2| = |E_1| + |E_2|$, implying that $|\mathcal{B}_i| = |E_i|$ for $i = 1, 2$. Therefore, x_i is a vertex of $\mathcal{S}(G_i)$ and thus is $\frac{1}{d}$-integral, for $i = 1, 2$. This shows that x is $\frac{1}{d}$-integral.

The proof is identical when G_1 and G_2 have one node in common. □

THEOREM 3.2 *Let G_1 and G_2 be two graphs having an edge in common. If G_1 is $\frac{1}{d}$-integral and G_2 is integral, then their 2-sum is $\frac{1}{d}$-integral.*

PROOF. Take $G_i = (V_i, E_i)$, for $i = 1, 2$, and let f denote the common edge of G_1 and G_2. Let $G = (V, E)$ denote the 2-sum of G_1 and G_2. We show that G is $\frac{1}{d}$-integral, i.e. that every vertex of $\mathcal{S}(G)$ is $\frac{1}{d}$-integral.

Let x be a vertex of $\mathcal{S}(G)$ and let x_{E_i} denote the projection of x on $\Re^{E_i}$, for $i = 1, 2$. If $x_f = 0$ or 1, then we can contract the edge f. Namely, then x is a trivial extension of a vertex y of $\mathcal{S}(G/f)$. But, the graph G/f can be seen as the 0-sum of the graphs G_1/f and G_2/f. By Theorem 3.1, y is $\frac{1}{d}$-integral. Therefore, x is $\frac{1}{d}$-integral.

We can now assume that $x_f \neq 0, 1$. Let $\mathcal{B}$ be a family of $|E|$ linearly independent equalities from the system (6), (7) satisfied by x. Let $\mathcal{B}_i$ denote the subset of $\mathcal{B}$ consisting of those equalities that are supported by G_i, for $i = 1, 2$. Since $0 < x_f < 1$, the families $\mathcal{B}_1$ and $\mathcal{B}_2$ are disjoint and, thus, $|E| = |\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| = |E_1| + |E_2| - 1$. Therefore, $|E_i| - 1 \leq |\mathcal{B}_i| \leq |E_i|$, for $i = 1, 2$. We distinguish two cases.

First, suppose that $|\mathcal{B}_2| = |E_2|$. Then, x_{E_2} is a vertex of $\mathcal{S}(G_2)$ and, thus, since G_2 is integral, x_{E_2} is 0,1-valued, in contradiction with the assumption that $x_f \neq 0, 1$.

Suppose now that $|\mathcal{B}_2| = |E_2| - 1$. Then, $\mathcal{B}_1 = |E_1|$; hence, x_{E_1} is a vertex of $\mathcal{S}(G_1)$ and, thus, is $\frac{1}{d}$-integral. On the other hand, since it satisfies $|E_2| - 1$ linearly independent equalities, x_{E_2} can be written as the convex combination of two vertices of $\mathcal{S}(G_2)$. Hence, $x_{E_2} = \alpha\chi^{\delta(A)} + \beta\chi^{\delta(B)}$, where $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and $\delta(A)$, $\delta(B)$ are two cuts in G_2. Then, $x_f = \alpha$, or $x_f = \beta$; hence, α, β and, thus x_{E_2}, are $\frac{1}{d}$-integral. Therefore, x is $\frac{1}{d}$-integral. □

COROLLARY 3.3 *Every subdivision of a $\frac{1}{d}$-integral graph is $\frac{1}{d}$-integral.*

PROOF. Let e be an edge of G which should be subdivided. Consider the 2-sum of G with a triangle along the edge e. Then delete the edge e from the 2-sum. The resulting graph is the required subdivision of G. It is $\frac{1}{d}$-integral by Theorem 3.2 and Proposition 2.4. □

REMARK 3.4 *The 2-sum operation does not preserve $\frac{1}{d}$-integrality in general.*

As counterexample, consider the graph G obtained by taking the 2-sum of two copies of K_5; K_5 is $\frac{1}{3}$-integral, but we construct a $\frac{1}{6}$-integral vertex of $\mathcal{S}(G)$.

We use the following notation. If $K_{S,T}$ denotes the complete bipartite graph with node sets S, T, then $x(K_{S,T})$ takes the value $\frac{1}{3}$ on the edges of $K_{S,T}$ and the value $\frac{2}{3}$ on the other edges. Recall that $x(K_{S,T})$ is a vertex of $\mathcal{MP}_n$, $n = |S| + |T| \geq 5$ (Avis 1980a).

Consider two copies G_1 and G_2 of K_5 defined, respectively, on the node sets $\{1,2,3,4,5\}$ and $\{1,2,6,7,8\}$. G is their 2-sum along the edge $(1,2)$. We define $y \in \mathcal{S}(G)$ as follows: its projection on the edge set of G_1 is $x(K_{\{1,5\},\{2,3,4\}})$ and its projection on the edge set of G_2 is $\frac{1}{2}(x(K_{\{1,2,8\},\{6,7\}}) + \chi^{\delta(\{1,2,6\})})$. So, y takes the values $\frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}$. It is easy to check that y is a vertex of $\mathcal{S}(G)$. Indeed, there are altogether 19 triangle equalities satisfied by y (10 on G_1 and 9 on G_2) and they are linearly independent. □

We say that a triangle (i,j,k) supports a triangle equality for a vector x if at least one of the four inequalities (1) or (2) is satisfied as equality by x.

DEFINITION 3.5 *Call a graph G rich if, for every vertex x of $\mathcal{S}(G)$, each triangle of G supports at least one triangle equality for x.*

Clearly, every subgraph of a rich graph is rich. For example, K_6 is rich (see section 5). Therefore, every graph on at most 6 nodes is rich.

Also, every integral graph is rich (in fact, for every vertex, each triangle supports three triangle equalities !).

Note that a $\frac{1}{3}$-integral graph G is rich if no vertex x of $\mathcal{S}(G)$ satisfies $x_{ij} = x_{ik} = x_{jk} = \frac{1}{3}$, or $x_{ij} = x_{ik} = \frac{2}{3}, x_{jk} = \frac{1}{3}$, for some triangle (i,j,k) of G.

REMARK 3.6 *It follows easily from the proofs of Theorems 3.1 and 3.2 that the 0- and 1-sums of rich $\frac{1}{d}$-integral graphs are $\frac{1}{d}$-integral and rich, while the 2-sum of a rich $\frac{1}{d}$-integral graph and an integral graph is $\frac{1}{d}$-integral and rich.*

We see below that Theorem 3.2 can be extended to the 3-sum case, if we make the additional assumption that the graphs are rich.

THEOREM 3.7 *Let G_1 and G_2 be two graphs having a triangle in common. If G_1 is $\frac{1}{d}$-integral and rich, and if G_2 is integral, then their 3-sum is $\frac{1}{d}$-integral and, moreover, rich.*

PROOF. Take $G_i = (V_i, E_i)$, for $i = 1, 2$, and denote by $\Delta = (1, 2, 3)$ the common triangle to G_1 and G_2. Let $G = (V, E)$ denote the 3-sum of G_1 and G_2. We show that every vertex of $\mathcal{S}(G)$ is $\frac{1}{d}$-integral.

Let x be a vertex of $\mathcal{S}(G)$ and let x_{E_i} denote the projection of x on $\Re^{E_i}$, for $i = 1, 2$. If $x_e = 0$ or 1 for some edge of Δ, then, by contraction of this edge, we can apply Theorem 3.2 on the 2-sum and deduce that x is $\frac{1}{d}$-integral. Hence, we can now assume that $x_e \neq 0, 1$ for each edge $e \in \Delta$.

Let $\mathcal{B}$ be a family of $|E|$ linearly independent equalities for x and let $\mathcal{B}_i$ denote the subset of the equalities in $\mathcal{B}$ that are supported by G_i, for $i = 1, 2$. We distinguish two cases depending whether Δ supports a triangle equality for x or not.

We first suppose that Δ supports a triangle equality for x. W.l.o.g. we can assume that $x_{12}+x_{13}+x_{23} = 2$ (if not, apply switching). We suppose that this equality belongs to $\mathcal{B}$. Hence, $|E| = |\mathcal{B}| = |\mathcal{B}_1|+|\mathcal{B}_2|-1 = |E_1|+|E_2|-3$, implying that $|E_i|-2 \leq |\mathcal{B}_i| \leq |E_i|$, for $i = 1, 2$. But $|\mathcal{B}_2| \neq |E_2|$, else x_{E_2} would be a vertex of $\mathcal{S}(G_2)$ and, thus, x_{E_2} would be integral.

If $|\mathcal{B}_2| = |E_2| - 1$, then x_{E_2} is the convex combination of two vertices of $\mathcal{S}(G_2)$, $x_{E_2} = \alpha\chi^{\delta(A)} + \beta\chi^{\delta(B)}$, where $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and $\delta(A), \delta(B)$ are two cuts in G_2. Both cuts $\delta(A), \delta(B)$ satisfy the triangle equality: $x_{12}+x_{13}+x_{23} = 2$. Hence, at least one edge e of Δ belongs to both cuts $\delta(A), \delta(B)$, implying that $x_e = 1$, a contradiction.

If $|\mathcal{B}_2| = |E_2| - 2$, then $|\mathcal{B}_1| = |E_1|$; hence, x_{E_1} is a vertex of $\mathcal{S}(G_1)$ and, thus, x_{E_1} is $\frac{1}{d}$-integral. On the other hand, x_{E_2} is the convex combination of three vertices of $\mathcal{S}(G_2)$, $x_{E_2} = \alpha\chi^{\delta(A)} + \beta\chi^{\delta(B)} + \gamma\chi^{\delta(C)}$, where $\alpha, \beta, \gamma \geq 0$, $\alpha + \beta + \gamma = 1$ and $\delta(A), \delta(B), \delta(C)$ are cuts in G_2. From the fact that the three cuts $\delta(A), \delta(B), \delta(C)$ satisfy the equality: $x_{12} + x_{13} + x_{23} = 2$ and that $x_e \neq 0, 1$ for each edge $e \in \Delta$, we deduce that $\delta(A) \cap \Delta = \{12, 13\}$, $\delta(B) \cap \Delta = \{12, 23\}$ and $\delta(C) \cap \Delta = \{13, 23\}$. Hence, $x_{12} = \alpha + \beta$, $x_{13} = \alpha + \gamma$ and $x_{23} = \beta + \gamma$. Setting $x_{12} = \frac{a}{d}$, $x_{13} = \frac{b}{d}$, $x_{23} = 2 - \frac{a+b}{d}$ for some integers a, b, we obtain that $\alpha = \frac{a+b}{d} - 1$, $\beta = 1 - \frac{b}{d}$ and $\gamma = 1 - \frac{a}{d}$. Therefore, x_{E_2} and, thus, x, are $\frac{1}{d}$-integral.

We now suppose that Δ does not support any triangle equality for x. Hence, $|E| = |\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| = |E_1| + |E_2| - 3$, implying that $|E_i| - 3 \leq |\mathcal{B}_i| \leq |E_i|$, for $i = 1, 2$. But, $|\mathcal{B}_2| \neq |E_2|$, since $x_e \neq 0, 1$ for each edge $e \in \Delta$, and $|\mathcal{B}_1| \neq |E_1|$, since G_1 is rich (else, x_{E_1} would be a vertex of $\mathcal{S}(G_1)$ with the triangle Δ supporting no equality for x_{E_1}). Hence, $|\mathcal{B}_2| = |E_2| - 1$ or $|E_2| - 2$.

If $|\mathcal{B}_2| = |E_2| - 1$, then x_{E_2} is the convex combination of two cuts in G_2, implying easily that $x_e = 0$ or 1 for some edge $e \in \Delta$.

If $|\mathcal{B}_2| = |E_2| - 2$, then x_{E_2} is the convex combination of three vertices of $\mathcal{S}(G_2)$, $x_{E_2} = \alpha\chi^{\delta(A)} + \beta\chi^{\delta(B)} + \gamma\chi^{\delta(C)}$, where $\alpha, \beta, \gamma \geq 0$, $\alpha + \beta + \gamma = 1$ and $\delta(A), \delta(B), \delta(C)$ are cuts in G_2. Since $x_e \neq 0, 1$ for each edge $e \in \Delta$, no edge of Δ belongs to all three cuts, and every edge belongs to at least one of them. Hence, we have (up to permutation) only the following two possibilities:

- either, $\delta(A) \cap \Delta = \emptyset$, $\delta(B) \cap \Delta = \{12, 13\}$, $\delta(C) \cap \Delta = \{12, 13\}$; then, $x_{12} = \beta + \gamma$, $x_{13} = \beta$, $x_{23} = \gamma$, implying that $x_{12} - x_{13} - x_{23} = 0$.

- or, $\delta(A) \cap \Delta = \{12, 13\}$, $\delta(B) \cap \Delta = \{12, 23\}$, $\delta(C) \cap \Delta = \{13, 23\}$; then, $x_{12} = \alpha + \beta$, $x_{13} = \alpha + \gamma$, $x_{23} = \beta + \gamma$, implying that $x_{12} + x_{13} + x_{23} = 2$.

In both cases, we have a contradiction with our assumption that Δ supports no triangle equality for x. This concludes the proof that G is $\frac{1}{d}$-integral.

Finally, we verify that G is rich, i.e. that, for each vertex x of $\mathcal{S}(G)$, every triangle supports an equality for x. Take a vertex x of $\mathcal{S}(G)$. Looking through the above proof, we see that, either x is some trivial extension, or x_{E_2} is the convex combination of three cuts of G_2 while x_{E_1} is a vertex of $\mathcal{S}(G_1)$. Hence, each triangle of G supports an equality for x; in the first case, apply Remark 3.6 and, in the second case, check it directly.

□

The motivation for the notion of rich graphs comes from the 3-sum operation. Namely, we have the following result.

PROPOSITION 3.8 *Let G be a $\frac{1}{3}$-integral graph. If G is not rich, then the 3-sum of G with K_4 is not $\frac{1}{3}$-integral.*

PROOF. If G is not rich, then there exists a vertex x of $\mathcal{S}(G)$ and a triangle $\Delta = (1, 2, 3)$ of G which supports no equality for x. Up to switching, we can suppose that $x_{12} = x_{13} = x_{23} = \frac{1}{3}$. Consider K_4 on the node set $\{1, 2, 3, u_0\}$ where $u_0 \notin V(G)$. Let H denote the 3-sum of G and K_4 along Δ. Let $y \in \mathcal{S}(H)$ be defined by: $y_e = x_e$ for every edge e of G and $y_{u_0 1} = y_{u_0 2} = y_{u_0 3} = \frac{1}{6}$. Then, y is a vertex of $\mathcal{S}(H)$ which is not $\frac{1}{3}$-integral. □

As an application of the 3-sum operation, we obtain that the ΔY-operation preserves $\frac{1}{d}$-integral rich graphs. The ΔY-operation consists of replacing a triangle $\Delta = (1, 2, 3)$ in a graph by a claw, i.e. deleting the triangle Δ from G and adding a new node u_0 to G adjacent to the nodes $1, 2$ and 3.

COROLLARY 3.9 *The ΔY-operation preserves the class of $\frac{1}{d}$-integral rich graphs.*

PROOF. Let G be a $\frac{1}{d}$-integral rich graph and let $\Delta = (1,2,3)$ be a triangle of G. Consider K_4 defined on the node set $\{1,2,3,u_0\}$. By Theorem 3.7, the 3-sum of G and K_4 along the triangle Δ is $\frac{1}{d}$-integral and rich. Then, delete the edges (u_0, i) for $i = 1,2,3$. The resulting graph is $\frac{1}{d}$-integral and rich; it is precisely the ΔY-transform of G. □

4 Forbidden minors for $\frac{1}{3}$-integrality

The purpose of this section is to present some minimal forbidden minors for $\frac{1}{3}$-integrality. As a consequence, we can classify the $\frac{1}{3}$-integral graphs up to seven nodes. We also give the full description of the metric polytope $\mathcal{MP}_n$ for $n \leq 6$.

4.1 small metric polytopes

We recall the description of the metric polytopes of small dimension.

For $n = 4$, $\mathcal{MP}_4$ has $8 = 2^3$ vertices, all of them integral.

For $n = 5$, $\mathcal{MP}_5$ has 32 vertices consisting of 2^4 integral vertices and 2^4 $\frac{1}{3}$-integral vertices obtained by switching of $(\frac{2}{3}, \ldots, \frac{2}{3})$.

For $n = 6$, $\mathcal{MP}_6$ has 544 vertices consisting of 2^5 integral vertices, 2^5 $\frac{1}{3}$-integral vertices obtained by switching of $(\frac{2}{3}, \ldots, \frac{2}{3})$ and 480 vertices which are the trivial extensions of the $\frac{1}{3}$-integral vertices of $\mathcal{MP}_5$.

For $n = 7$, Grishukhin (1989) has computed all the extreme rays of the metric cone $\mathcal{MC}_7$. He found that there are 13 distinct classes (up to permutation and switching) of extreme rays. We do not know the complete description of the vertices of $\mathcal{MP}_7$.

Clearly, every extreme ray of the metric cone $\mathcal{MC}_n$ determines a vertex of the metric polytope $\mathcal{MP}_n$ which is the intersection of the ray with some triangle facet (2). In (Laurent and Poljak 1992), we conjecture that every vertex of $\mathcal{MP}_n$ can be obtained, up to switching, in this way. Equivalently, we conjecture that every fractional vertex of $\mathcal{MP}_n$ is adjacent to some integral vertex. This conjecture holds for $\mathcal{MP}_n$, $n \leq 6$, and for several classes of graphical vertices of $\mathcal{MP}_n$ constructed in (Laurent 1991).

It follows from the explicit description of $\mathcal{MP}_n$, $n = 5,6$, that K_5 and K_6 are $\frac{1}{3}$-integral and rich. Therefore, every graph on at most six nodes is $\frac{1}{3}$-integral and rich. As a consequence, any graph on 7 nodes which has a node of degree at most 3 is $\frac{1}{3}$-integral and rich (from Remark 3.6 and Theorem 3.7). K_7 is not rich; many examples of vertices of $\mathcal{MP}_7$, for which some triangle exists which supports no equality, can be found in the list of vertices from (Grishukhin 1989).

By Corollary 3.9, using the ΔY operation, new examples of rich $\frac{1}{3}$-integral graphs can be constructed, starting, for example, from K_6. One such graph is the Petersen graph.

We conclude with a remark on the possible denominators for the fractional vertices of the metric polytope. By Corollary 2.8, no vertex of $\mathcal{MP}_n$ has denominator 2. On the other hand, vertices can be constructed with arbitrary denominator $d \geq 3$.

PROPOSITION 4.1 *For every $d \geq 3$ and for every n sufficiently large, e.g. $n \geq 3d-1$, there exists a vertex of $\mathcal{MP}_n$ with denominator d.*

PROOF. We first recall a construction due to Avis (1980a). Let $G = (V, E)$ be a graph and $G' = (V', E')$ be a copy of G, where $V = \{1, \ldots, n\}$ and $V' = \{1', \ldots, n'\}$. Consider the graph G^* with node set $V \cup V' \cup \{u_e : e \in E\}$ constructed as follows. The edge set of G^* consists of the edges of G, the edges of G' and the following new edges. Join each node $i \in V$ to its twin $i' \in V'$. For each edge $e = (i, j)$ of G with $i < j$, join i and j' to u_e.

Let d_G denote the path metric of G, where $d_G(i, j)$ is the length of a shortest path from i to j in G, for $i, j \in V$. Set $\tau(G) = \max(d_G(i,j) + d_G(i,k) + d_G(j,k) : 1 \leq i < j < k \leq n)$. Define similarly d_{G^*} and $\tau(G^*)$. It is easy to check that $\tau(G^*) = \tau(G) + 2$ holds.

Define the vector $x_{G^*} \in \mathcal{MP}_N$, $N = 2n + |E|$, by $x_{G^*} = \frac{2}{\tau(G^*)} d_{G^*}$. Then, it follows from (Avis 1980a) that x_{G^*} is a vertex of $\mathcal{MP}_N$. Its denominator is $\tau(G) + 2$ or $\frac{\tau(G)+2}{2}$, according to the parity of $\tau(G)$.

Let $d \geq 3$ be an integer. Let G be a path on d nodes, then $\tau(G) = 2(d-1)$ and, therefore, x_{G^*} is a vertex of $\mathcal{MP}_{3d-1}$ with denominator d. Trivial extensions of x_{G^*} are vertices of $\mathcal{MP}_n$ with denominator d for all $n \geq 3d-1$. $\square$

For instance, the polytope $\mathcal{MP}_7$ has vertices with denominators 3, 4, 5, 6 and 7.

4.2 Forbidden minors

We have shown in Proposition 2.4 that $\frac{1}{3}$-integrality is preserved by taking minors. Robertson and Seymour have proved that, for every minor closed class of graphs, there are only finitely many minimal forbidden minors. Thus arises the problem of finding minimal forbidden minors for the class of $\frac{1}{3}$-integral graphs. We present four of them. This yields the classification of $\frac{1}{3}$-integrality for graphs up to 7 nodes.

We first give some preliminary results.

LEMMA 4.2 *Let G be a graph and let x be a fully fractional $\frac{1}{3}$-integral vertex of $\mathcal{S}(G)$. The only inequalities (6) which are satisfied as equality by x are for C triangle of G.*

PROOF. Let F, C be such that the inequality (6) is satisfied as equality by x. Let a (resp. b) denote the number of edges $e \in F$ (resp. $e \in C - F$) for which $x_e = \frac{1}{3}$. From the equality $x(F) - x(C-F) = |F| - 1$, we deduce that $\frac{1}{3}a + \frac{2}{3}(|F| - a) - \frac{1}{3}b - \frac{2}{3}(|C| - |F| - b) = |F| - 1$.

We obtain that $|F| = 2|C| + a - b - 3$. But, $a \geq 0$ and $b \leq |C| - |F|$, from which we deduce that $|C| \leq 3$, i.e. C is a triangle. □

LEMMA 4.3 *Let G be a graph and let x be a fully fractional vertex of $\mathcal{S}(G)$. For each cycle C of G, at most one of the inequalities (6) supported by C is satisfied as equality by x.*

PROOF. Let C be a cycle of G and let F, F' be two distinct subsets of C of odd cardinality. Let $x \in \mathcal{S}(G)$ satisfy the equalities $x(F) - x(C-f) = |F|-1$ and $x(F') - x(C-F') = |F'|-1$. We obtain that $|F \cap F'| - x(F \cap F') + \frac{1}{2}(|F \Delta F'| - 2) + x(C - F \cup F') = 0$. Therefore, $|F \cap F'| = x(F \cap F')$, $|F \Delta F'| = 2$ and $x(C - F \cup F') = 0$. This implies that $x_e = 1$ for $e \in F \cap F'$ and $x_e = 0$ for $e \in C - F \cup F'$. If x is fully fractional, then $F \cap F' = \emptyset$, $C = F \cup F'$, implying that $|C| = 2$, a contradiction. □

COROLLARY 4.4 *Let $G = (V, E)$ be a $\frac{1}{3}$-integral graph on 7 nodes. If G has at most $|E|$ distinct triangles, then G is rich.*

PROOF. Let x be a vertex of $\mathcal{S}(G)$. We show that each triangle of G supports an equality for x. Suppose first that $x_e = 0$ or 1 for some edge $e \in E$. Let Δ be a triangle of G. If Δ contains the edge e, then Δ trivially supports an equality for x. Otherwise Δ is a triangle in the graph G/e, obtained by contracting the edge e. Since G/e is on 6 nodes, it is rich. Hence, Δ supports an equality for the projection of x on G/e. Therefore, Δ also supports an equality for x. We suppose now that x is fully fractional. From Lemmas 4.2 and 4.3, we deduce that G has exactly $|E|$ triangles and each of them supports an equality for x. This shows that G is rich. □

In the following result, we classify the graphs on 7 nodes that are $\frac{1}{3}$-integral. If E is a subset of edges of K_7, $K_7 - E$ denotes the graph obtained by deleting from K_7 the edges of E. For example, $K_7 - (C_4 + P_3)$ is $K_7 - E$ where E is the disjoint union of a cycle of length 4 and a path on 3 nodes; $K_7 - (C_4 + P_3)$ is obtained by taking the 3-sum of two copies of K_5 along a triangle and then deleting two edges of this triangle. Also, $K_8 - (K_{3,3} + K_2)$ is $K_8 - E$ where E is the disjoint union of $K_{3,3}$ and an edge; so $K_8 - (K_{3,3} + K_2)$ is obtained by taking the 2-sum of two copies of K_5 along an edge and then deleting this edge.

THEOREM 4.5 *(i) The graphs $K_7 - C_7$, $K_7 - C_5$, $K_7 - (C_4 + P_3)$ and $K_8 - (K_{3,3} + K_2)$ are minimal forbidden minors for the class of $\frac{1}{3}$-integral graphs.*

(ii) Every graph on 7 nodes not containing $K_7 - C_7$, $K_7 - C_5$, or $K_7 - (C_4 + P_3)$ is $\frac{1}{3}$-integral and, moreover, rich.

PROOF. The proof of (i) relies partly on computer check. Namely, we checked by computer that

- $K_7 - C_7$, $K_7 - C_5$, $K_7 - (C_4 + P_3)$ are, respectively, $\frac{1}{5}$, $\frac{1}{5}$, $\frac{1}{4}$-integral
- $K_7 - C_3$ is $\frac{1}{3}$-integral.

For each of the graphs $K_7 - C_7$, $K_7 - C_5$, $K_7 - (C_4 + P_3)$, $K_8 - (K_{3,3} + K_2)$, we give a vertex x of $\mathcal{S}(G)$ which is not $\frac{1}{3}$-integral.

Let $x \in \Re^{\binom{7}{2}}$ such that $x_{14} = x_{15} = x_{36} = x_{37} = \frac{1}{5}$, $x_{13} = x_{24} = x_{27} = x_{46} = x_{57} = \frac{2}{5}$, $x_{16} = x_{35} = \frac{3}{5}$, $x_{25} = x_{26} = x_{47} = \frac{4}{5}$. Then, x is a vertex of $\mathcal{S}(K_7 - C_7)$ where C_7 is the cycle (1,2,3,4,5,6,7).

Let $x_{12} = x_{23} = x_{34} = x_{45} = x_{15} = x_{67} = \frac{4}{5}$, $x_{i6} = x_{i7} = \frac{3}{5}$ for $1 \leq i \leq 5$. Then, x is a vertex of $\mathcal{S}(K_7 - C_5)$ where C_5 is the cycle (1,2,3,4,5).

Let $x_{13} = x_{14} = x_{25} = x_{36} = x_{46} = \frac{1}{4}$, $x_{12} = x_{34} = x_{67} = \frac{2}{4}$ and $x_{15} = x_{23} = x_{24} = x_{37} = x_{47} = x_{57} = \frac{3}{4}$. Then, x is a vertex of $\mathcal{S}(K_7 - (C_4 + P_3))$, where C_4 is the cycle (1,7,2,6) and P_3 is the path (3,5,4).

The graph $K_8 - K_{3,3}$ is obtained by taking the 2-sum of two copies of K_5 along an edge e. We gave in Remark 3.4 a $\frac{1}{6}$-integral vertex x of the polytope $\mathcal{S}(K_8 - K_{3,3})$. In fact, if we project out the edge e, the projection of x remains a vertex of $\mathcal{S}(K_8 - (K_{3,3} + e))$. Therefore, $K_8 - (K_{3,3} + e)$ is not $\frac{1}{3}$-integral. On the other hand, it is easily seen that every minor of $K_8 - (K_{3,3} + e)$ is $\frac{1}{3}$-integral.

We now verify that, for each of the graphs $G = K_7 - C_7$, $K_7 - C_5$, $K_7 - (C_4 + P_3)$, every minor of G is $\frac{1}{3}$-integral. This is clear for every contraction minor, since it is a subgraph of K_6. Let $G - e$ be a deletion minor. If the deleted edge e is adjacent to a node of degree at most 4 in G, then $G - e$ has a node of degree at most 3 and, hence, is $\frac{1}{3}$-integral. Therefore, every minor of $K_7 - C_7$ is $\frac{1}{3}$-integral, since $K_7 - C_7$ is regular of degree 4. All nodes of $K_7 - C_5$ have degree 4 except two adjacent nodes which have degree 5. If e is the edge joining them, then $K_7 - (C_5 + e)$ is planar and, therefore, is $\frac{1}{3}$-integral. All the nodes of $K_7 - (C_4 + P_3)$ have degree 4 except two adjacent nodes which have degree 5. If e is the edge joining them, then $K_7 - (C_4 + P_3 + e)$ is contained in $K_7 - C_3$ and, therefore, is $\frac{1}{3}$-integral. This shows the part (i) of Theorem 4.5.

We prove (ii). Let G be a graph on 7 nodes that does not contain any of $K_7 - C_7$, $K_7 - C_5$, $K_7 - (C_4 + P_3)$ as a subgraph. If G has a node of degree at most 3, then G is $\frac{1}{3}$-integral and rich. So we can suppose that all the nodes of G have degree at least 4 in G. Hence, all nodes have degree at most 2 in the complement $\bar{G}$ of G, i.e. $\bar{G}$ is a disjoint union of cycles. Since $\bar{G} \not\subseteq C_7$, $\bar{G}$ contains a cycle. If $\bar{G}$ contains a cycle of length 3, then G is contained in $K_7 - C_3$ and, therefore, G is $\frac{1}{3}$-integral. If $\bar{G}$ contains a cycle of length 4, then $\bar{G} = C_4 + C_3$, since $\bar{G}$ is not contained in $C_4 + P_3$. Therefore, G is again contained in $K_7 - C_3$. If $\bar{G}$ contains a cycle of length 5, then $\bar{G} = C_5 + K_2$. Therefore, G is integral since it is planar. If $\bar{G}$ contains a cycle of length 6, then $G = K_7 - C_6$ is $\frac{1}{3}$-integral. Indeed, $K_7 - C_6$ has 14 chordless cycles (including 11 triangles and 3 cycles of length 4) and 15 edges. By Lemma 4.3, every vertex of $\mathcal{S}(K_7 - C_6)$ has some integral coordinate and thus is $\frac{1}{3}$-integral, since it is the trivial extension of a vertex of the cycle polytope of a graph on 6 nodes.

In order to conclude the proof of (ii), we must show that G is rich. By the above argument, it suffices to verify that both $K_7 - C_3$ and $K_7 - C_6$ are rich. The graph $K_7 - C_6$ has 11 triangles; therefore, it is rich, by Corollary 4.4. We cannot apply Corollary 4.4 for showing that $K_7 - C_3$ is rich since this graph has 22 triangles and 18 edges. But it can be checked directly as follows.

Let $G = K_7 - C_3$ be defined on the nodes $\{1,2,3,4,5,6,7\}$ and the deleted triangle C_3 be (5,6,7). Let x be a vertex of $\mathcal{S}(G)$. If x has some integral component, then every triangle of G supports an equality for x. Let x be fully fractional, so its components are $\frac{1}{3}$, $\frac{2}{3}$. Call a triangle Δ of G *bad* if it supports no equality for x, i.e. x takes the values $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, or $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ on the edges of Δ. At most 4 triangles of G are bad. There are 4 triangles on the nodes $\{1,2,3,4\}$. Among them, the number of bad triangles can be 0, 2 or 4. If the 4 triangles on $\{1,2,3,4\}$ are bad, then $x_{12} = x_{13} = x_{14} = x_{23} = x_{24} = x_{34} = \frac{1}{3}$ (up to switching). Clearly, no such x exists for which all the remaining 18 triangles of G support an equality. If two of the triangles on $\{1,2,3,4\}$ are bad, then e.g. $x_{12} = x_{13} = x_{14} = x_{23} = x_{24} = \frac{1}{3}$, $x_{34} = \frac{2}{3}$ (up to switching). It is again impossible to find such x for which at most two of the remaining 18 triangles are bad. Let the four triangles on $\{1,2,3,4\}$ support an equality for x, i.e. $x_{12} = x_{13} = x_{14} = x_{23} = x_{24} = x_{34} = \frac{2}{3}$ (up to switching). We look at the possibilities for x_{ij}, $1 \leq i \leq 4$, $5 \leq j \leq 7$. Fix $j \in \{5,6,7\}$. If $x_{ij} = \frac{1}{3}$ for exactly one of the edges $1j, 2j, 3j, 4j$, say $x_{1j} = \frac{1}{3}$, then no triangle equality covers the edge $1j$, contradicting the fact that x is a vertex. The same holds if $x_{ij} = \frac{1}{3}$ for three of the edges $1j, 2j, 3j, 4j$. If $x_{ij} = \frac{1}{3}$ for two (resp. four) of the edges $1j, 2j, 3j, 4j$, then four (resp. six) of the six triangles going through node j are bad. This contradicts the fact that x is a vertex since the equalities supported by triangles on $\{1,2,3,4,5,6,7\} - \{j\}$ have rank at most 14. □

5 Box $\frac{1}{3}$-integral graphs

We have seen that the 2-sum operation does not preserve $\frac{1}{3}$-integrality. This led us to the study of a stronger notion, box $\frac{1}{3}$-integrality, which is preserved by 2-sums. Box $\frac{1}{3}$-integrality is a stronger property than $\frac{1}{3}$-integrality. Namely, we ask not only that the polytope $\mathcal{S}(G)$ has all its vertices $\frac{1}{3}$-integral, but also that each slice of $\mathcal{S}(G)$ determined by adding the box constraints: $l_e \leq x_e \leq u_e$ for $e \in E$, has only $\frac{1}{3}$-integral vertices, for all choices of $\frac{1}{3}$-integral bounds l and u.

DEFINITION 5.1 *The graph G is box $\frac{1}{3}$-integral if the polytope $\mathcal{S}(G) \cap \{x : l_e \leq x_e \leq u_e,\ e \in E\}$ is empty or has only $\frac{1}{3}$-integral vertices, for all $l,\ u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^E$.*

Equivalently, the graph $G = (V, E)$ is box $\frac{1}{3}$-integral if, for every $l, u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^E$ such that $\mathcal{MP}_n \cap \{x : l_e \leq x_e \leq u_e\} \neq \emptyset$, and for every objective function c supported by G, the linear program: $\max(c^T x : x \in \mathcal{MP}_n, l_e \leq x_e \leq u_e \ for\ e \in E)$ admits a $\frac{1}{3}$-integral optimizing vector.

We are able to characterize the class of box $\frac{1}{3}$-integral graphs. Recall that a graph G is called series parallel if G is a subgraph of a graph which can be obtained by iterated 2-sums of a collection of copies of K_3. Equivalently, G is series parallel if G does not contain any K_4 minor.

THEOREM 5.2 *A graph G is box $\frac{1}{3}$-integral if and only if G is series parallel.*

The proof of Theorem 5.2 consists of the following steps:

- box $\frac{1}{3}$-integrality is preserved by 0-, 1- and 2-sums.
- K_3 is box $\frac{1}{3}$-integral, but K_4 is not box $\frac{1}{3}$-integral.

The fact that 0- and 1-sums preserve box $\frac{1}{3}$-integrality is proved in the same way as for $\frac{1}{3}$-integrality. The result about the 2-sum needs two preliminary lemmas.

In the next lemma, we show that every point in a slice of the metric polytope can be rounded to a $\frac{1}{3}$-integral point of the slice.

LEMMA 5.3 *Take $l, u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{\binom{n}{2}}$ such that $\mathcal{MP}_n \cap \{x : l \leq x \leq u\} \neq \emptyset$. Given $x \in \mathcal{MP}_n \cap \{x : l \leq x \leq u\}$, there exists $y \in \mathcal{MP}_n \cap \{x : l \leq x \leq u\}$ such that y satisfies:*

(i) $y_e = x_e$ *if* $x_e \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

(ii) $y_e = \frac{1}{3}$ *if* $0 < x_e < \frac{1}{3}$

(iii) $y_e = \frac{2}{3}$ *if* $\frac{2}{3} < x_e < 1$

(iv) $y_e \in \{\frac{1}{3}, \frac{2}{3}\}$ *if* $\frac{1}{3} < x_e < \frac{2}{3}$.

PROOF. We will proceed by induction on $n \geq 2$. The statement holds trivially if $n = 2$. Let $n \geq 3$ be given. We distinguish two cases.

Assume first that $0 < x_e < 1$ for all edges. Then, we define y by:

$$y_e = \begin{cases} \frac{1}{3} & \text{if } 0 < x_e \leq \frac{1}{3} \\ \frac{2}{3} & \text{if } \frac{2}{3} \leq x_e < 1 \\ \frac{1}{3} \text{ or } \frac{2}{3} & \text{if } \frac{1}{3} < x_e < \frac{2}{3} \end{cases}$$

Clearly, $y \in \mathcal{MP}_n$ and satisfies $l \leq y \leq u$.

Assume now that $x_e = 0, 1$ for some edge e; we can consider only the case of $x_e = 0$ due to switching. Let $e = (1, n)$. Since $x_{1n} = 0$, $x_{1i} = x_{in}$ for all $2 \leq i \leq n$. Set $l'_{1i} = \max(l_{1i}, l_{in})$ and $u'_{1i} = \min(u_{1i}, u_{in})$ for $2 \leq i \leq n$, and $l'_{ij} = l_{ij}$, $u'_{ij} = u_{ij}$ otherwise. Let x' denote the projection of x in $\mathcal{MP}_{n-1}$. Clearly, x' satisfies $l' \leq x' \leq u'$. By the induction hypothesis, there exists y' satisfying the statement for x' and the bounds l' and u'. Let y be the 0-extension of y'. Then, y satisfies the statement for x and the bounds l and u. □

The following lemma deals with sensitivity of optimization over slices of the metric polytope when the objective function varies on a single edge.

LEMMA 5.4 *Take l and $u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{\binom{n}{2}}$ such that $\mathcal{MP}_n \cap \{x : l \leq x \leq u\} \neq \emptyset$ and $c \in \Re^{\binom{n}{2}}$. For $t \in \Re$, define $c(t) \in \Re^{\binom{n}{2}}$ by $c(t)_e = c_e$ for all edges e except $c(t)_f = c_f + t$ for a fixed edge f. For $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, we define the set M_α consisting of the scalars $t \in \Re$ for which the linear program $\max(c(t)^T x : x \in \mathcal{MP}_n$ and $l \leq x \leq u)$ admits a $\frac{1}{3}$-integral optimizing vector x satisfying $x_f = \alpha$. Then, the set M_α is a closed interval.*

PROOF. We show that M_α is convex. Let $t, t + s \in M_\alpha$ and $0 \leq \lambda \leq 1$ be given. We show that $t + \lambda s \in M_\alpha$.

Let C_0 (respectively, C_1, C) denote the maximum value for the objective function $c(t)^T x$ (respectively, $c(t+s)^T x$, $c(t + \lambda s)^T x$) optimized over $\mathcal{MP}_n \cap \{x : l \leq x \leq u\}$ and let x_0 (respectively, x_1, x) denote the corresponding optimizing vectors. By assumption, we can suppose that $x_0(f) = x_1(f) = \alpha$.

First, note that, for any $y \in \Re^{\binom{n}{2}}$, $c(t + \lambda s)^T y = c(t)^T y + \lambda s y_f$, and $c(t + \lambda s)^T y = c(t + s)^T y - (1 - \lambda) s y_f$.

In particular, $c(t + \lambda s)^T x_0 = C_0 + \lambda s \alpha$, and $c(t + \lambda s)^T x_1 = C_1 - (1 - \lambda) s \alpha$, implying that $(1 - \lambda) C_0 + \lambda C_1 \leq C$.

On the other hand, we have that: $C = c(t + \lambda s)^T x = c(t)^T x + \lambda x_f \leq C_0 + \lambda x_f$, and $C = c(t+\lambda)^T x = c(t+s)^T x - (1-\lambda) x_f \leq C_1 - (1-\lambda) x_f$, implying that $(1-\lambda) C_0 + \lambda C_1 \geq C$.

Therefore, the equality $(1-\lambda)C_0 + \lambda C_1 = C$ holds and, in consequence, each of the vectors x_0 and x_1 is an optimizing vector for the program $\max(c(t+\lambda s)^T x : x \in \mathcal{MP}_n, l \leq x \leq u)$. Hence, $t + \lambda s \in M_\alpha$.

Using compactness of the set $\mathcal{MP}_n \cap \{x : l \leq x \leq u,\ x(f) = \alpha\}$, it is easy to see that the set M_α is closed. □

THEOREM 5.5 *The k-sum operation, $k = 0, 1, 2$, preserves box $\frac{1}{3}$-integrality.*

PROOF. For $k = 0, 1$, the proof is identical to that of Theorem 3.1.

We now show that the 2-sum operation preserves box $\frac{1}{3}$-integrality. Take two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, having a common edge f and denote their 2-sum by $G = (V, E)$. We suppose that G_i is box $\frac{1}{3}$-integral, for $i = 1, 2$, and we show that G is box $\frac{1}{3}$-integral. Take $l, u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^E$ such that $\mathcal{MP}_n \cap \{x : l_e \leq x_e \leq u_e, e \in E\} \neq \emptyset$ and $c \in \Re^E$. Let y be an optimizing vector for the program:

$(P)\ \max(c^T x : x \in \mathcal{MP}_n, l_e \leq x_e \leq u_e, e \in E)$.

Observe, first, that we may assume that each interval $[l_e, u_e]$ is tight for y, i.e. satisfies: $l_e = u_e = y_e$ if $y_e \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and $u_e - l_e = \frac{1}{3}$ otherwise. Indeed, if it is not the case, define l', u' by the above conditions; then, y is also an optimizing vector for the program $\max(c^T x : x \in \mathcal{MP}_n, l'_e \leq x_e \leq u'_e, e \in E)$, and the bounds l', u' are tight for y.

Define $c_i \in \Re^{E_i}$ by: $c_i(e) = c(e)$ for all edges $e \in E_i$, except $c_1(f) = c(f)$ and $c_2(f) = 0$.

Let us first suppose that $l_f = u_f := \alpha$. By assumption, the program $\max(c_i^T x : x \in \mathcal{MP}(V_i), l_e \leq x_e \leq u_e, e \in E_i)$ admits a $\frac{1}{3}$-integral optimizing vector z_i, for $i = 1, 2$. Since $z_1(f) = z_2(f) = \alpha$, we can construct the 2-union z of z_1 and z_2. Then, z is a $\frac{1}{3}$-integral optimizing vector for the program (P).

We can now assume that (l_f, u_f) is $(0, \frac{1}{3})$, or $(\frac{1}{3}, \frac{2}{3})$, or $(\frac{2}{3}, 1)$. For $t \in \Re$, we consider the translate $c_i(t)$ of the objective function c_i defined by: $c_i(t)(e) = c_i(e)$ for all edges $e \in E_i$, except $c_1(t)(f) = c_1(f) + t$ and $c_2(t)(f) = c_2(f) - t$. For $i = 1, 2$, and for $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, we define the set M^i_α, consisting of the scalars $t \in \Re$ for which the program $\max(c_i(t)^T x : x \in \mathcal{MP}(V_i), l_e \leq x_e \leq u_e, e \in E_i)$ admits a $\frac{1}{3}$-integral optimizing vector taking the value α on the edge f. Hence, $M^i_\alpha = \emptyset$ if $\alpha \neq l_f, u_f$ and, by Lemma 5.4, $M^i_{l(f)}$ and $M^i_{u(f)}$ are two closed intervals covering $\Re$, for $i = 1, 2$.

Consider the program $\max(c_1(t)^T x : x \in \mathcal{MP}(V_1), l_e \leq x_e \leq u_e, e \in E_1)$ for large t, $t \to +\infty$, and for small t, $t \to -\infty$, respectively. Hence, if $MP(V_1) \cap \{x : l_e \leq x_e \leq u_e, e \in E_1, x(f) = l(f)\} \neq \emptyset$, then $M^1_{l(f)} \neq \emptyset$ and, if $MP(V_1) \cap \{x : l_e \leq x_e \leq u_e, e \in E_1, x(f) = u(f)\} \neq \emptyset$, then $M^1_{u(f)} \neq \emptyset$; in fact, any t small enough belongs to $M^1_{l(f)}$ and any t large enough belongs to $M^1_{u(f)}$. In the same way, if $MP(V_2) \cap \{x : l_e \leq x_e \leq u_e, e \in E_2, x(f) = l(f)\} \neq \emptyset$, then $M^2_{l(f)} \neq \emptyset$ (it contains any t large enough) and, if

$MP(V_2) \cap \{x : l_e \le x_e \le u_e, e \in E_2, x(f) = u(f)\} \ne \emptyset$, then $M^2_{u(f)} \ne \emptyset$ (it contains any t small enough). Therefore, we can always find some $t \in M^1_\alpha \cap M^2_\alpha$ for $\alpha = l(f)$ or $u(f)$, except in the cases when $M^1_{u(f)} = M^2_{l(f)} = \emptyset$, or $M^1_{l(f)} = M^2_{u(f)} = \emptyset$. But these two cases cannot occur; to see it, we use Lemma 5.3.

Indeed, if $(l_f, u_f) = (0, \frac{1}{3})$, then, by Lemma 5.3, we can find $y \in \mathcal{MP}(V) \cap \{x : l \le x \le u\}$ such that $y_f = \frac{1}{3}$. By the above observations, we deduce that $M^1_{u(f)}$ and $M^2_{u(f)}$ are both nonempty. Similarly, if $(l_f, u_f) = (\frac{2}{3}, 1)$, then Lemma 5.3 produces y with $y_f = \frac{2}{3}$ and, thus, both sets $M^1_{l(f)}$ and $M^2_{l(f)}$ are nonempty. Also, in the case $(l_f, u_f) = (\frac{1}{3}, \frac{2}{3})$, we have such y with, say, $y_f = \frac{1}{3}$ and, then, $M^1_{l(f)}$ and $M^2_{l(f)}$ are nonempty.

In consequence, we can always find some $t \in M^1_\alpha \cap M^2_\alpha$, for $\alpha = l(f)$ or $u(f)$. Then, for such t, there exists a $\frac{1}{3}$-integral optimizing vector z_i for the program $\max(c_i(t)^T x : x \in \mathcal{MP}(V_i), l_e \le x_e \le u_e, e \in E_i)$ and satisfying $z_i(f) = \alpha$. Therefore, we can construct the 2-union z of z_1 and z_2 which is a $\frac{1}{3}$-integral optimizing vector for the program (P). □

LEMMA 5.6 *K_3 is box $\frac{1}{3}$-integral.*

PROOF. We show that the polytope $\mathcal{MP}_3 \cap \{x : l \le x \le u\}$ has only $\frac{1}{3}$-integral vertices for every $l, u \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^3$. Let x be a vertex of the polytope $\mathcal{MP}_3 \cap \{x : l \le x \le u\}$ and let $\mathcal{B}$ be a set of three linearly independent active constraints at x. $\mathcal{B}$ contains some triangle equalities and some bounding equalities: $x_e = l_e$, or $x_e = u_e$.

- If $\mathcal{B}$ contains three triangle equalities, then x is a vertex of $\mathcal{MP}_3$ and, thus, x is 0-1 valued.
- If $\mathcal{B}$ contains two triangle equalities, then we deduce that $x_e = 0$ or 1, for some edge e; but $\mathcal{B}$ contains another bounding equality, say on edge f, $f \ne e$. Then, two coordinates of x are $\frac{1}{3}$-integral and, thus, the third one too.
- If $\mathcal{B}$ contains only one triangle equality and two bounding equalities, or if $\mathcal{B}$ contains three bounding equalities, then x is clearly $\frac{1}{3}$-integral.

□

REMARK 5.7 *The graph K_4 is not box $\frac{1}{3}$-integral. For example, consider the vector $x \in \mathcal{MP}_4$ defined by: $x_{12} = x_{13} = x_{14} = \frac{1}{6}$ and $x_{23} = x_{24} = x_{34} = \frac{1}{3}$. Then, x is a vertex of the polytope $\mathcal{MP}_4 \cap \{x : 0 \le x_{ij} \le \frac{1}{3}, 1 \le i < j \le 4\}$.*

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